Elect. Comm. in Probab. **5** (2000) 85–90

**ELECTRONIC COMMUNICATIONS in PROBABILITY**

# ON THE COVER TIME OF PLANAR GRAPHS

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*submitted* February 29, 2000 *Final version accepted* May 5, 2000

AMS 1991 Subject classification: 60J10, 52C15
effective resistance, commute time, hitting time, difference time, circle packing, triangulation

*Abstract*
*The cover time of a finite connected graph is the expected number of steps needed for a simple random walk on the graph to visit all the vertices. It is known that the cover time on any $n$-vertex, connected graph is at least $\big(1+o(1)\big)n\log n$ and at most $\big(1+o(1)\big)\frac{4}{27}n^3$. This paper proves that for bounded-degree planar graphs the cover time is at least $cn(\log n)^2$, and at most $6n^2$, where $c$ is a positive constant depending only on the maximal degree of the graph. The lower bound is established via use of circle packings.*

## 1 Introduction

Let $G=(V,E)$ be a finite, connected, $n$-vertex graph and let $\{X_k\}_{k=0}^{\infty}$ be a simple random walk on $G$. For each $v \in V$, set $T_v = \min\{k \in \mathbb{N} : X_k = v\}$ and let $C = \max_{v\in V} T_v$ be the *cover time*. We are primarily interested in the expected cover time $\mathbf{E}_v C$, where $\mathbf{E}_v$ denotes expectation with respect to the probability measure of the random walk starting at $X_0 = v$. In words, $\mathbf{E}_v C$ is the expected time taken for the random walk starting at $v$ to visit every vertex of the graph.
Over the last decade or so, much work has been devoted to finding the expected cover time for different graphs and to giving general upper and lower bounds of the cover time. For an introduction, we refer the reader to the draft book by Aldous and Fill [2], in particular to Chapters 3, 5 and 6. It has been shown by Feige [9, 8] that

$$(1+o(1))n\log n \le \mathbf{E}_v C \le (1+o(1))\frac{4}{27}n^3,$$

and these bounds are tight.
In this paper, we show that for bounded-degree planar graphs, one has better bounds, namely,

THEOREM 1.1 *Let $G=(V,E)$ be a finite connected planar graph with $n$ vertices and maximal degree $M$. Then for every vertex $v \in V$,*

$$cn(\log n)^2 < \mathbf{E}_v C < 6n^2\,,$$

*where $c$ is a positive constant depending only on $M$.*

This generalizes a result of Zuckerman [19] showing that $\min_v \mathbf{E}_v C \ge cn(\log n)^2$ for bounded-degree trees on $n$ vertices. If $G = \mathbb{Z}^d \cap [-m,m]^d$, a finite portion of the $d$-dimensional integer lattice, then $\mathbf{E}_v C$ is $\Theta(n^2)$ for $d=1$, $\Theta(n(\log n)^2)$ for $d=2$ and $\Theta(n\log n)$ for $d \ge 3$ [1,20]. Here,

$n = (2m+1)^d = |V|$. The cases $d = 1$ and $d = 2$ show that Theorem 1.1 is tight (up to the constants). The case $d = 3$ shows that the planarity assumption is necessary.
The upper bound in Theorem 1.1 is quite easy. The lower bound will be based on Koebe's [12] Circle Packing Theorem (CPT):

THEOREM 1.2 *Let $G = (V, E)$ be a finite planar graph. Then there is a disk packing $\{C_v : v \in V\}$ in $\mathbb{R}^2$, indexed by the vertices of $G$, such that $C_v \cap C_u \neq \emptyset$ iff $\{v, u\} \in E$.*

Koebe's proof relies on complex analysis, but recently several new proofs have been discovered. See, for example, [4] for a geometric, combinatorial proof.
Some fascinating relations between the CPT and analytic function theory have been studied in the last decade. Additionally, the CPT became a tool for studying planar graphs in general, and random walks on planar graphs in particular [15, 13, 10, 3, 17]. In these applications, as well as here, the CPT is useful because it endows the graph with a geometry that is better, for many purposes, than the usual graph-metric.

