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A SIMPLE FLUCTUATION LOWER BOUND FOR A DISORDERED MASSLESS RANDOM CONTINUOUS SPIN MODEL IN $D = 2$

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Abstract
We prove a finite volume lower bound of the order $\sqrt{\log N}$ on the delocalization of a disordered continuous spin model (resp. effective interface model) in $d = 2$ in a box of size N. The interaction is assumed to be massless, possibly anharmonic and dominated from above by a Gaussian. Disorder is entering via a linear source term. For this model delocalization with the same rate is proved to take place already without disorder. We provide a bound that is uniform in the configuration of the disorder, and so our proof shows that disorder will only enhance fluctuations.

1 Introduction

Our model is given in terms of the formal infinite-volume Hamiltonian

$$H[\eta](\varphi) = \frac{1}{2}\sum_{i,j} p(i-j)V(\varphi_i - \varphi_j) - \sum_i \eta_i \varphi_i \tag{1}$$

where the pair potential $V(t)$ is assumed to be twice continuously differentiable with an *upper* bound $V''(t) \leq c$ and $V(t) = V(-t)$, i.e symmetric. A configuration $\varphi = (\varphi_i)_{i\in\Lambda} \in \mathbb{R}^\Lambda$ can be viewed either as a continuous spin configuration or as an "effective interface". The $\eta = (\eta_i)_{i\in\Lambda}$ denotes an arbitrary fixed configuration of external fields.
We do not assume a lower bound on the curvature of the potential, in particular it might change sign and $V(t)$ might possess several minima. This is identical to [9] and unlike to

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results based on the Brascamp-Lieb inequalities [3, 4] which need the curvature to be strictly positive.
Our result will be valid for all choices of the potential $V(t)$ and the random field configurations η for which the integrals in finite volume are well-defined. For simplicity let us assume that V grows faster than linear to infinity, i.e. $\lim_{|x|\uparrow\infty}\frac{V(x)}{|x|^{1+\epsilon}}=\infty$. This guarantees the existence of the finite volume measure for all arbitrary fixed choices of $\eta\in\mathbb{R}^\Lambda$.
Finally $p(\cdot)$ is the transition kernel of an aperiodic, irreducible random walk X on $\mathbb{Z}^d$, assumed to be symmetric and, for simplicity, finite range.
Define, correspondingly the quenched finite volume Gibbs measures $\mu_N^{\hat\varphi}[\eta]$, in a square $\Lambda\equiv\Lambda_N$ of sidelength $2N+1$, centered at the origin to be

$$\begin{aligned}&\mu_N^{\hat\varphi}[\eta](F(\varphi))\\&:=\frac{\int d\varphi_\Lambda F(\varphi_\Lambda,\hat\varphi_{\Lambda^c})e^{-\frac{1}{2}\sum_{i,j\in\Lambda}p(i-j)V(\varphi_i-\varphi_j)-\sum_{i\in\Lambda,j\in\Lambda^c}p(i-j)V(\varphi_i-\hat\varphi_j)+\sum_{i\in\Lambda}\eta_i\varphi_i}}{Z_\Lambda^{\hat\varphi}[\eta]}\end{aligned}\tag{2}$$

where $\hat\varphi$ is a boundary condition, η a fixed "frozen" configuration of random fields in Λ and $Z_\Lambda^{\hat\varphi}$ is the normalization factor.
What kind of behavior of delocalization resp. localization is expected to occur in a massless disordered model in dimension $d=2$? As a motivation, consider the Gaussian nearest neighbor case first, i.e. $V(x)=\frac{x^2}{2}$ and $p(i-j)=\frac{1}{2d}$ for i and j nearest neighbors. Then, for any fixed configuration η_Λ, the measure $\mu_N^{\hat\varphi}[\eta]$ is a Gaussian measure with covariance matrix $(-\Delta_\Lambda)^{-1}$ and expectation

$$\int\mu_N^{\hat\varphi}[\eta](d\varphi_x)\varphi_x=\sum_{y\in\Lambda}(-\Delta_\Lambda)^{-1}_{x,y}\eta_y+\sum_{y\in\Lambda^c,|x-y|=1}(-\Delta_\Lambda)^{-1}_{x,y}\hat\varphi_y.\tag{3}$$

For every x and y in $\mathbb{Z}^d$, $d\ge 3$, the limit of $(-\Delta_\Lambda)^{-1}_{x,y}$ as $\Lambda\nearrow\mathbb{Z}^d$ exists and it is finite, diverges like $\log N$ in $d=2$. Taking for simplicity the random fields η_y to be i.i.d. standard Gaussians, denote their expectations by $\mathbb{E}$, we see that mean at the site 0 of the random interface is itself a Gaussian variable as a linear combination of Gaussians and has variance

