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GEODESICS AND RECURRENCE OF RANDOM WALKS IN DISORDERED SYSTEMS

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Abstract
In a first-passage percolation model on the square lattice $\mathbb{Z}^2$, if the passage times are independent then the number of geodesics is either 0 or $+\infty$. If the passage times are stationary, ergodic and have a finite moment of order $\alpha > 1/2$, then the number of geodesics is either 0 or $+\infty$. We construct a model with stationary passage times such that $E[t(e)^\alpha] < \infty$, for every $0 < \alpha < 1/2$, and with a unique geodesic. The recurrence/transience properties of reversible random walks in a random environment with stationary conductances $(a(e); e$ is an edge of $\mathbb{Z}^2)$ are considered.

1 Introduction.

In a first-passage percolation model, a sequence of non-negative stationary random variables $(t(e); e$ is an edge of $\mathbb{Z}^2)$ is given.

A finite path γ, in the square lattice $\mathbb{Z}^2$, from x to y is a finite sequence of neighboring vertices of $\mathbb{Z}^2$ $x = x_0, x_1, \ldots, x_n = y$ and the *passage time* of the path is defined by

$$T(\gamma) = \sum_{i=1}^{n} t(x_{i-1}, x_i)$$

where $t(u,v)$ is the passage time of the edge between the neighboring vertices u and v. For two vertices x and y of $\mathbb{Z}^2$, define

$$T(x,y) := \inf\{T(\gamma); \gamma \text{ is a finite path from } x \text{ to } y\} .$$

Although $T(x,y)$ is often interpreted as the travel time between x and y, as in the seminal paper of Hammersley and Welsh in 1965 [9], the interpretation of T as a random distance is more appropriate here. The triangle inequality reflects the subadditivity of T : for all $x, y, z \in \mathbb{Z}^2$,

$$T(x,z) \leq T(x,y) + T(y,z).$$

A doubly infinite path γ in $\mathbb{Z}^2$ is called a *geodesic* if for all vertices x and y of γ,

$$T(x,y) = T(\gamma(x,y))$$

where $\gamma(x,y)$ is the finite path from x to y along γ. The question of the existence of geodesics is a natural one. However there is also a motivation from statistical physics since it is related to the existence of non-constant ground states for the two-dimensional Ising model in a random environment (see [14, Chapter 1]).

Note that if the sequence $(t(e); e \text{ is an edge of } \mathbb{Z}^2)$ is ergodic, then the number of geodesics is non random.

Our first goal is to extend theorem 1 of Wehr [16], a $0-\infty$ law, two different ways. In theorem 1, we consider stationary sequences of passage times satisfying an appropriate moment condition. Then to show that this condition is tight, in section 2, we construct a stationary sequence of passages times such that $E[t(e)^\alpha] < \infty$ for all $0 < \alpha < 1/2$ and for which, almost surely, there exists exactly one geodesic. In theorem 2, we consider passage times that are independent and we show that no moment condition is needed to prove the $0 - \infty$ law.

There is a strong analogy between these facts and the recurrence/transience properties of reversible random walks on $\mathbb{Z}^d$ in a random environment. In section 3, we recall these properties and then we give an example of stationary conductances with finite moments of order α for all $0 < \alpha < 1$ and such that the reversible random walks on $\mathbb{Z}^2$ in a random environment are transient and examples of reversible random walks on $\mathbb{Z}^d$, $d \geq 3$, with stationary conductances that are recurrent but have a finite moment of order less than $1/(d-1)$.

Theorem 1 *If $(t(e); e$ is an edge of $\mathbb{Z}^2)$ is a stationary and ergodic sequence of positive random variables such that*

$$E[t(e)^\alpha] < \infty \quad \textit{for some} \quad \alpha > 1/2,$$

then the number of geodesics is 0 or $+\infty$.

The stationarity of the sequence of passage times $(t(e); e \text{ is an edge of } \mathbb{Z}^2)$, that is, the finite distributions are translation invariant by each element of $\mathbb{Z}^2$, is equivalent to the existence of a group $(S_x; x \in \mathbb{Z}^2)$ of measure preserving transformations of a probability space $(\Omega, \mathcal{F}, P)$ and of two non-negative random variables t_1 and t_2 such that $S_{(0,0)}$ is the identity on Ω and for all $x, y \in \mathbb{Z}^2$ and all $A \in \mathcal{F}$,

$$S_x \circ S_y = S_{x+y}, \quad P(S_{-x}A) = P(A) \quad \text{and} \quad t\left(x, x+e_\ell, \omega\right) = t_\ell\left(S_x\omega\right), \quad \ell = 1, 2,$$

where $\{e_1, e_2\}$ is the canonical basis of $\mathbb{R}^2$ (see for instance [2, propositions 6.9 and 6.12] or [11, section 1.4]).
With this representation, we have that $(S_x; x \in \mathbb{Z}^2)$ is ergodic if and only if $(t(e); e$ is an edge of $\mathbb{Z}^2)$ is ergodic [2, proposition 6.18].

For two vertices $x, y \in \mathbb{Z}^2$, $y \sim x$ means that x and y are neighbors in the square lattice. For A a finite subset of $\mathbb{Z}^2$, $\partial A := \{x \notin A \; ; \text{ there exists } y \sim x \; , \; y \in A\}$ and $\overline{A} = A \cup \partial A$.

Proof of theorem 1. Suppose that N, the number of geodesics, is $0 < N < +\infty$.

By stationarity, $\eta := P(x$ belongs to a geodesic) does not depend on the particular vertex x and $\eta > 0$ since $N > 0$.

Since the random variables $t(e)$ are positive, there is $\delta > 0$ such that $P\left(t_\ell < \delta\right) < \eta/9N$ for $\ell = 1, 2$, where t_1 and t_2 are the random variables of the representation given above.

Let Q_n be the set of vertices of $\mathbb{Z}^2$ in the square $]-n, n]^2$ and $\mathcal{Q}_n$ be the set of edges with at least one vertex in Q_n.

