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# EXCURSIONS INTO A NEW DUALITY RELATION FOR DIFFUSION PROCESSES

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*Abstract*
*This work was motivated by the recent work by H. Dette, J. Pitman and W.J. Studden on a new duality relation for random walks [1]. In this note we consider the diffusion process limit of their result, and use the alternative approach of Itô excursion theory. This leads to a duality for Itô excursion rates.*

## 1 A Duality for Excursion Rates of a Brownian Motion with Non-Uniform Drift

In a recent paper [1] a new duality relation for random walks was introduced, and at the end of that paper a brief mention was made of a corresponding result conjectured for diffusion processes. In this note we show how the diffusion process limit of the random walk result is very much easier to prove directly. This leads to a duality for Itô excursion rates.

For a good review of Itô excursion theory see the book by Rogers and Williams [2], or for the bare bones of excursion theory, which is sufficient for the current work see Dean and Jansons [3], who use it to consider polymer solutions in straining flows.

Let $u : \mathbb{R} \mapsto \mathbb{R}$ be locally integrable, and let $u^*(x) = -u(x)$. Fixing $a < b$ and $\lambda > 0$ until further notice, consider two diffusion processes $X$ and $X^*$ with values in $[a, b]$, satisfying

$$dX_t = dB_t + u(X_t)dt, \quad dX_t^* = dB_t + u^*(X_t^*)dt, \tag{1}$$

with both $X$ and $X^*$ reflected instantaneously at $a$ and with $X$ killed at $b$ and $X^*$ reflected instantaneously at $b$. We shall refer to any process satisfying an Itô SDE of the form of those given in equation (1) as "Brownian motion with non-uniform drift". Define for $a \leq x \leq b$

$$\psi(x) = \mathbb{E}^x \exp(-\lambda H_a), \quad \psi^*(x) = \mathbb{E}^x \exp(-\lambda H_a^*),$$

where $H_a = \inf\{t : X_t = a\}$ and $H_a^* = \inf\{t : X_t^* = a\}$. Then it is well known that both $\psi$ and $\psi^*$ are strictly decreasing in $[a, b]$, $\psi(a) = \psi^*(a) = 1$ and $\psi(b) = D\psi^*(b) = 0$. Both $\psi$ and

$\psi^*$ are strictly positive in $[a,b)$, and satisfy

$$\begin{aligned}\mathcal{G}\psi &\equiv \tfrac{1}{2}D^2\psi + uD\psi = \lambda\psi,\\ \mathcal{G}^*\psi^* &\equiv \tfrac{1}{2}D^2\psi^* + u^*D\psi^* = \lambda\psi^*.\end{aligned}$$

The main result of this paper, before translation into excursion theoretic language, is as follows.

**Proposition.** *Using the above notation and definitions*

$$2\lambda = \frac{D\psi}{\psi}\cdot\frac{D\psi^*}{\psi^*}.$$

*Proof.* Define a function $\phi$ by

$$\frac{D\phi}{\phi} = 2\lambda\frac{\psi}{D\psi},$$

in $(a,b]$, with $\phi(a)=1$. Firstly note that $\phi$ is well defined, as it easily follows from the definition of $\psi$ that $D\psi$ is continuous and strictly negative in $[a,b]$. Secondly routine calculus proves that $\phi$ solves the same ODE as $\psi^*$ with the same boundary conditions, therefore $\phi=\psi^*$. ■

We now translate the above proposition into the language of excursion theory, which leads to the nicest and most natural form of the duality. Using the semi-martingale normalization of local time, let $n_x$ and $n^*_x$ be respectively the excursion measures for upward excursions of the processes $X$ and $X^*$ from $x\in[a,b)$.

**Theorem 1** *Using the above definitions and assumptions, suppose that $X$ is a Brownian motion with non-uniform drift $u(x)$ on $[a,b]$, killed at $b$, and $X^*$ is a Brownian motion with non-uniform drift $-u(x)$, reflected at $b$. If the processes are marked by the points of an independent homogeneous Poisson point process of rate $\lambda$, then their excursion measures satisfy:*

$$n_x(\text{marked or killed})\; n^*_x(\text{marked}) = \tfrac{1}{2}\lambda, \tag{2}$$

