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HEAT KERNEL ASYMPTOTICS ON THE LAMPLIGHTER GROUP

DAVID REVELLE [1]
Department of Statistics, # 3860
University of California, Berkeley
367 Evans Hall
Berkeley, CA 94720-3860
United States of America
email: revelle@stat.berkeley.edu

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Abstract
We show that, for one generating set, the on-diagonal decay of the heat kernel on the lamplighter group is asymptotic to $c_1 n^{1/6} \exp[-c_2 n^{1/3}]$. We also make off-diagonal estimates which show that there is a sharp threshold for which elements have transition probabilities that are comparable to the return probability. The off-diagonal estimates also give an upper bound for the heat kernel that is uniformly summable in time. The methods used also apply to a one dimensional trapping problem, and we compute the distribution of the walk conditioned on survival as well as a corrected asymptotic for the survival probability. Conditioned on survival, the position of the walker is shown to be concentrated within $\alpha n^{1/3}$ of the origin for a suitable α.

1 Introduction

Let μ be a symmetric, finitely supported probability measure whose support generates a group G. The measure μ generates a random walk X_n on G whose transition probabilities are given by $p(x, y) = \mu(x^{-1}y)$. There are a number of connections between the geometry of G and the long term behavior of $\mathbb{P}(X_n = x)$. For example, if G has polynomial volume growth, that is if the volume of a ball of radius n grows like n^d for some d, then for any irreducible, aperiodic walk there is a constant C such that

$$C^{-1} n^{-d/2} \exp\left[-\frac{C|x|^2}{n}\right] \le \mathbb{P}(X_n = x) \le C n^{-d/2} \exp\left[-\frac{|x|^2}{Cn}\right] \tag{1}$$

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where $|x|$ is the word length of x [9]. Another result along these lines is that if G is amenable, which is equivalent to $p^n(x,x)$ decaying subexponentially, then for any irreducible, aperiodic walk [3]

$$\lim_{n\to\infty} \frac{\mathbb{P}(X_n = x)}{\mathbb{P}(X_n = e)} = 1. \tag{2}$$

For amenable groups with exponential volume growth, however, no lower bound of the form

$$C^{-1} p_n(e,e) \exp\left[-\frac{C|x|^2}{n}\right] \tag{3}$$

is possible because summing (3) over the ball of radius $[-n \log p_n(e,e)]^{1/2}$ gives a quantity that is unbounded in n, while $\sum_{x\in G} \mathbb{P}(X_n = x) = 1$. A more complex bound is therefore necessary.

In this paper, we will compute precise asymptotics for the return probability on the lamplighter group as well as bounds for the distribution of the position of a random walk away from the identity. Much of the behavior that we obtain appears to be new in that it does not occur in the case of groups with polynomial volume growth.

The lamplighter group is a name often given to the wreath product $G \wr \mathbb{Z}$. The elements of the group are pairs of the form (η, y), where $y \in \mathbb{Z}$ and η is an element of $\sum_{\mathbb{Z}} G$, that is to say, $\eta : \mathbb{Z} \to G$ is a function such that $\eta(x) = e$ for all but finitely many $x \in \mathbb{Z}$. Multiplication is given by $(\eta_1, y_1)(\eta_2, y_2) = (\eta, y_1 + y_2)$, where $\eta(x) = \eta_1(x)\eta_2(x - y_1)$. This multiplication rule means that $\mathbb{Z}$ acts by coordinate shift on $\sum_{\mathbb{Z}} G$, and $G \wr \mathbb{Z} = \sum_{\mathbb{Z}} G \rtimes \mathbb{Z}$ is the resulting semi-direct product. These groups are known as lamplighter groups for the following reason: imagine that there is a lamp located at each vertex of the Cayley graph of $\mathbb{Z}$ and that there is a lamplighter who moves from lamp to lamp. Let a_0 denote the element of $\sum_{\mathbb{Z}} G$ that is a at 0 and the identity elsewhere, and let id denote the identity element in $\sum_{\mathbb{Z}} G$. In $G \wr \mathbb{Z}$, multiplying by $(a_0, 0)$ corresponds to changing the current lamp by a, and multiplying by $(id, \pm 1)$ corresponds to the lamplighter moving left or right. The second coordinate y denotes the position of the lamplighter, while the configuration $\eta \in \sum_{\mathbb{Z}} G$ represents the brightness of each lamp. The case of $\mathbb{Z}_2 \wr \mathbb{Z}$ corresponds to an infinite street of lamps, finitely many of which are on.