By the multidimensional pointwise ergodic theorem [11, p.205], for n large enough, the number of vertices x in Q_n such that $t(x, x+e_1) < \delta$ or $t(x, x+e_2) < \delta$ is less than $\eta(2n)^2/9N$. Thus, the number of edges e in $\mathcal{Q}_n$ such that $t(e) < \delta$ is less than $2\eta(2n)^2/9N$.
Furthermore, for n large enough, one of the geodesics, denoted by γ in the sequel, must contain at least $\eta(2n)^2/2N$ vertices of Q_n. Thus, it must contain at least $\eta(2n)^2/4N$ edges of $\mathcal{Q}_n$. Let v and w be respectively the first and the last vertex of γ that are in ∂Q_{n-1}.
Since the passage time between v and w along γ is less than the passage time along a portion of ∂Q_{n-1}, for n large enough and for all $0 < \alpha \leq 1$ and $\varepsilon > 0$, we have

$$\left(\frac{\eta}{4N}(2n)^2 - \frac{2\eta}{9N}(2n)^2\right)\delta \leq T(v,w) \leq \sum_{x,y \in \partial Q_{n-1},\; x \sim y} t(x,y) \tag{1}$$

$$\leq \left(\sum_{x,y \in \partial Q_{n-1},\; x \sim y} t(x,y)^\alpha\right)^{1/\alpha} \leq (n-1)^{(1+\varepsilon)/\alpha}\left((n-1)^{-(1+\varepsilon)} \sum_{x \in \partial Q_{n-1}} (t_1 + t_2)^\alpha \circ S_x\right)^{1/\alpha}$$

which leads to a contradiction if $(1+\varepsilon)/\alpha \leq 2$, since, almost surely,

$$\begin{aligned}
\lim_{K \to +\infty} (2K)^{-2} \sum_{x \in Q_K} (t_1 + t_2)^\alpha \circ S_x &= \lim_{K \to +\infty} (2K)^{-2} \sum_{n=2}^{K} \sum_{x \in \partial Q_{n-1}} (t_1 + t_2)^\alpha \circ S_x \\
&= \mathrm{E}[(t_1 + t_2)^\alpha] < \infty \,,
\end{aligned}$$

and therefore

$$\liminf_{n \to +\infty} \quad (n-1)^{-(1+\varepsilon)} \sum_{x \in \partial Q_{n-1}} (t_1 + t_2)^\alpha \circ S_x = 0 \,.$$

•

Since there is an equivalence between the existence of nonconstant ground states in a disordered Ising ferromagnetic model and the existence of geodesics in the corresponding first-passage percolation model (see [12], [14, Chapter 1] or [16]), it follows that if the interactions are stationary, ergodic and with finite moment of order $\alpha > 1/2$ then P-a.s., there are either two ground states or an infinity of ground states.
In dimension $d > 2$, the notion of geodesic can be replaced by the interface of a non constant ground state for an Ising ferromagnetic model in a random environment. In this case, with a similar argument, one proves that if the interactions are stationary, ergodic and with finite moment of order $\alpha > (d-1)/d$ then P-a.s., there are either two or an infinity of ground states.
In Licea and Newman [12], the non existence of $(\hat{x}, \hat{y})$-bigeodesics is proved for distributions without atoms but there are no moment conditions. Related arguments can be found in [3].

Theorem 2 *If $(t(e); e$ is an edge of $\mathbb{Z}^2)$ is a sequence of independent positive random variables then the number of geodesics is 0 or $+\infty$.*

The properties of open paths that we need to prove theorem 2 without assuming a moment condition, can be obtained by elementary arguments on a slightly modified model of oriented percolation (see [7, section 10.10]).
Consider a percolation model on the lattice $\mathbb{Z}^2$. Let p be the probability that an edge of $\mathbb{Z}^2$ is open and let P_p be the product measure on the configurations. Let $\mathcal{C}$ be the set of vertices $x \in \mathbb{Z}^2 \cap \{(i,j); 0 \leq j \leq i\}$ that can be reached from 0 by at least one open path $0 = x_0, x_1, \ldots, x_n = x$ such that for all k, $0 \leq k < n$, if $x_k = (i_k, j_k)$ and $x_{k+1} = (i_{k+1}, j_{k+1})$ then the edge connecting x_k and x_{k+1} is open, $i_k \leq i_{k+1}$, $j_k \leq j_{k+1}$ and $j_k \leq i_k$. Then consider $\vec{\theta}(p) = P_p(\sharp\mathcal{C} = +\infty)$ where $\sharp\mathcal{C}$ is the number of vertices in $\mathcal{C}$. To prove theorem 2, we only need to know that $\vec{\theta}(p) > 0$ for some $p < 1$.

First we prove the theorem using the following two lemmas.

Lemma 1 *As $p \uparrow 1$ then $\vec{\theta}(p) \uparrow 1$.*

This lemma is used to prove the existence of a sequence of circuits with linearly growing passage times.

Lemma 2 *Let p, $0 < p \leq 1$, be the probability that an edge of $\mathbb{Z}^2$ is open. If $\vec{\theta}(p) > 0$, then P_p-almost surely, there are infinitely many n such that 0 is surrounded by an open circuit in $]-7^n; 7^n[^2 \backslash]-7^{n-1}; 7^{n-1}[^2$ consisting of less than $56 \cdot 7^{n-1}$ edges.*

Proof of theorem 2. By lemma 1, there is $p_0 < 1$ such that $\vec{\theta}(p) > 0$ for all $p > p_0$. Take $M < \infty$ large enough so that $P(t(e) < M) > p_0$ and call an edge e open when $t(e) < M$. Then by lemma 2 with $p = P(t(e) < M)$, P-almost surely, there are infinitely many n, such that 0 is surrounded by an open circuit π_n in $]-7^n; 7^n[^2 \backslash]-7^{n-1}; 7^{n-1}[^2$ such that $T(\pi_n) < 56 \cdot 7^{n-1} M$. Resume the proof of theorem 1 up to equation (1) with Q_n replaced by $Q_{7^{n-1}}$ as the only modification.
Then let v_n and w_n be respectively the first and the last vertex of γ that are on π_n.
Since the passage time between v_n and w_n along γ is less than the passage time along a portion of π_n, for infinitely many n we have that

$$\left(\frac{\eta}{4N}(2 \cdot 7^{n-1})^2 - \frac{2\eta}{9N}(2 \cdot 7^{n-1})^2\right) \delta \leq T(v_n, w_n) < 56 \cdot 7^{n-1} M$$