We conjecture that Theorem 1.1 holds with $c = c'/\log(M+2)$, where $c' > 0$ is a positive constant. For example, this is true for trees, since in a tree one can easily find a set of at least $n^{1/2}$ vertices with pairwise distances at least $\log n / \big(2\log(M+2)\big)$. As we shall see, this implies that the expected cover time is bounded below by a constant times $n(\log n)^2/\log(M+2)$.

## 2 Preliminaries

For a simple random walk on the graph $G = (V, E)$ we define for every ordered pair $(u, v)$ of vertices, the *hitting time* as $H(u, v) := \mathbf{E}_u T_v$. The *commute time* is given by $C(u, v) := H(u, v) + H(v, u)$ and the *difference time* is given by

$$D(u, v) := H(u, v) - H(v, u).$$

From the so called cyclic tour property of reversible Markov chains it follows that difference times are additive (see [6]):

$$D(u, v) + D(v, w) = D(u, w). \tag{2.1}$$

Commute times are closely related to *effective resistances* in electrical networks: Regard each edge of $G$ as a unit resistor and define for each pair $(u, v)$ of vertices the effective resistance $R(u, v)$ between them as $i^{-1}$ where $i$ is the current flowing into $v$ when grounding $v$ and applying a 1 volt potential to $u$. In mathematical terms, $R(u, v)$ can be defined as

$$R(u, v) := \sup \frac{\big(f(v) - f(u)\big)^2}{\mathcal{D}(f)},$$

where $\mathcal{D}(f)$ is the Dirichlet energy of $f$,

$$\mathcal{D}(f) := \sum_{\{a,b\}\in E} \big(f(a) - f(b)\big)^2,$$

and the sup is with respect to all $f : V \to \mathbb{R}$ such that $\mathcal{D}(f) > 0$. (If $u$ and $v$ are in distinct components of $G$, then $R(u, v) = \infty$.) It is an immediate consequence from this definition that when $G$ is a subgraph of another graph $G'$, and $u, v$ are vertices in $G$, then the effective resistance between $u$ and $v$ in $G'$ is bounded from above by the effective resistance between them in $G$. It is well known that resistances satisfy the triangle inequality

$$R(u, w) \leq R(u, v) + R(v, w), \tag{2.2}$$

which follows from the following useful formula from [5]:

$$C(u, v) = 2\,|E|\, R(u, v). \tag{2.3}$$

There is also a formula from [18] for $H(u, v)$ in terms of resistances, but it is more complicated:

$$H(u, v) = \frac{1}{2} \sum_{w \in V} d_w \big(R(u, v) + R(v, w) - R(u, w)\big), \tag{2.4}$$

where $d_w$ is the degree of $w$.
The main lemma in the proof of Theorem 1.1 involves estimating the resistances. A combination of the above identities will then yield lower bounds for the hitting times. We then need some way to estimate the cover time from the hitting times. For this *Matthews' method* [14] will prove useful.

LEMMA 2.1 *Let $G(V,E)$ be a finite graph. Then*

$$\max_{v\in V} \mathbf{E}_v C \le h_{n-1} \max\{H(u,v) : v,u \in V\},$$

*where $h_k$ denotes the harmonic series $\sum_{i=1}^k i^{-1}$. Furthermore,*

$$\min_{v\in V} \mathbf{E}_v C \ge h_{|V_0|-1} \min\{H(u,v) : u,v \in V_0,\, u \ne v\}$$

*holds for every subset $V_0 \subset V$.*

A proof can be found in [14] or [2]. The proof relies on one ingenious trick, namely, to assign a uniformly chosen random order to $V$ independent of the random walk.