*for $x\in[a,b)$.*

*Proof.* Take any $c\in[a,b)$ and notice that, with semi-martingale local time,

$$\exp\left[-\lambda\int_0^t I_{(c,b]}(X_s)ds - \tfrac{1}{2}\frac{D\psi}{\psi}(c)L(t,c)\right]\psi(X_t\vee c)$$

is a martingale, where $L$ is the local-time process of $X$. Then if we time change by $T_{\mathcal{L}}=\inf\{t : L(t,c)>\mathcal{L}\}$ we obtain another martingale

$$\exp\left[-\lambda\int_0^{T_{\mathcal{L}}} I_{(c,b]}(X_s)ds - \tfrac{1}{2}\frac{D\psi}{\psi}(c)\mathcal{L}\right], \tag{3}$$

ignoring the multiplicative constant $\psi(c)$. Equation (3) identifies $-\frac{1}{2}\frac{D\psi}{\psi}(c)$ as the (local-time) rate of excursions of $X$ into $(c,b]$ which are $\lambda$-marked or killed. Similarly we can identify $-\frac{1}{2}\frac{D\psi^*}{\psi^*}(c)$ with the rate of excursions of $X^*$ into $(c,b]$ which are $\lambda$-marked (as there is no killing). Then the above proposition gives the required result. ■

We now see that the choice of $a$ is arbitrary (and irrelevant). Also, imposing a finite boundary $b$ is not necessary; if we let $b \to \infty$ the corresponding excursion rates converge and the duality relation still holds.

Example 1. Consider the simplest example of Brownian motion with constant drift $u$ on $[0,\ell]$, with $\ell > 0$. For $x \in [0,\ell)$ let

$$\mathsf{a}(x) = n_x(\text{marked or killed}), \quad \mathsf{a}^*(x) = n_x^*(\text{marked})$$

and $\nu = \sqrt{2\lambda}$, then

$$\mathsf{a}(x) = \tfrac{1}{2}\left(\nu^2+u^2\right)^{\frac{1}{2}} \coth\left(\left(\nu^2+u^2\right)^{\frac{1}{2}}(\ell-x)\right) + \tfrac{1}{2}u.$$

In this case, the duality implies that

$$\mathsf{a}^*(x) = \frac{\frac{1}{2}\nu^2}{\left(\nu^2+u^2\right)^{\frac{1}{2}} \coth\left(\left(\nu^2+u^2\right)^{\frac{1}{2}}(\ell-x)\right) + u}.$$

Example 2. Consider an $N$-layer system, with layer $i$ having uniform drift $u_i$ and length $\ell_i$. Then for each layer define the following excursion rates:

$$\begin{aligned}
\mathsf{a}_+^i &= \tfrac{1}{2}\left(\nu^2+u_i^2\right)^{\frac{1}{2}} \coth\left(\left(\nu^2+u_i^2\right)^{\frac{1}{2}}\ell_i\right) + \tfrac{1}{2}u_i,\\
\mathsf{a}_-^i &= \tfrac{1}{2}\left(\nu^2+u_i^2\right)^{\frac{1}{2}} \coth\left(\left(\nu^2+u_i^2\right)^{\frac{1}{2}}\ell_i\right) - \tfrac{1}{2}u_i,\\
\mathsf{b}_+^i &= \tfrac{1}{2}\left(\nu^2+u_i^2\right)^{\frac{1}{2}} \exp\left(u_i\ell_i\right) \operatorname{cosech}\left(\left(\nu^2+u_i^2\right)^{\frac{1}{2}}\ell_i\right),\\
\mathsf{b}_-^i &= \tfrac{1}{2}\left(\nu^2+u_i^2\right)^{\frac{1}{2}} \exp\left(-u_i\ell_i\right) \operatorname{cosech}\left(\left(\nu^2+u_i^2\right)^{\frac{1}{2}}\ell_i\right),
\end{aligned}$$

The $\mathsf{a}_+$ and $\mathsf{a}_-$ rates are the excursion rates for *crossing a layer or being marked*, in the $+$ and $-$ directions, and the $\mathsf{b}_+$ and $\mathsf{b}_-$ rates are the excursion rates for *crossing a layer without being marked*, in the $+$ and $-$ directions.