which leads to a contradiction. •

Proof of lemma 1. As in [7, section 1.4], if $\sharp\mathcal{C} < \infty$, consider the circuit π that surrounds $\mathcal{C}$ in $\mathbb{L}^2$, the dual lattice with vertices in $\mathbb{Z}^2 + (1/2, 1/2)$. An edge of $\mathbb{L}^2$ is said to be closed if it intersects a closed edge of $\mathbb{Z}^2$. Suppose that π consists of n edges. Then $n \geq 4$ and at least $n/4$ of these edges must be closed.
To see this, for $j \geq 0$, let $-1 \leq i_1 < ... < i_k < ... < i_{v(j)}$ be such that the horizontal edge from (i_k, j) to $(i_k + 1, j)$ intersects a vertical edge of π. If $k \leq v(j)$ is even then this edge must be closed while if k is odd it might not be. Since π is a circuit, $v(j)$ is even. Therefore $v(j)$ vertical edges of π intersect the horizontal line $(\cdot, j)$ and at least $v(j)/2$ of them are closed. Now to count how many horizontal edges of π are closed, for $i \geq 0$, let $h(i)$ be the number of horizontal edges of π which intersect the vertical line $(i, \cdot)$. Since π is a circuit, $h(i)$ is even. Let $-1 \leq j_1 < ... < j_k < ... < j_{h(i)}$ be such the vertical edge from (i, j_k) to $(i, j_k + 1)$ intersects a horizontal edge of π. If $k \leq h(i)$ is even then this edge must be closed unless $k = h(i)$ and $j_k = i$. Let d be the number of i for which this last possibility occurs, that is, $(i, i) \in \mathcal{C}$. Then, for d vertical lines $(i, \cdot)$, at least $(h(i)/2) - 1$ of horizontal edges of π are closed and for all other vertical lines, at least $h(i)/2$ of them are closed.
Finally, note that $\sum_{i=0}^{\infty} h(i) + \sum_{j=0}^{\infty} v(j) = n$ and $d \leq n/4$. Then the number of closed edges in π is at least $\displaystyle\frac{1}{2}\sum_{j=0}^{\infty} v(j) + \frac{1}{2}\sum_{i=0}^{\infty} h(i) - d \geq \frac{n}{2} - \frac{n}{4} = \frac{n}{4}$.

Therefore, if $\rho(n)$ is the number of circuits in $\mathbb{L}^2$ which have length n and which contain 0,

$$
\begin{aligned}
1 - \vec{\theta}(p) &= \sum_{n=4}^{\infty} P_p(\mathcal{C} \text{ the circuit } \pi \text{ that surrounds } \mathcal{C} \text{ consists of } n \text{ edges}) \\
&\leq \sum_{n=4}^{\infty} \rho(n)(1-p)^{n/4} \\
&\leq \frac{5^4(1-p)}{1 - 5(1-p)^{1/4}} \to 0 \qquad \text{as } p \to 1 ,
\end{aligned}
$$

since $\rho(n) \leq n3^n \leq 5^n$. •

Proof of lemma 2. For $n \geq 1$, let us introduce

$v_{n,1} = (-3{\cdot}7^{n-1}; 7^{n-1})$; $v_{n,2} = (3{\cdot}7^{n-1}; 7^{n-1})$; $v_{n,3} = (3{\cdot}7^{n-1}; 7^n)$; $v_{n,4} = (-3{\cdot}7^{n-1}; -7^n)$;

$v_{n,5} = (5{\cdot}7^{n-1}; -7^n)$; $v_{n,6} = (3{\cdot}7^{n-1}; -7^{n-1})$; $v_{n,7} = (-3{\cdot}7^{n-1}; -7^{n-1})$; $v_{n,8} = (-5{\cdot}7^{n-1}; 7^n)$.

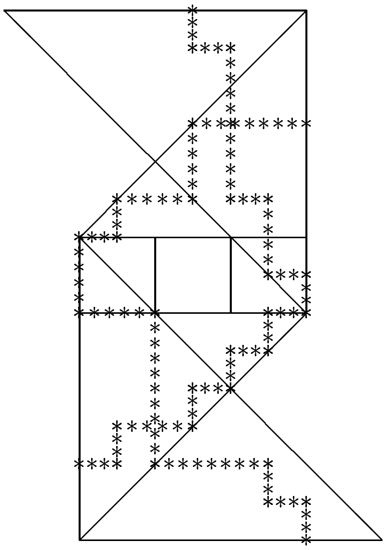
and consider

$A_n(1)$, the event that there is an open path, consisting of less than $12 \cdot 7^{n-1}$ edges, inside the right-angle triangle $(v_{n,1}; v_{n,2}; v_{n,3})$ from $v_{n,1}$ to the opposite side $[v_{n,2}; v_{n,3}]$,

$A_n(2)$, the event that there is an open path, consisting of less than $16 \cdot 7^{n-1}$ edges, inside the right-angle triangle $(v_{n,1}; v_{n,4}; v_{n,5})$ from $v_{n,1}$ to the opposite side $[v_{n,4}; v_{n,5}]$,

$A_n(3)$, the event that there is an open path, consisting of less than $12 \cdot 7^{n-1}$ edges, inside the right-angle triangle $(v_{n,6}; v_{n,7}; v_{n,4})$ from $v_{n,6}$ to the opposite side $[v_{n,7}; v_{n,4}]$,

$A_n(4)$, the event that there is an open path, consisting of less than $16 \cdot 7^{n-1}$ edges, inside the right-angle triangle $(v_{n,6}; v_{n,3}; v_{n,8})$ from $v_{n,6}$ to the opposite side $[v_{n,3}; v_{n,8}]$.



