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OPTION PRICE WHEN THE STOCK IS A SEMIMARTINGALE

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Abstract

The purpose of this note is to give a PDE satisfied by a call option when the price process is a semimartingale. The main result generalizes the PDE in the case when the stock price is a diffusion. Its proof uses Meyer-Tanaka and occupation density formulae. Presented approach also gives a new insight into the classical Black-Scholes formula. Rigorous proofs of some known results are also given.

1 Introduction

In this note we are concerned with pricing a call option. It is well known, see e.g. Shiryaev (1999), that in a complete market its price at time zero is given by

$$C = C(S, K, T) = e^{-rT} E((S_T - K)^+ | S_0 = S), \tag{1}$$

where T is maturity, S_t, $0 \le t \le T$ is the price of the stock, K is the strike and r is the riskless interest rate. In the classical model of Black and Scholes S_T is a LogNormal random variable $LN(\mu, \sigma^2 T)$, and in this case equation (1) gives

$$C = S\Phi(h) - Ke^{-rT}\Phi(h - \sigma\sqrt{T}), \quad h = \frac{\ln(S/K) + (r + \frac{1}{2}\sigma^2 T)}{\sigma\sqrt{T}}. \tag{2}$$

Auxiliary results are as follows. $\Delta = \frac{\partial C}{\partial S}$ is known as the delta of the option (it gives the number of shares in a self-financing portfolio replicating the option). In the above formula $\frac{\partial C}{\partial S} = \Phi(h)$, which is rather strange, as S enters the formula in two places in a nonlinear way. We show that this is true in general, and that in the most general situation $C = \Delta S + \kappa K$, which can be seen as an analogue of the Black-Scholes formula. One practically does not need any assumptions for this result besides that the expectation exists.
It is well-known that in order to price options by no-arbitrage arguments the market model should not admit arbitrage opportunities. A necessary condition for this is that the price

process must be a semimartingale, see e.g. Shiryaev (1999). The main result shows that when the price process is a continuous semimartingale, then the price of an option satisfies a partial differential equation (PDE). It is well-known that when the price process is a diffusion, then a PDE holds in the backward variables t and x, but surprisingly for a general continuous semimartingale it is possible to have a PDE in the variables T and K. It turned out that such a PDE is known in the literature when the price process is a martingale diffusion, i.e. when $dS_t = b(S_t,t)\,dB_t$ for some deterministic function $b(x,t)$, Dupire (1997) and Bartells (2000). Their method relies on diffusion properties of the process, the sketch of the proof given there uses forward equations for the transition density. The proof given here is direct and appears to be new. In the case of diffusions it recovers the result under less stringent conditions. The result also gives a new insight into the pricing formula.

2 Results

Before stating the main result we need two auxiliary propositions, which are of interest in their own right.

Theorem 1 *Assume that $ES_T^+ < \infty$ and set $F(y) = e^{-rT}E(S_T/S - y)^+$. Assume that $F(y)$ is differentiable in y in an open interval I. Then in the domain $K/S \in I$ the option price as a function of the present stock price S and strike K satisfies*

$$C = S\frac{\partial C}{\partial S} + K\frac{\partial C}{\partial K}. \tag{3}$$

Theorem 2 *Let $\Delta = \frac{\partial C}{\partial S}$ and $\kappa = \frac{\partial C}{\partial K}$. Under the conditions of the previous theorem, Δ and κ of the option satisfy*

$$\Delta = \frac{\partial C}{\partial S} = \frac{1}{S}e^{-rT}E(S_T I(S_T > K)) \tag{4}$$

$$\kappa = \frac{\partial C}{\partial K} = -e^{-rT}P(S_T > K). \tag{5}$$

From equations (3) and (5) we have

Corollary 3 *Under the conditions of Theorem 1*
(a) price of the option satisfies

$$C = \Delta S + \kappa K.$$

(b) $\frac{\partial^2 C}{\partial K^2}$ exists if and only if S_T has a density f_T, moreover

$$f_T(K) = e^{rT}\frac{\partial^2 C}{\partial K^2}. \tag{6}$$

Proof of Theorem 1. The arbitrage-free price of the option is given by

$$\begin{aligned} C &= e^{-rT}E((S_T - K)^+|S_0 = S) \\ &= Se^{-rT}E((S_T/S - K/S)^+|S_0 = S) = SF(K/S) \end{aligned} \tag{7}$$

Using (7) it is easy to see that the PDE (3) is satisfied. □
Note here that on the other hand, a general solution to (3) is given by (7) for *some* function F.