Now define corresponding rates for layers $i$ to $N$ taken together, which we denote by upper case letters. A simple excursion argument gives

$$\mathsf{A}_+^i = \mathsf{a}_+^i - \frac{\mathsf{b}_+^i\mathsf{b}_-^i}{\mathsf{a}_-^i + \mathsf{A}_+^{i+1}}, \tag{4}$$

$$\mathsf{A}_-^i = \mathsf{A}_-^{i+1} - \frac{\mathsf{B}_+^{i+1}\mathsf{B}_-^{i+1}}{\mathsf{a}_-^i + \mathsf{A}_+^{i+1}}, \tag{5}$$

$$\mathsf{B}_+^i = \frac{\mathsf{b}_+^i\mathsf{B}_+^{i+1}}{\mathsf{a}_-^i + \mathsf{A}_+^{i+1}}, \tag{6}$$

$$\mathsf{B}_-^i = \frac{\mathsf{b}_-^i\mathsf{B}_-^{i+1}}{\mathsf{a}_-^i + \mathsf{A}_+^{i+1}}. \tag{7}$$

Clearly, for layer $N$, we need

$$\mathsf{A}_+^N = \mathsf{a}_+^N,\ \mathsf{A}_-^N = \mathsf{a}_-^N,\ \mathsf{B}_+^N = \mathsf{b}_+^N,\ \mathsf{B}_-^N = \mathsf{b}_-^N. \tag{8}$$

Equations (4–8) naturally lead to continued fractions for the rates for the whole $N$-layer system.
Now applying the general duality (2), we find

$$\mathsf{A}_+^i = \frac{\frac{1}{4}\nu^2}{\left(\mathsf{A}_+^i - \frac{\mathsf{B}_+^i\mathsf{B}_-^i}{\mathsf{A}_-^i}\right)^*}.$$

where $(\cdot)^*$ denotes the dual of $(\cdot)$, which in this case means replacing $u_i$ by $-u_i$ throughout. We can find other similar relations using the general duality, but these are left to the reader! Furthermore, this example can be used to recover the corresponding random walk results [1], by considering the process each time it is at an end point of one of the layers.

## 2 Connection with the Duality of Dette et al.

We now put the duality into the form conjectured by Dette *et al* [1]. Without loss of generality, take $a = 0$. Consider the processes $X$ and $X^*$, and recall that both processes have a reflecting boundary condition at the origin. The assumption that $u$ is locally integrable implies that $X$ and $X^*$ have well-behaved transition densities $p$ and $p^*$, with respect to Lebesgue measure. The semi-martingale normalization of local time implies that

$$\begin{aligned} \mathbb{E}\left\{\int_0^\infty \exp\left(-\lambda t\right) L(t,0)dt\right\} &= \int_0^\infty \exp(-\lambda t)\int_0^t p(s,0,0)ds\,dt & (9)\\ &= \lambda^{-1}\int_0^\infty \exp(-\lambda t)p(t,0,0)dt; & (10)\end{aligned}$$

furthermore, using standard excursion theoretic reasoning,

$$\mathbb{E}\left\{\int_0^\infty \lambda\exp\left(-\lambda t\right) L(t,0)dt\right\} = \left(n_0(\text{marked or killed})\right)^{-1}. \tag{11}$$

Rearranging equation (2) and setting $x = 0$, we find

$$2\lambda^{-1}n_0^*(\text{marked}) = \left(n_0(\text{marked or killed})\right)^{-1},$$

which on combining with equations (9–11) gives

$$2\lambda^{-1}n_0^*(\text{marked}) = \int_0^\infty \exp(-\lambda t)p(t,0,0)dt. \tag{12}$$

Since

$$n_0^*(\text{marked}) = \int_0^\infty \lambda\exp(-\lambda t)n_0^*(\text{duration} > t)dt,$$

taking inverse Laplace transforms of equation (12) gives

$$n_0^*(\text{duration greater than } t) = \tfrac{1}{2}p(t,0,0). \tag{13}$$

Similarly, we find

$$n_0(\text{duration greater than } t \text{ or killed}) = \tfrac{1}{2}p^*(t,0,0),$$

i.e. the dual of equation (13).
The corresponding results for random walks can be obtained by embedding them in a non-uniform drifting Brownian motion, as mentioned at the end of example 2. Also the continued fractions that appear in Dette *et al* [1] all have very simple excursion theoretic interpretations (compare them to the continued fractions that result from expanding equations (4–8)).

# 3 Conclusion

First, equation (2) is at the heart of several known dualities, see Siegmund [4] and Dette *et al* [1], and the references therein. Second, equation (2) is very easy to prove directly.
It does not look likely that any form of this duality will have one side that is much easier to calculate than the other. However, as a user of excursion theory, and someone who spends a great deal of time calculating excursion rates, anything that halves the work and halves the size of my notebook of excursion rates is welcome.
In many polymer applications [3] it is necessary to calculate excursion rates for a diffusion process on a network, so it will be interesting to see if this duality generalizes to networks. Higher dimensional forms of this duality do not appear to exist.

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