Representation of the four right-angle triangles
and possible paths corresponding to $A_n(i)$, $i = 1, \ldots, 4$

By symmetries, for each $n \geq 1$ and $0 < p < 1$, we have that

$$P_p(A_n(1)) = P_p(A_n(3)) \geq \vec{\theta}(p) \quad \text{and} \quad P_p(A_n(2)) = P_p(A_n(4)) \geq \vec{\theta}(p) \ .$$

And since these four events are increasing, by Harris FKG inequality (see [7, section 2.2]), for each $n \geq 1$ and $0 < p < 1$,

$$P_p(A_n(1) \cap A_n(2) \cap A_n(3) \cap A_n(4)) \geq \vec{\theta}(p)^4.$$

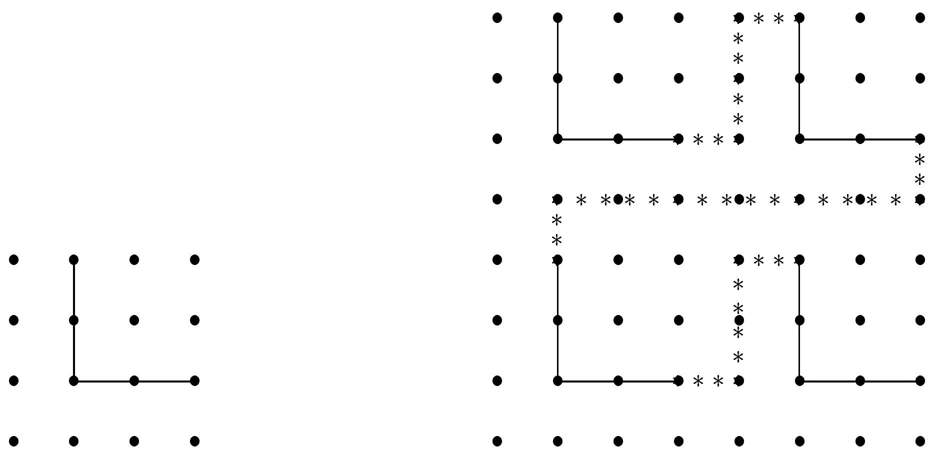
Therefore, by Borel-Cantelli, if $\vec{\theta}(p) > 0$, the four events occur simultaneously for infinitely many n. And for each n for which $A_n(1) \cap A_n(2) \cap A_n(3) \cap A_n(4)$ occurs, there is an open circuit in $]-7^n; 7^n[^2 \backslash]-7^{n-1}; 7^{n-1}[^2$ consisting of less than $56 \cdot 7^{n-1}$ edges and surrounding 0. $\bullet$

2 A counterexample for $0 < \alpha < 1/2$.

In this section, we define a stationary and ergodic sequence of positive random variables $(t(e); e$ is an edge of $\mathbb{Z}^2)$ with finite moments of order α for any $0 < \alpha < 1/2$, such that, almost surely, there exists exactly one geodesic. For two integers a, b such that $a \leq b$, denote $\{k \in \mathbb{Z}; a \leq k \leq b\}$ by $[a, b]$ and denote $\{k \in \mathbb{Z}; k \geq a\}$ by $[a, \infty[$.

First of all, take a path Γ in $[1, \infty[^2$ with the following property : if Γ_n is the restriction of Γ to the square $[2, 2^{n+1}] \times [2, 2^{n+1}]$, then the translates, $\Gamma_n + 2^{n+1}e_1$, $\Gamma_n + 2^{n+1}e_2$ and $\Gamma_n + 2^{n+1}(e_1 + e_2)$ are the restrictions of Γ to the respective translated squares.

The restrictions Γ_1 and Γ_2 of the path Γ that we use, have this shape :



Representation of Γ_1 and Γ_2

For $n \geq 1$, Γ_{n+1} is obtained from Γ_n by connecting the four translated paths with paths similar to the star-studded in the representation of Γ_2 given above. These paths correspond to $\gamma_1^{(n+1)} \cup \gamma_2^{(n+1)}$ in the notations below.
Some randomness will be introduced in this process. To do so we will use a sequence $(\vec{X}_n; n \geq 1)$ of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $\vec{X}_1$ has uniform distribution on the vectors $\{(i,j)\ ;\ 0 \leq i,j \leq 3\}$ and, for $n > 1$, $\vec{X}_n$ has uniform distribution on $\{(0,0),(2^n,0),(0,2^n),(2^n,2^n)\}$.

The sequence of random passage times will be defined so that almost surely, there exists only one geodesic $\gamma = \gamma(\omega)$ and that it looks like Γ.

(2.1) *Definition of* $(t(e)\ ;\ e$ *is an edge of* $\mathbb{Z}^2)$.

The passage times are defined on $(\Omega, \mathcal{F}, P)$ using the sequence $(\vec{X}_n; n \geq 1)$.
Most of the edges not in $\gamma(\omega)$ constitute barriers that force the way along the geodesic. These edges correspond to

$$(x, x+e_\ell) \quad x \in \mathcal{B}_\ell^{(n)} + \vec{m} \cdot (\vec{X}_1, \ldots, \vec{X}_n)\ ,\ \vec{m} \in \mathbb{Z}^n\ ,\ n \geq 1\ ,\ \ell = 1,2\ ,$$

with the notations below.

Set

$$\begin{array}{rclcrcl} \gamma_1^{(1)} & = & \{(2,2),(3,2)\} & & \gamma_2^{(1)} & = & \{(2,2),(2,3)\} \\ \mathcal{B}_1^{(1)} & = & \{(2,3),(2,4)\} & & \mathcal{B}_2^{(1)} & = & \{(3,2),(4,2)\} \end{array}$$

and, for $n \geq 2$,

$$\gamma_1^{(n)} = \{(2^n, 2^n+2),(2^n+1, 2^{n+1})\} \ \cup\ [2, 2^{n+1}-1] \times \{2^n+1\} \ \cup\ \{(2^n,2),(2^n+1,2^n)\}$$
$$\gamma_2^{(n)} = \{2^n+1\} \times [2^n+2, 2^{n+1}-1] \ \cup\ \{(2^{n+1}, 2^n+1),(2,2^n)\} \ \cup$$

$$\begin{aligned}
& \{2^n+1\}\times[2,2^n-1] \\
\mathcal{B}_1^{(n)} &= \{2^n\}\times[2^n+3,2^{n+1}]\ \cup\ \{2^n+1\}\times[2^n+2,2^{n+1}-1]\cup \\
& \{2^n\}\times[3,2^n]\ \cup\ \{2^n+1\}\times[2,2^n-1] \\
\mathcal{B}_2^{(n)} &= [2,2^{n+1}-1]\times\{2^n+1\}\ \cup\ [3,2^{n+1}]\times\{2^n\}.
\end{aligned}$$

At stage $n \geq 1$, for $\ell = 1, 2$, set

$$t(x, x+e_\ell) = 1 \quad \text{for all } \ x \in\ \cup_{\vec{m}\in\mathbb{Z}^n}\gamma_\ell^{(n)} + \vec{m}\cdot(\vec{X}_1,\dots,\vec{X}_n).$$

These edges will be part of the geodesic.
Then, for $\ell = 1, 2$, set

$$t(x, x+e_\ell) = 9\cdot 2^{2n} \quad \text{for all } \ x \in\ \cup_{\vec{m}\in\mathbb{Z}^n}\mathcal{B}_\ell^{(n)} + \vec{m}\cdot(\vec{X}_1,\dots,\vec{X}_n).$$

The values $\alpha > 0$ such that $E[t(e)^\alpha] < \infty$ depend on how large these passage times have to be.
The passage time of any other edge is set equal to 0. Actually, any other constant will do.