Our goal is to understand the long term behavior of the transition probabilities for a random walk on $G \wr \mathbb{Z}$. In contrast to current on-diagonal bounds for the heat kernel, which use fairly general geometric arguments (see e.g., [5]), our off-diagonal methods involve a more direct computation and are limited to one specific generating set. More precisely, consider the random walk associated with the measure $\nu * \mu * \nu$, where μ is a simple random walk by the lamplighter, and ν is a measure causing the lamplighter to randomize the current lamp. That is to say, $\mu(id, \pm 1) = 1/2$, and $\nu(a_0, 0) = 1/|G|$ for all $a \in G$. In terms of the lamplighter description, this convolution measure corresponds to the following walk: at each time, the lamplighter randomizes the current lamp, moves to an adjacent lamp, and randomizes the new lamp. Varopoulos [17] made the first on-diagonal estimates for the heat kernel on $\mathbb{Z}_2 \wr \mathbb{Z}$ also using a convolution measure, but he used the measure $\mu * \nu * \mu$. Switching to the measure $\nu * \mu * \nu$ simplifies some of the computations–this particular convolution has now become more common and has been used in [12], [14], and [7].

The reason why these generators are relatively nice is because any lamp that has been visited is randomized, so the analysis is reduced to studying visited sites by simple random walk on $\mathbb{Z}$. For a given element of the group $g = (\eta, x)$, let $F_R(g) = \max\{i : \eta(i) \neq e\} \vee x$ and $F_L(g) = -(\min\{i : \eta(i) \neq e\} \wedge x)$. Let π be the natural projection from $G \wr \mathbb{Z}$ onto $\mathbb{Z}$ given by

$\pi(g) = x$. In terms of the lamplighter representation, F_R is the coordinate of the rightmost lamp that is on (or the lamplighter's position if that is farther from 0), and F_L is a similar quantity for the leftmost lamp. Finally, let $\phi(g) = F_R(g) + F_L(g)$ be the number of lamps that need to be visited to get from the identity to g. We can then compute precise asymptotics for the return probability as well as find which other elements have a comparable proability of being hit at time n.

Theorem 1. *For a random walk on $G \wr \mathbb{Z}$, where G is finite and the walk is generated by the measure $\nu * \mu * \nu$ given above, the asymptotics for the return probability are given by*

$$\lim_{n\to\infty} \frac{\mathbb{P}(X_{2n} = e)}{(2n)^{1/6} \exp\left[-c_1(|G|)(2n)^{1/3}\right]} = c_2(|G|) \tag{4}$$

$$\textit{where} \quad c_1(|G|) = \frac{3}{2}(\pi \log |G|)^{2/3}, \quad c_2(|G|) = \frac{2(|G|-1)^2 \pi^{5/6}}{|G|(\log |G|)^{2/3}} \sqrt{\frac{2}{3}}.$$

Moreover, let $\alpha = (\pi^2/\log|G|)^{1/3}$. For a given $\epsilon > 0$, let $U_n(\epsilon) = \{g : \phi(g) < (1-\epsilon)\alpha n^{1/3}, x \in 2\mathbb{Z}\}$ and let $V_n(\epsilon) = \{g : \phi(g) > (1+\epsilon)\alpha n^{1/3}\}$. Then

$$\lim_{n\to\infty} \frac{\inf_{g\in U_{2n}(\epsilon)} \mathbb{P}(X_{2n} = g)}{\mathbb{P}(X_{2n} = e)} > 0 \tag{5}$$

$$\lim_{n\to\infty} \frac{\sup_{g\in V_{2n}(\epsilon)} \mathbb{P}(X_{2n} = g)}{\mathbb{P}(X_{2n} = e)} = 0. \tag{6}$$