# 3 Proof of Theorem 1.1

We start with the easy proof of the upper bound. It is a well known consequence of Euler's formula $|V| - |E| + |F| = 2$ (see [7, Theorem 4.2.7]) that the average degree $\bar{d}$ in a finite planar graph is less than 6. By [2, Chapter 6, Theorem 1], $\max_v \mathbf{E}_v C \le \bar{d}n(n-1) < 6n^2$, which gives the upper bound.
Let us now turn to the lower bound. The main tool in the proof of Theorem 1.1 is the following lemma:

LEMMA 3.1 *There exist positive constants $c = c(M)$ and $r = r(M)$ such that for every planar connected graph $G = (V,E)$ with maximum degree $M$ and every set of vertices $W \subset V$ there is a subset $V' \subset W$ with $|V'| \ge |W|^c$ and $R(u,v) \ge r\log|W|$ for every $u \ne v$, $u,v \in V'$.*

*Proof of Theorem 1.1 from Lemma 3.1.* The strategy is to convert the information Lemma 3.1 gives about resistances to information about hitting times $H(v,u)$, and then use the second part of Lemma 2.1.
Let $a \in V$ be some vertex, and let $\{v_1, v_2, \ldots, v_n\}$ be an ordering of $V$ such that $i \le j$ implies $D(a,v_i) \le D(a,v_j)$. Then we have

$$i \le j \Rightarrow D(v_i,v_j) \ge 0 \tag{3.1}$$

for all $i,j \in \{1,\ldots,n\}$, by (2.1). Let $k = [n/2]$, the largest integer in $[0,n/2]$. We now consider several distinct cases.

**Case 1:** there are some $i < j$ in $\{1,2,\ldots,n\}$ such that $H(v_j,v_i) \ge n(\log n)^2/2$. Observe that for all $v \in V$, we have $\mathbf{E}_v C \ge \min\{H(v_j,v_i), H(v_i,v_j)\}$, for the random walk starting at $v$ must either visit $v_j$ before $v_i$ or visit $v_i$ before $v_j$. Consequently, (3.1) completes the proof in this case.

**Case 2:** $D(v_1,v_k) \ge n(\log n)^3$, and Case 1 does not hold. By (3.1) and (2.1), we then have $D(v_1,v_j) \ge n(\log n)^3$ for all $j \ge k$. By (2.4) and (2.2) we have

$$\begin{aligned} H(v_n,v_1) &\ge \frac{1}{2}\sum_{j=k}^{n}\Big(R(v_1,v_n) + R(v_1,v_j) - R(v_n,v_j)\Big) \\ &= \frac{1}{4|E|}\sum_{j=k}^{n}\Big(C(v_1,v_n) + C(v_1,v_j) - C(v_n,v_j)\Big) \qquad \text{(by (2.3))} \\ &> \frac{1}{12n}\sum_{j=k}^{n}\Big(C(v_1,v_n) + C(v_1,v_j) - D(v_j,v_n) - 2H(v_n,v_j)\Big), \end{aligned}$$

since $|E| < 3|V|$ for planar graphs and $C(v_n, v_j) = 2H(v_n, v_j) + D(v_j, v_n)$. Consequently, since Case 1 does not hold,

$$\begin{aligned} H(v_n, v_1) &> \frac{1}{12n} \sum_{j=k}^{n} \Big( C(v_1, v_n) + C(v_1, v_j) - D(v_j, v_n) - n(\log n)^2 \Big) \\ &\geq \frac{1}{12n} \sum_{j=k}^{n} \Big( D(v_1, v_n) + D(v_1, v_j) - D(v_j, v_n) - n(\log n)^2 \Big) \\ &= \frac{1}{12n} \sum_{j=k}^{n} \Big( 2D(v_1, v_j) - n(\log n)^2 \Big) \geq \frac{1}{12} n(\log n)^3, \end{aligned}$$

for all sufficiently large $n$, since we have $D(v_1, v_j) \geq n(\log n)^3$ for all $j \geq k$. However, $H(v_n, v_1) \geq \frac{1}{12} n(\log n)^3$ brings us back to Case 1.