Almost surely, in any square, the proportion of edges with passage time equal to $9\cdot 2^{2n}$ is less than $8\cdot 2^n\cdot 2^{-2(n+1)} = 8\cdot 2^{-n}$. Therefore $E[t(e)^\alpha] < \infty$ for all $\alpha < \frac{1}{2}$ whether e is an horizontal or a vertical edge.

For $\omega\in\Omega$, let $\gamma(\omega) = \{e \text{ an edge of } \mathbb{Z}^2\ ;\ t(e) = 1\}$.

The remainder of this section is to show that $\gamma(\omega)$ is the unique geodesic.

(2.2) *Almost surely, $\gamma(\omega)$ is a geodesic :*

For each $n \geq 1$, the set of edges with passage time defined up to stage n and equal to 1, is a union of vertex-disjoint finite paths in $\mathbb{Z}^2$, each one being a translation of a fixed path Γ_n contained in $[2,2^{n+1}]\times[2,2^{n+1}]$. Therefore $\gamma(\omega)$ is a path in $\mathbb{Z}^2$.

Let x and y be two vertices of $\gamma(\omega)$ and denote the finite path between x and y along $\gamma(\omega)$ by $\gamma(x,y)$. Our goal is to show that for any path $\pi(x,y)$ from x to y, $T(\gamma(x,y)) \leq T(\pi(x,y))$.
Let $N \geq 1$ be the greatest integer such that at stage N, the passage time of at least one edge of $\gamma(x,y)$ is not yet defined.
First, since all passage times of $\gamma(x,y)$ are defined at stage $N+1$,

$$T(\gamma(x,y)) \leq \left(2^{N+2}\right)^2.$$

Secondly, since the passage time of some edge of $\gamma(x,y)$ is not defined at stage N, x and y cannot belong to $\mathbb{Z}^2\cap[2,2^N]\times[2,2^N]+\vec{m}\cdot(\vec{X}_1,\dots,\vec{X}_N)$ for the same index $\vec{m}$. Therefore, any finite path $\pi(x,y)$ from x to y contains an edge whose passage time is not defined at stage N. But all passage times defined at a later stage are either 1 or are $\geq 9\cdot 2^{2(N+1)}$.
In particular, if $\pi(x,y)$ is edge disjoint from $\gamma(x,y)$, then

$$T(\pi(x,y)) \geq 9\cdot 2^{2N+2} > T(\gamma(x,y)).$$

(2.3) *Almost surely, $\gamma(\omega)$ is the unique geodesic :*

Suppose that, for some $\omega \in \Omega$, there is a second geodesic, distinct from $\gamma(\omega)$. Then on this second geodesic, there is a finite path $\pi(x,y)$, $x \neq y$, that is edge-disjoint from $\gamma(\omega)$.
By (2.2), at most one vertex of $\pi(x,y)$ can belong to $\gamma(\omega)$. Therefore, there is at least one vertex, $z \in \pi(x,y)$ that does not belong to $\gamma(\omega)$. But the only vertices z' of $\mathbb{Z}^2$ that do not belong to $\gamma(\omega)$ verify $z' \in \cup_{m\in\mathbb{Z}}\{3,4\} \times \{3,4\} + m\vec{X}_1$. And since this is the case for z, any self-avoiding path of the form $x_{-4}, \ldots, x_{-1}, z, x_1 \ldots, x_4$ contains at least two distinct vertices of $\gamma(\omega)$, in contradiction with (2.2).

3 Recurrence of reversible random walks on $\mathbb{Z}^d$.

As in a first-passage percolation model, we are given a sequence of positive random variables $(a(e) : e$ is an edge of $\mathbb{Z}^d)$ which are now interpreted as the electrical conductance of the edges. $a(e)^{-1}$ is called the resistance of the edge. Almost surely, there is an associated random walk, $(\xi_k; k \geq 0)$, on $\mathbb{Z}^d$ whose transition probabilities are given by

$$P_\omega(\xi_{k+1} = y | \xi_k = x) := a(x,y)/a(x) \quad \text{if} \quad x \sim y \tag{2}$$

where $a(x) := \sum_{y\sim x} a(x,y)$.
$(\xi_k; k \geq 0)$ is a reversible random walk with $a(x)$ as an invariant measure.

Recall that in a finite graph with conductances given by a sequence $a(e)$, e an edge of the graph, the effective resistance, $R_a(x,V)$, between a vertex x and a set of vertices V not containing x is the intensity of the electric current needed to maintain a unit potential difference between x and V (cf. [5] or [15, Chapters 8 and 9] for example). It has the following probabilistic interpretation :

$$R_a(x,V)^{-1} = a(x)P_\omega(\tau_x^+ > \tau_V | \xi_0 = x)$$

where $\tau_x^+ = \inf\{k > 0; \xi_k = x\}$ and $\tau_V = \inf\{k \geq 0; \xi_k \in V\}$.
Therefore, a.s., the random walk on $\mathbb{Z}^d$ is recurrent if and only if

$$R_a(0, \partial Q_n) \to +\infty \quad \text{as} \quad n \to \infty \quad a.s. \tag{3}$$

where Q_n is the set of vertices of $\mathbb{Z}^d$ in the cube $]-n,n]^d$.
It follows from the Rayleigh's monotonicity principle (see [15, Theorem 8.5]), that if the conductances are bounded, but not necessarily stationary, then the associated reversible random walk on $\mathbb{Z}^2$ is recurrent while if the conductances are bounded below away from 0 then the associated random walk on $\mathbb{Z}^d$, $d \geq 3$, is transient (for instance [5], [6] or [15]).
For the one-dimensional walk, it is simple to see, using Poincaré recurrence theorem for instance, that if the conductances $(a(e); e$ is an edge of $\mathbb{Z})$ form a stationary sequence of positive random variables then, a.s., the associated reversible random walk is recurrent.