$$\sigma_0^2=\sum_{y\in\Lambda}((-\Delta_\Lambda)^{-1}_{0,y})^2.\tag{4}$$

This should diverge as $\int^N r(\log r)^2dr\sim N^2(\log N)^2$ when the sidelength N of the box diverges to infinity. In dimension $d>2$, we have $\int^N r^{d-1}(r^{-(d-2)})^2dr$, so the interface stays bounded in $d>4$.
In particular the explicit computation shows that delocalization is enhanced by randomness in the Gaussian model. It is however not clear whether this phenomenon is still present in an anharmonic model where a separation of the influence caused by the η_i's is not possible and the minimizer of the Hamiltonian cannot be computed in a simple way. A priori one might not exclude the possibility that, depending on the interaction V, a symmetrically distributed random field possibly stabilizes the interface.
We show in this note that this is not the case and the divergence is at least as strong as in the model without disorder, for any fixed field configuration. The method is typically two-dimensional. Hence it does *not* show in the present form that in three or four dimensions disorder will cause an anharmonic localized interface to diverge. The latter would be a continuous spin-analogue of the result in [2] obtained for discrete disordered SOS-models. In

that paper the existence of stable two-dimensional SOS-interfaces was excluded, using a soft martingale argument in the spirit of [1]. A disadvantage of that method however lies in the inability to give explicit fluctuation lower bounds on the behavior of the interface in finite volume.
The present proof is based on a "two-dimensional" Mermin-Wagner type argument involving the entropy inequality (see [9]). The result is a quenched result, uniformly for all (and not only almost all) configuration of the disorder fields. We stress that such a "quenched instability" at any field configuration can only hold in $d = 2$, as the Gaussian interface shows. Indeed, for the Gaussian interface the instability of the interface is caused by fluctuations w.r.t. disorder *of* the groundstate, while the Gibbs fluctuations *relative to* the groundstate stay bounded. So the dimensionality of our result is correct.

1.1 Result and proof

Theorem 1.1 *Suppose $d = 2$. Suppose that $\eta \in \mathbb{R}^\Lambda$ is an arbitrary fixed configuration of fields. Then there exists a constant c, independent of η, such that*

$$\mu^0_N[\eta]\Big(|\varphi_0| \geq T\sqrt{\log N}\Big) \geq e^{-cT^2}. \tag{5}$$

Remark: This generalizes the inequality of [9] to the case of arbitrary linear disorder fields. We thus see that the interface is to (at least) one side "at least as divergent" as in the case without disorder.
Remark 2: Let us suppose that η are symmetrically distributed random variables, possibly non-i.i.d. with *any* dependence structure. Then we have as a corollary the averaged one-sided bound

$$\int \mathbb{P}(d\eta)\mu^0_N[\eta]\Big(\varphi_0 \geq T\sqrt{\log N}\Big) \geq e^{-cT^2}/2. \tag{6}$$

This follows immediately from the Theorem, by symmetry. Note that no integrability assumptions on the distribution of the random fields are needed, given the lower bound on the potential we assume.

Proof: As in [9] we take a test-configuration $\bar\varphi$, to be chosen later, that interpolates between $\bar\varphi_0 = R$ and $\bar\varphi_x \equiv 0$ for $x \in \Lambda^c_N$. We define the *shifted measure* Êto be $\mu^0_{N;\bar\varphi}[\eta](\cdot) = \mu^0_N[\eta](\cdot+\bar\varphi)$. Note that $\bar\varphi$ does not depend on η.
Let us drop the boundary condition from our notation and write $\mu_N[\eta] \equiv \mu^0_N[\eta]$ in the following. Using the entropy-inequality we have

$$\begin{aligned}
&\mu_N[\eta](|\varphi_0| \geq R)\\
&= \sum_{s=\pm 1} \mu_N[\eta](s\varphi_0 \geq R)\\
&= \sum_{s=\pm 1} \mu_N[s\eta](\varphi_0 \geq R)\\
&= \sum_{s=\pm 1} \mu_{N;\bar\varphi}[s\eta](\varphi_0 \geq 0)\\
&\geq \sum_{s=\pm 1} \mu_N[s\eta](\varphi_0 \geq 0)\exp\Big(-\frac{1}{\mu_N[s\eta](\varphi_0 \geq 0)}\Big(H(\mu_{N;\bar\varphi}[s\eta]|\mu_N[s\eta]) + e^{-1}\Big)\Big).
\end{aligned} \tag{7}$$

It remains to control the relative entropy

$$H(\mu_{N;\bar\varphi}[s\eta]|\mu_N[s\eta]) = \int \mu_{N;\bar\varphi}[s\eta](d\varphi)\log\Big(\frac{d\mu_{N;\bar\varphi}[s\eta]}{d\mu_N[s\eta]}(\varphi)\Big). \tag{8}$$

The strategy of the proof is to show that we may choose $R = R(N)$ diverging with N so that $\inf_{\substack{\bar\varphi:\bar\varphi_0=R \text{ and} \\ \bar\varphi_x\equiv 0 \text{ for } x\in\Lambda_N^c}} H(\mu_{N;\bar\varphi}[s\eta]|\mu_N[s\eta]) \le \text{Const}$, uniformly in N. This is identical to the case without disorder. Further we show below that the bound is also uniform in the field configuration η.
Turning to the relative entropy we note that the appearing partition functions cancel and so

$$\frac{d\mu_{N;\bar\varphi}[s\eta]}{d\mu_N[s\eta]}(\varphi) = \exp\Big(-H^0_\Lambda[s\eta](\varphi-\bar\varphi) + H^0_\Lambda[s\eta](\varphi)\Big). \tag{9}$$