**Case 3:** $D(v_1, v_k) \leq n(\log n)^3$. Set $W = \{v_1, \ldots, v_k\}$, and let $V' \subset W$ be as in Lemma 3.1. Let $m := |V'| \geq n^c$, and let $i_1 < i_2 < \cdots < i_m$ be those indices $i \in \{1, \ldots, k\}$ such that $v_i \in V'$. Set $s := [\sqrt{m}] - 1$. Since

$$\sum_{j=1}^{s-1} D(v_{i_{js}}, v_{i_{(j+1)s}}) = D(v_{i_s}, v_{i_{s^2}}) \leq D(v_1, v_k) \leq n(\log n)^3,$$

there is some $t \in \{1, \ldots, s-1\}$ such that $D(v_{i_{ts}}, v_{i_{(t+1)s}}) \leq n(\log n)^3/(s-1) = o(n)$. Set $V_0 := \{v_{i_{ts}}, v_{i_{ts+1}}, \ldots, v_{i_{(t+1)s}}\}$. Then $|V_0| \geq n^{c'}$ for some constant $c' > 0$ and $D(u, w) \leq o(n)$ for $u, w \in V_0$, if $n$ is large. However, we have $C(u, w) = 2|E|R(u, w) \geq rn \log n$ for $u, w \in V_0$, since $V_0 \subset V'$. Because $2H(u, w) = C(u, w) - D(w, u)$, this gives $H(u, w) \geq (r/3)n \log n$ for $u, w \in V_0$, provided that $n$ is large. Now the second part of Lemma 2.1 completes the proof. □

**Remark.** The recent preprint by Kahn et. al. [11] gives an estimate (Prop. 1.2 and Thm. 1.3) of the expected cover time in terms of the commute times. This result could be used to simplify the above argument (but was not available at the time of writing of the first draft of the current paper).

*Proof of Lemma 3.1.* We first consider the case where $G$ is a triangulation of the sphere. This means that $G$ is a graph embedded in $S^2$ with the property that every connected component of $S^2 \setminus G$ has precisely 3 edges of $G$ as its boundary.

The Circle Packing Theorem implies the existance of a disk packing $\big\{C_v : v \in V\big\}$ indexed by the vertices of $G$, such that each $C_v$ is a closed round disk in $\mathbb{R}^2$ and $C_v \cap C_u \neq \emptyset$ iff $\{v, u\} \in E$. Moreover (by normalizing by a Möbius transformation), we assume with no loss of generality that the outer three disks in the packing all have radius 1.

The Ring Lemma from [16] implies that there is a constant $A = A(M)$ such that

$$\{v, u\} \in E \Rightarrow r_v < Ar_u, \tag{3.2}$$

where $r_v$ denotes the radius of $C_v$. It then follows that there is another constant $A' := A'(M) > 0$ such that

$$\{v, u\} \notin E \Rightarrow \operatorname{dist}(C_v, C_u) \geq A' r_u, \tag{3.3}$$

where $\operatorname{dist}(C_v, C_u) := \inf\{|p - q| : p \in C_v,\ q \in C_u\}$, because the disks around $C_u$ separate $C_v$ from $C_u$, since $G$ is assumed to be a triangulation.

Most important for us is the following lower bound for the resistance

$$R(w, u) \geq A'' \log\big(\operatorname{dist}(C_w, C_u)/r_u\big), \tag{3.4}$$

for some constant $A'' := A''(M) > 0$. Similar estimates appear in [10] and in [3]. For completeness, we include a quick proof here. For each $v \in V$ let $z_v$ be the center of the disk $C_v$. Set $a := \log r_u$ and $b := \log |z_w - z_u|$. Consider $F(z) := \log(z - z_u)$ as a map from $\mathbb{C} \setminus \{z_u\} = \mathbb{R}^2 \setminus \{z_u\}$ to the cylinder $\mathbb{R} + i(\mathbb{R}/2\pi\mathbb{Z})$. Set $f(v) := \min\{\operatorname{Re}F(z_v), b\}$, for $v \neq u$ and $f(u) := \log r_u = a$. The inequality $\mathrm{dist}(C_u, C_v) \geq A' r_v$ implies that $\mathrm{area}\big(F(C_v)\big)/\mathrm{diam}\big(F(C_v)\big)^2$ is bounded above and below by positive constants. For neighbors $v_1$ and $v_2$ we have