The next four remarks gather together the recurrence/transience properties of the reversible random walks on $\mathbb{Z}^d$, $d \geq 2$ with random conductances, independent or not.

Remark 1 *Let $(a(e); e$ an edge of $\mathbb{Z}^2)$ be a sequence of non-negative conductances.*

If $\sup_e E[a(e)] < \infty$, *then, in almost all environments, the associated reversible random walk is recurrent.*

This is due to Y. Peres [1, lemma 4.3]. Example (3.3) below shows that this moment condition is tight. Then in his study of the random walk on the infinite cluster of long-range percolation, Berger [1, Theorem 1.9] considered conductances on $\mathbb{Z}^2$ that are independent and identically distributed. Using the exponential decay of the radius of an open cluster in the subcritical phase in bond percolation (see [7, p.46]), he proved that

Remark 2 *If* $(a(e); e$ *an edge of* $\mathbb{Z}^2)$ *is a sequence of independent non-negative conductances then, in almost all environments, the associated reversible random walk is recurrent.*

No moment condition on the conductances is needed. Note that this could also be obtained by lemma 2 and Nash-Williams criterion (see [15, Corollary 9.2]).
The most useful expression of the effective resistance to study the transience properties of reversible random walks on $\mathbb{Z}^d$ for $d \geq 3$, is given by Thomson's Principle [15, theorem 8.4].

$$R_a(0, \partial Q_n) = \inf \left\{ \sum_e a(e)^{-1}[J(e)]^2; \quad J(e) \ \text{ is a unit flow from 0 to } \partial Q_n \right\} \tag{4}$$

where the sum is over the edges of $\mathcal{Q}_n$, that is the edges with at least one vertex in Q_n.
Using this variational principle, the proof of Peres [1, lemma 4.3] shows that if the conductances on $\mathbb{Z}^d$, $d \geq 3$, are such that $\sup_e E[a(e)^{-1}] < \infty$, then, in almost all environments, the associated reversible random walk is transient. However, it is possible to weaken this integrability condition.

Remark 3 *Let* $(a(e); e$ *an edge of* $\mathbb{Z}^d)$, $d \geq 3$, *be a sequence of positive conductances.*
If $\sup_e E[a(e)^{-\alpha}] < \infty$, *for some* $\alpha > \dfrac{d}{2(d-1)}$ *then, in almost all environments, the associated reversible random walk is transient.*

Proof. Let $J(e)$ be a unit flow from 0 to ∞ in $\mathbb{Z}^d$ such that for some constant $c_1 < \infty$, for all $x \neq 0$ and $i = 1, \ldots, d$,

$$|J(x, x + e_i)| \leq c_1 |x|^{1-d} \tag{5}$$

where for two vertices $x \sim y$, $J(x, y)$ denotes the flow in the edge from x to y (see [15, example 9.4 and exercise 11.4] or [13]).
Then, if $0 < \alpha < 1$, there is a constant $c_2 < \infty$ such that

$$\begin{aligned}
E\left[R_a(0, \partial Q_n)^\alpha\right] &\leq E\left[\left(\sum_{e \in \mathcal{Q}_n} a(e)^{-1} J(e)^2\right)^\alpha\right] \\
&\leq E\left[\sum_{e \in \mathcal{Q}_n} a(e)^{-\alpha} J(e)^{2\alpha}\right] \\
&\leq \sup_e E\left[a(e)^{-\alpha}\right] \sum_{e \in \mathcal{Q}_n} J(e)^{2\alpha} \\
&\leq c_2 \sup_e E\left[a(e)^{-\alpha}\right] \sum_{n=1}^{\infty} n^{d-1} n^{2\alpha(1-d)}
\end{aligned}$$

which is finite if $\sup_e E[a(e)^{-\alpha}] < \infty$ and if $d - 1 + 2\alpha(1-d) < -1$.
Thus, we obtain in this case

$$E\left[R_a(0 \leftrightarrow \infty)\right] < \infty$$

giving that

$$R_a(0 \leftrightarrow \infty) < \infty \quad a.s..$$

•

Finally, the independent case follows immediately from the transience of the reversible random walk on the infinite open cluster in the supercritical case proved in [8]. It suffices to take $\varepsilon > 0$ small enough so that $P(a(e) < \varepsilon) < p_c(\mathbb{Z}^d)$, the critical probability of bond percolation on $\mathbb{Z}^d$. Then set $\tilde{a}(e) = \begin{cases} 0 & \text{if } a(e) < \varepsilon \\ \varepsilon & \text{if } a(e) \geq \varepsilon \end{cases}$. By the monotonicity principle, $R_a(0, \partial Q_n) \leq R_{\tilde{a}}(0, \partial Q_n)$ which remains bounded as $n \to \infty$ since the random walk on the open cluster is transient.

Remark 4 *Let $(a(e); e$ is an edge of $\mathbb{Z}^d)$, $d \geq 3$, be a sequence of independent non-negative conductances. If $P(a(e) = 0) < p_c(\mathbb{Z}^d)$ then, in almost all environments, the associated reversible random walk is transient.*

While example (3.3) shows that the integrability condition in remark 1 for the recurrence of the reversible random walk with stationary conductances is tight, example (3.4) below shows that the appropriate moment condition in remark 3 is in between $1/(d-1)$ and $d/(2(d-1))$.

Further remarks A transience criterion is given in Durrett [6, theorem 1]. Note however that the random walks studied in [6, theorems 2 and 3] are reversible, with stationary transition probabilities but the conductances are not stationary. In [4], it is proved that if $E[a(e)] < \infty$ and $E[a(e)^{-1}] < \infty$, then the central limit theorem holds in measure. If the conductivies are bounded away from 0 and $+\infty$, then there is a strong law of large number [10, p.84]. Under an additional hypothesis on the range of the conductivities, the central limit theorem holds a.s. [10, p.117].

(3.3) EXAMPLE of a transient reversible random walk on $\mathbb{Z}^2$ with stationary and ergodic conductances with finite moments of order $\alpha < 1$.