Proof of Theorem 2. Let $h(x,K) = (x-K)^+$, then

$$C = e^{-rT} Eh(S_T, K). \tag{8}$$

Using $\frac{\partial}{\partial K} h = -I(x > K)$ at any point $x \neq K$, we find by changing expectation and derivative (using dominated convergence) that (5) holds.
We have also from (8)

$$C = e^{-rT} ES_T I(S_T > K) - e^{-rT} KP(S_T > K), \tag{9}$$

where $I(A)$ is the indicator of set A. Using (3)

$$S\Delta = e^{-rT} E(S_T I(S_T > K)). \tag{10}$$

The expressions for Δ and κ now follow. □

Remark. The formula (6) with an indication of proof is given in Breeden and Litzenberger (1978) and Dupire (1994) in the context of diffusion models for the stock price. Here a rigorous proof is given under minimal assumptions. The next Theorem is the main result of the note.

Theorem 4 *Let S_t be a continuous semimartingale, such that there exists a martingale measure Q under which the process $S_t e^{-rt}$ is an H^1 martingale on $[0, \top]$. Suppose also the conditions of Theorem 1, that S_T possesses a density for any $T \leq \top$, and that $\langle S \rangle_T$ is differentiable. Then the following PDE is satisfied*

$$\frac{\partial C}{\partial T} = \frac{1}{2} H_T(K) \frac{\partial^2 C}{\partial K^2} - rK \frac{\partial C}{\partial K}, \tag{11}$$

where

$$H_T(K) = E\left(\frac{d\langle S \rangle_T}{dT} | S_T = K\right), \tag{12}$$

and the initial condition is given by $C(K, 0) = (S_0 - K)^+$.

Remark. The assumptions of the theorem are rather general.
1. Existence of the martingale measure Q is a sufficient condition for the model to be free of arbitrage; in some models it is also a necessary condition.
2. If filtration is generated by Brownian motion then $\langle S \rangle_T$ is absolutely continuous due to the predictable representation property.

Proof of Theorem 4: It is easy to see that under the risk-neutral measure $S_t = S_0 + \int_0^t rS_u du + M_t$, where M_t is an H^1 martingale. Let L_T^K be the local time of S at K. By the definition of the local time or Meyer-Tanaka's formula (see e.g. Protter (1992) or Klebaner (1998))

$$(S_T - K)^+ = (S_0 - K)^+ + \int_0^T I(S_t > K)\, dS_t + \frac{1}{2} L_T^K$$

We can see by using the Davis inequality (ibid) that $\int_0^T I(S_t > K) dM_t$ is a martingale, and we have from the representation of S_t above by taking expectation

$$E(S_T - K)^+ = E(S_0 - K)^+ + rE \int_0^T I(S_t > K) S_t\, dt + \frac{1}{2} E(L_T^K). \tag{13}$$

Next we show that

$$\frac{\partial}{\partial T}E(L_T^K) = f_T(K)H_T(K), \tag{14}$$

where $f_T(K)$ is the density of S_T. The occupation density formula (ibid) used for a positive bounded function g gives

$$\int L_T^K g(K)\,dK = \int_0^T g(S_t)\,d\langle S\rangle_t. \tag{15}$$

Taking expectations in the above formula

$$\int E(L_T^K)g(K)\,dK = E\int_0^T g(S_t)\,d\langle S\rangle_t = \int_0^T E(g(S_t)\frac{d\,\langle S\rangle_t}{dt})\,dt.$$

Proceeding

$$\int E(L_T^K)g(K)\,dK = \int_0^T E(g(S_t)E(\frac{d\langle S\rangle_t}{dt}|S_t))\,dt = \int_0^T \int g(K)H_t(K)f_t(K)\,dK\,dt.$$