$$|f(v_1) - f(v_2)| \leq \mathrm{diam}\big(F(C_{v_1})\big) + \mathrm{diam}\big(F(C_{v_2})\big) \leq O(1)\mathrm{diam}\big(F(C_{v_1})\big)$$

and $f(v_1) - f(v_2) = 0$ unless $\mathrm{dist}(C_u, C_{v_1} \cup C_{v_2}) \leq |z_w - z_u| - r_u$. Consequently, $\mathcal{D}(f) \leq O(1) \sum_v \mathrm{diam}\big(F(C_v)\big)^2$, where the sum extends over all $v \neq u$ such that $F(C_v)$ intersects the cylinder $[a, b] + i(\mathbb{R}/2\pi\mathbb{Z})$. All these sets $F(C_v)$ are contained in the cylinder $\big[a, b + O(1)\big] + i(\mathbb{R}/2\pi\mathbb{Z})$, and their interiors are disjoint. Since the area of each $F(C_v)$ is proportional to the square of its diameter, we find that

$$\mathcal{D}(f) \leq O(1)\mathrm{area}\big([a + O(1)] + i(\mathbb{R}/2\pi\mathbb{Z})\big) = O(1)(b - a + 1).$$

The inequality (3.4) now follows from the definition of the effective resistance.
Fix a small $s > 0$ (which will be specified later), and set $n = |W|$. For $j \in \mathbb{Z}$, let

$$W_j := \big\{v \in W : r_v \in (n^{s(j-1)}, n^{sj}]\big\}.$$

Then $W = \bigcup_{j \in \mathbb{Z}} W_j$. For $n$ so large that $n^s \geq A$ we have by (3.2) that if $u \in W_j$, $v \in W_k$ and $k - j \geq 2$ then $\{u, v\} \notin E$, and by (3.3) and (3.4), $R(u, v) \geq A'' \log(\mathrm{dist}(C_u, C_v)/r_u) \geq A'' \log(A' r_v / r_u) \geq \frac{1}{2} A'' s \log n$, when $n$ is large.
Now either $\left|\bigcup_{j \text{ odd}} W_j\right| \geq n/2$ or $\left|\bigcup_{j \text{ even}} W_j\right| \geq n/2$. Let us assume the latter case, noting that the former is treated similarly.
For each even $j$, let $Z_j$ be a maximal subset of vertices of $W_j$ such that

$$u, v \in Z_j,\ u \neq v \quad \Rightarrow \quad \mathrm{dist}(C_u, C_v) \geq n^{s(j+1)},$$

and note that by the definition of $W_j$ and (3.4), $R(u, v) \geq A'' \log(n^{s(j+1)}/n^{sj}) = A'' s \log n$ for all $u, v \in Z_j$, $u \neq v$. Since for any $v \in W_j$ the disk of radius $3n^{s(j+1)}$ centered at $z_v$, the center of $C_v$, does not contain more than $\big(3n^{s(j+1)}/n^{s(j-1)}\big)^2 = 9n^{4s}$ disks $C_u$ with $u \in W_j$, it follows that $|Z_j| \geq n^{-4s}|W_j|/9$. Now put $V' = \bigcup_{j \text{ even}} Z_j$. Then $|V'| \geq n^{1-5s}$ for $n$ large enough and when $v \neq v'$ are in $V'$ we have $R(u, v) \geq \frac{1}{2} A'' s \log n$. The result for $G$ a triangulation of $S^2$ follows by choosing $s = 1/6$, say.
Now consider the case where $G$ is not a triangulation of $S^2$. It is easy then to construct a triangulation $T$ of the sphere with maximum degree at most $3M$ which contains $G$ as a subgraph. The effective resistence $R_G(u, v)$ in $G$ between two vertices $u, v$ in $G$ is at least $R_T(u, v)$, their effective resistance in $T$. Consequently, this case follows from the previous.

□

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