The conductances are defined so that from every vertex of $\mathbb{Z}^2$ there is a path to infinity whose edges have increasing conductances. One expects that if they increase fast enough, then the walk will be transient. The example shows that this can happen for conductances with finite moments of order $\alpha < 1$.

For $n \geq 2$ even, let

$$\mathcal{T}_1^{(n)} = \{(2^{n-1}, 2^{n-2}), (2^{n-1}, 3 \cdot 2^{n-2})\} \quad \text{and} \quad \mathcal{T}_2^{(n)} = \{2^{n-1}\} \times [1, 2^n - 1]$$

and for $n \geq 3$ odd, let

$$\mathcal{T}_1^{(n)} = [1, 2^n - 1] \times \{2^{n-1}\} \quad \text{and} \quad \mathcal{T}_2^{(n)} = \{(2^{n-2}, 2^{n-1}), (3 \cdot 2^{n-2}, 2^{n-1})\}$$

As in section 2, we use a sequence $(\vec{X}_n; n \geq 2)$ of independent random variables defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $\vec{X}_n$ has uniform distribution on

$$\{(0,0), (2^n, 0), (0, 2^n), (2^n, 2^n)\}\ .$$

The random conductances, $(a(e); e$ is an edge of $\mathbb{Z}^2)$, are defined on Ω as follows :
for $n \geq 2$ and $\ell = 1, 2$, set

$$a\,(x, x + e_\ell, \omega) = n^2 2^n, \quad \text{for all} \quad x \in \cup_{\vec{m} \in \mathbb{Z}^n} \mathcal{T}_\ell^{(n)} + \vec{m} \cdot (\vec{X}_2, \ldots, \vec{X}_n).$$

and for all the other edges of $\mathbb{Z}^2$, set $a(e) = 1$.

Almost surely, in any square, the proportion of edges with conductance equal to $n^2 2^n$ is less than $2^n \cdot 2^{-2n}$, for all $n \geq 2$ and $\ell = 1, 2$, $E[a(e)^\alpha] < \infty$ whether e is a horizontal or a vertical edge.

For all $\omega \in \Omega$, there is an infinite path γ in $\mathbb{Z}^2$ starting from $(0,0)$ whose successive vertices can be labeled

$$\gamma : \quad (0,0) = x_0^{(1)}, x_1^{(1)}, \ldots, x_{k_1}^{(1)}, x_1^{(2)}, \ldots, x_{k_2}^{(2)}, x_1^{(3)}, \ldots, x_{k_3}^{(3)}, \ldots$$

such that for all $n \geq 1$, $k_n \leq 2^n$, and for all $n \geq 2$,

$$a(x_m^{(n)}, x_{m+1}^{(n)}, \omega) = n^2 2^n \quad \text{for } 1 \leq m < k_n, \text{ and} \quad a(x_{k_n}^{(n)}, x_1^{(n+1)}, \omega) = n^2 2^n.$$

For $N \geq 1$, let R_N be the effective resistance between the origin and ∂Q_N along the path γ, that is, by setting the resistance of all edges of $\mathbb{Z}^2$ equals to $+\infty$ except those of γ whose values are left unchanged.
The walk is transient since by the monotonicity principle,

$$R_a(0, \partial Q_N) < R_N < \sum_{n=1}^{\infty} k_n (n^2 2^n)^{-1} = \sum_{n=1}^{\infty} n^{-2} < \infty.$$

(3.4) EXAMPLE. Let $d \geq 3$. For each α, $0 < \alpha < 1/(d-1)$, there is a sequence of stationary and ergodic conductances such that $E[a(e)^{-\alpha}] < \infty$ and, for almost all environments, the reversible random walk on $\mathbb{Z}^d$ is recurrent.

Let $(Z(x); x \in \mathbb{Z}^d)$ be a sequence of independent identically distributed random variables with distribution

$$P(Z(x) = n) = p_n, \quad \text{for } n \in \mathbb{N} \qquad \text{and} \quad \sum_{n \geq 0} p_n = 1.$$

Two other sequences of positive real numbers are used to define the conductances of the edges: $(a_n; n \geq 0)$ and $(r_n; n \geq 0)$ such that, $a_0 = 1$, $r_0 = 1$ and, as $n \to \infty$, a_n decreases to 0 and r_n increases to ∞. Then the conductance of the edge between two neighboring vertices y and z of $\mathbb{Z}^d$ is

$$a(y, z) \quad = \inf\{ \quad a_n \ ; \ n = Z(x) \ \ \text{for some } x \in \mathbb{Z}^d \text{ such that}$$

$$-1 < |y - x| - r_n < 1 \text{ or } -1 < |z - x| - r_n < 1\} \tag{6}$$

where $|\cdot|$ is the euclidean norm in $\mathbb{R}^d$, or, $a(y,z) = 1$ if the infimum is over an empty set. Consequently, if x is a vertex where $Z(x) = n$, the conductance of every edge near the boundary of the ball $B_n(x)$, of radius r_n and centered at x, is at most a_n.
The idea is to choose the sequences $(p_n), (a_n)$ and (r_n) such that, almost surely, the origin, and therefore every vertex, belongs to an increasing sequence of balls and the effective resistances between 0 and the boundary of these balls are unbounded.
A first condition must be given to insure that the conductivities are well defined and positive. To do so, let

$$D_n = \{\text{there is a vertex } x \ ; \ r_n - 2 < |x| < r_n + 2, \ Z(x) = n\}.$$

Since, for some constant $c_1 > 0$, $P(D_n^c) \geq (1 - p_n)^{c_1 r_n^{d-1}}$ then,

$$\sum_n P(D_n) \leq \sum_n [1 - (1 - p_n)^{c_1 r_n^{d-1}}]$$

which converges if

$$\sum_n p_n r_n^{d-1} < \infty. \tag{7}$$

And therefore, by Borel-Cantelli lemma, the infimum in (6) is over a finite set of positive numbers.
The next step is to prove that, almost surely, the origin belongs to an increasing sequence of balls such that the conductances of the edges near the boundary decreases to 0. Introduce a new sequence $(\rho_n; n \geq 0)$ such that $\rho_n \uparrow \infty$ and consider the events

$$A_n = \{\text{there is a vertex } x \ ; \ \rho_n < |x| < \rho_{n+1}, Z(x) = n\}.$$