Changing the order of integration

$$\int E(L_T^K)g(K)\,dK = \int g(K)\int_0^T f_t(K)H_t(K)\,dt\,dK.$$

It follows (since g is arbitrary) that $E(L_T^K) = \int_0^T H_t(K)f_t(K)\,dt$, and (14) follows.
Using the notation $h(x) = (x-K)^+$ we see from (13) that $Eh(S_T)$ is differentiable in T, moreover with (5) and (6) we have

$$\begin{aligned}\frac{\partial Eh(S_T)}{\partial T} &= rEh(S_T) + rKP(S_T > K) + \frac{1}{2}f_T(K)H_T(K)\\ &= rEh(S_T) - rKe^{rT}\frac{\partial C}{\partial K} + \frac{1}{2}e^{rT}H_T(K)\frac{\partial^2 C}{\partial K^2}.\end{aligned}$$

As $C = e^{-rT}Eh(S_T)$, the stated PDE now follows. □

Using the relation between solutions of PDE's and expectations of functions of diffusions we obtain that the option price can be calculated as if the strike K were a diffusion process depreciating at rate r and diffusion term H.

Corollary 5 *Assume the conditions of Theorem 4. Then*

$$C(K,T) = E((S_0 - K_T)^+|K_0 = K), \tag{16}$$

where K_t solves the following SDE on $[0,T]$

$$dK_t = -rK_t\,dt + \sqrt{H_T(K_t)}\,dB_t, \ \ \textit{with}\ \ K_0 = K. \tag{17}$$

Proof of Corollary 5:
Let $\top \geq T$ be fixed. Introduce the following notations: $t = \top - T$, $C'(K,t) := C(K,T) = C(K, \top - t)$, $H_t'(K) := H_{\top - t}(K) = H_T(K)$. Then $\frac{\partial C}{\partial T} = -\frac{\partial C'}{\partial t}$. Thus the PDE for C' is

$$\frac{1}{2}H_t'(K)\frac{\partial^2 C'}{\partial K^2} - rK\frac{\partial C'}{\partial K} + \frac{\partial C'}{\partial t} = 0.$$

This is the backward equation if K_t is a diffusion with the generator

$$L_t f = \frac{1}{2} H_t'(K) \frac{\partial^2 f}{\partial K^2} - rK \frac{\partial f}{\partial K}. \tag{18}$$

The initial condition $C(K,0) = (S_0 - K)^+$ becomes the boundary condition $C'(K,\top) = (S_0 - K)^+$. Thus (see e.g. [7]) its solution can be represented as $C'(K,t) = E((S_0 - K_\top)^+ | K_t = K)$. But $C'(K,t) = C(K,T)$ for any $0 \le t \le \top$. Therefore taking $t = 0$ we have $T = \top$ and the result follows. □

In general the function $H_T(K)$ is hard to calculate. Below we give some simple examples. In the classical case this approach gives a representation for the pricing formula given by (19).

Example. Black-Scholes model. The dynamics for the stock price is assumed to be $dS_t = rS_t\,dt + \sigma S_t\,dB_t$. Hence we have $H_T(K) = E(\frac{d\langle S\rangle_T}{dT} | S_T = K) = E(\frac{d\int_0^T \sigma^2 S_u^2\,du}{dT} | S_T = K) = \sigma^2 K^2$. By Corollary 5

$$C(K,T) = E((S_0 - K_T)^+ | K_0 = K), \tag{19}$$

where K_t solves the SDE $dK_t = -rK_t\,dt + \sigma K_t\,dB_t$. In this case it is easy to see (using calculation for LogNormal variable K_T) that the classical Black-Scholes formula results.

Example. A Stochastic Volatility model of Bachelier type. The model for the stock price is $dS_t = v_t\,dB_t$, $dv_t = \sigma\,dB_t$, $r = 0$.
$H_T(K) = E(v_T^2 | S_T = K) = \sigma^2 T + 2\sigma K$. The PDE satisfied by the option is
$\frac{\partial C(K,T)}{\partial T} = (\frac{\sigma^2 T}{2} + \sigma K)\frac{\partial^2 C}{\partial K^2}$.

Remark The SDE for K_t above can also be obtained by the change of numéraire under the measure that renders e^{rt}/S_t a martingale, $K_t = K/S_t$ and the price of the option (in units of stock) is $E(1 - K/S_T)^+$. I thank the referee and Erik Schlögl for this remark. I also thank Jia-An Yan for useful comments.

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