Therefore

$$H(\mu_{N;\bar\varphi}[s\eta]|\mu_N[s\eta]) = \int \mu_N[s\eta](d\varphi)\Big(-H^0_\Lambda[s\eta](\varphi) + H^0_\Lambda[s\eta](\varphi+\bar\varphi)\Big). \tag{10}$$

We rewrite the integrand of (10) in the form

$$\begin{aligned}
&- H^0_\Lambda[s\eta](\varphi) + H^0_\Lambda[s\eta](\varphi+\bar\varphi)\\
&= \frac{1}{2}\sum_{i,j\in\Lambda} p(i-j)\Big(V(\varphi_i-\varphi_j) - V(\varphi_i-\varphi_j+\bar\varphi_i-\bar\varphi_j)\Big)\\
&+ \sum_{i\in\Lambda, j\in\Lambda^c} p(i-j)\Big(V(\varphi_i) - V(\varphi_i+\bar\varphi_i)\Big) - s\sum_{i\in\Lambda}\eta_i\bar\varphi_i.
\end{aligned} \tag{11}$$

We use now the symmetrization trick brought to our attention by Yvan Velenik (cf. [8, 5]) which here simply consists in estimating

$$H(\mu_{N;\bar\varphi}[s\eta]|\mu_N[s\eta]) \le \sum_{s'=\pm1} H(\mu_{N;\bar\varphi}[s'\eta]|\mu_N[s'\eta]). \tag{12}$$

We note that the s'-sum over the random potential term simply vanishes since it is independent of φ and hence

$$\sum_{s'=\pm1} s'\sum_{i\in\Lambda}\eta_i\bar\varphi_i = 0. \tag{13}$$

Finally, to estimate the other term we make apparent the quenched measure $\frac{\mu_N[\eta]+\mu_N[-\eta]}{2}$ and use its symmetry.
So we have that

$$\begin{aligned}
&2\int \frac{\mu_N[\eta]+\mu_N[-\eta]}{2}(d\varphi)\Big(V(\varphi_i-\varphi_j) - V(\varphi_i-\varphi_j+\bar\varphi_i-\bar\varphi_j)\Big)\\
&\le 2\int \frac{\mu_N[\eta]+\mu_N[-\eta]}{2}(d\varphi)V'(\varphi_i-\varphi_j)(\bar\varphi_i-\bar\varphi_j) + c(\bar\varphi_i-\bar\varphi_j)^2\\
&= c(\bar\varphi_i-\bar\varphi_j)^2.
\end{aligned} \tag{14}$$

This gives

$$H(\mu_{N;\bar\varphi}[s\eta]|\mu_N[s\eta]) \le \frac{c}{2}\sum_{i,j\in\Lambda} p(i-j)(\bar\varphi_i-\bar\varphi_j)^2 + c\sum_{i\in\Lambda, j\in\Lambda^c} p(i-j)\bar\varphi_i^2 \tag{15}$$

for both $s=\pm1$. Keeping only the s-term in inequality (7) for which $\mu_N[s\eta](\varphi_0\ge 0)\ge\frac{1}{2}$ one obtains in fact

$$\begin{aligned}&\mu_N[\eta](|\varphi_0|\ge R)\\ &\ge \frac{1}{2}\exp\Big(-2\Big(\frac{c}{2}\sum_{i,j\in\Lambda} p(i-j)(\bar\varphi_i-\bar\varphi_j)^2 + c\sum_{i\in\Lambda, j\in\Lambda^c} p(i-j)\bar\varphi_i^2 + e^{-1}\Big)\Big).\end{aligned} \tag{16}$$

This is exactly the same bound as in the case of vanishing η. It remains to choose $\bar\varphi$ optimal. Denoting by X a random walk with the transition kernel p, we choose as in [9], $\bar\varphi_i = R\mathbb{P}_i[T_{\{0\}}<\tau_{\Lambda_N}]$, where $\mathbb{P}_i$ is the measure of the random walk started in the point i, $T_{\{0\}}=\min\{n : X_n = 0\}$ and $\tau_{\Lambda_N}=\min\{n : X_n\notin\Lambda_N\}$. Taking into account the estimate [7]

$$\mathbb{P}_i[T_{\{0\}}<\tau_{\Lambda_N}] \simeq \frac{\ln(|i|+1)}{\ln(N+1)}$$

gives indeed

$$\inf_{\substack{\bar\varphi:\bar\varphi_0=R \text{ and}\\ \bar\varphi_x\equiv 0 \text{ for } x\in\Lambda_N^c}} \left(\frac{c}{2}\sum_{i,j\in\Lambda_N} p(i-j)(\bar\varphi_i-\bar\varphi_j)^2 + c\sum_{i\in\Lambda_N, j\in\Lambda_N^c} p(i-j)\bar\varphi_i^2\right) \le \mathrm{Const}\,\frac{R^2}{\log N}. \tag{17}$$

Choosing $R = T\sqrt{\log N}$ one obtains (5). $\square$

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