The second condition is to insure that A_n occurs infinitely often. If $\rho_{n+1} - \rho_n > 2$, then there is a constant $c_2 > 0$ such that, $P(A_n^c) < (1 - p_n)^{c_2(\rho_{n+1}^d - \rho_n^d)}$. Then, $\sum_n P(A_n)$ diverges if

$$\sum_n p_n(\rho_{n+1}^d - \rho_n^d) = +\infty. \tag{8}$$

And we can use the second Borel-Cantelli lemma to conclude
We also need the following property : if $(x_n; n \geq 1)$ is a sequence of vertices such that $\rho_n < |x_n| < \rho_{n+1}$ for all $n \geq 1$, then $0 \in B_n(x_n) \subset B_{n+1}(x_{n+1})$, for all n sufficiently large. This is the case if the next condition, a third one, is satisfied for all n sufficiently large :

$$r_{n+1} > \rho_{n+2} + \rho_{n+1} + r_n. \tag{9}$$

The next step is to verify that for almost all environments, the reversible random walk is recurrent. If conditions (7) to (9) are satisfied, then for almost all environments, there are sequences $(x_{n_k}; k \geq 1)$ and $(r_{n_k}; k \geq 1)$ such that for all $y \in \partial B_{n_k}(x_{n_k})$ and for all $z \sim y$, $a(y,z) \leq a_{n_k}$. Consider the network $\mathbb{Z}^d$ with the following conductances:

For $1 \leq i \leq d$, $\quad \tilde{a}(y, y + e_i, \omega) = \begin{cases} a_{n_k}, & \text{if } y \in \partial B_{n_k}(x_{n_k}) \text{ for some } k \geq 1; \\ \infty & \text{otherwise.} \end{cases}$

Then by Rayleigh's monotonicity principle, $R_a(0, \partial B_{n_k}) \geq R_{\tilde{a}}(0, \partial B_{n_k})$ where the latter is the effective resistance calculated with the conductances $\tilde{a}$. The edges around each sphere are in parallel and the spheres are in series, therefore, there is a constant $c_3 > 0$ such that

$$R_{\tilde{a}}(0, \partial B_{n_K}) \geq c_3 \sum_{k=1}^{K} \left(r_{n_k}^{d-1} a_{n_k} \right)^{-1}$$

which diverges if the following condition is satisfied

$$\limsup_n \; r_n^{d-1} a_n < \infty. \tag{10}$$

Then by Nash-Williams criterion, the random walk is recurrent.
Finally, we find that for a fixed edge e of $\mathbb{Z}^d$,
$E[a(e)^{-\alpha}] = \sum_{n=0}^{\infty} a_n^{-\alpha} P(a(e) = a_n) \leq \sum_{n=0}^{\infty} a_n^{-\alpha} P(D_n)$ which converges for all $\alpha > 0$ such that

$$\sum_n a_n^{-\alpha} p_n r_n^{d-1} < +\infty. \tag{11}$$

We see that it is possible to construct the example if $0 < \alpha < 1/(d-1)$. For example, one can take, for $n \geq 0$, $r_n = 2^n$ and $a_n = 2^{(1-d)n}$. Note that $\beta = 2^{(\alpha+1)(d-1)}$ verifies $2^{d-1} < \beta < 2^d$. Then, for $n \geq 0$, set $\rho_n = \beta^{n/d}$ and $p_n = c\beta^{-n}$ where c is the appropriate normalizing constant.
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References

[1] BERGER, N. *Transience, Recurrence and Critical Behavior for Long-Range Percolation.* Preprint.

[2] BREIMAN, L. (1968) Probability, *Addison-Wesley Series in Statistics.* Addison-Wesley Publishing Company, Reading, Mass.

[3] BURTON, R. and KEANE, M. (1989) Density and uniqueness in percolation. *Comm. Math. Phys.* **121** 501 - 505.

[4] DE MASI, A., FERRARI, P.A., GOLDSTEIN, S. and WICK, W.D. (1989) An invariance principle for reversible Markov processes. Applications to random motions in random environments. *J. Stat. Phys.* **55** 787-855.

[5] DOYLE, P. and SNELL, J.L. (1984) Random walks and electrical networks. *The Carus Mathematical Monographs*, 22 The Mathematical Association of America. John Wiley & Sons, New York.

[6] DURRETT, R. (1986) Multidimensional random walks in random environments with subclassical limiting behavior, *Comm. Math. Phys.* **104** 87 - 102.

[7] GRIMMETT, G. (1989) *Percolation.* Springer-Verlag. New York.

[8] GRIMMETT, G.R., KESTEN, H. and ZHANG, Y. (1993) *Random walk on the infinite cluster of the percolation model.* Probab. Theory Relat. Fields **96** 33-44.

[9] HAMMERSLEY, J.M. and WELSH, D.J.A. (1965) First-passage percolation, subadditive processes, stochastic networks, and generalized renewal theory. *Bernoulli-Bayes-Laplace Anniversary Volume*, J. Neyman and L.M. LeCam, eds. pp. 61-110, Springer-Verlag New York, N.Y.

[10] KOZLOV, S.M. (1985) The method of averaging and walks in inhomogeneous environments. *Russ. Math. Surv.* **40** 73-145 ; translation from *Usp. Mat. Nauk* **40** 61-120 (1985).

[11] KRENGEL, U. (1985) Ergodic Theorems. *De Gruyter Studies in Mathematics*, 6. Berlin-New York: Walter de Gruyter.

[12] LICEA, C. and NEWMAN, C. (1996) Geodesics in two-dimensional first-passage percolation. *Ann. Probab.* **24** 399 - 410.

[13] LYONS, T. (1983) A simple criterion for transience of a reversible Markov chain. *Ann. Probab.* **11** 393-402.

[14] NEWMAN, C. M. (1997) Topics in disordered systems. *Lectures in Mathematics*, ETH Zürich. Basel: Birkhäuser.

[15] PERES, Y. (1999) Probability on trees: An introductory climb. Bertoin, J. (ed.) et al., *Lectures on probability theory and statistics. Ecole d'été de Probabilités de Saint-Flour XXVII-1997*, Saint-Flour, France, July 7-23, 1997. Berlin: Springer. Lect. Notes Math. 1717, 193-280.

[16] WEHR, J. (1997) On the Number of Infinite Geodesics and Ground States in Disordered Systems. *J. Stat. Physics* **87** 439 -447.