In addition to the classical case of random walks on $\mathbb{Z}^d$, precise asymptotics for return probabilities are known on a variety of groups, many examples of which are discussed in Section 6 of the survey by Woess [18]. Although the asymptotic type of the return probability is a group invariant [13], so the $n^{1/3}$ in the exponential in (4) does not depend on the underlying measure, an example of Cartwright [4] shows that algebraic prefactor can depend on the underlying measure. As a result, it would be interesting to know how robust the factor of $n^{1/6}$ is on this example.

The more interesting part of the theorem is the second part of the theorem means that the transitions probabilities are comparable to the return probability for $\phi(g) < \alpha n^{1/3}$. This is somewhat similar to the fact that the transition probabilities are somewhat evenly distributed on a range of the order of $n^{1/2}$ in the classical case, but the sharp cut off at $\phi(g) = \alpha n^{1/3}$ does not occur in the classical case.

Heuristically, the reason for the cutoff at $\alpha n^{1/3}$ is as follows: the dominant contribution in the return probability is the probability that all of the lamps are off. Visiting a small number of lamps increases the chances that all of them are off, but the probability of having visited a small number of lamps decreases as the number tends to 0. Solving an optimization problem shows that the dominant contribution in the return probability is when $\alpha n^{1/3}$ lamps have been visited, at which point any configuration that requires visiting fewer than $\alpha n^{1/3}$ lamps is also quite likely, but a configuration that requires visiting more than $\alpha n^{1/3}$ lamps is unlikely to have been reached. A key geometric point about the sets U_n is that they are not balls: instead, they are a Følner sequence, meaning that the ratio of boundary to volume tends to zero. In fact, it turns out that for a fixed volume, the sets U_n have, up to a constant factor, the minimal possible ratio of boundary to volume for this group.

The lamplighter group is one of the examples for which the entire spectrum for some random walks is known. Grigorchuk and Zuk [8] computed the complete spectrum for the random

walk corresponding to a slightly different generating set on the lamplighter group, and the spectrum for some more general groups was computed by Dicks and Schick [6]. Asymptotics for the return probability can be computed directly from the spectrum, although we do not use that technique here. In particular, the probability of returning to the identity at time n is always exactly twice as high with the generators considered by Grigorchuk and Zuk as with the ones considered here.
The behavior away from the identity is described by the following theorem:

Theorem 2. *Consider the random walk as in Theorem 1. There exists a constant $C > 0$ such that for $\phi(g) > (\alpha/2)n^{1/3}$,*

$$\mathbb{P}(X_n = g) \leq C\left([\phi(g)]^2 n^{-5/2} + n^2[\phi(g)]^{-7}\right)\exp\left(-\phi(g)\log|G| - \frac{[\phi(g)]^2}{Cn} - \frac{n}{C[\phi(g)]^2}\right). \quad (7)$$

Because $\mathbb{P}(X_{2n} = g) \leq \mathbb{P}(X_{2n} = e)$, taking the minimum of (4) and (7) gives an upper bound for any $g \in G$. There are $(a+b+1)^2 \exp[(a+b+1)\log k]$ elements in the group with $\phi(g) = a+b$, so summing the upper bound over G gives a quantity that is uniformly bounded in n. Since $\sum_g \mathbb{P}(X_n = g) = 1$, one criterion for a good upper bound is that it should be uniformly bounded when summing over G. While we do not have a matching lower bound, (7) thus cannot be globally improved by multiplying by a function that uniformly tends to 0. Note that when $\phi(g) = o(n)$, the dominant term in the expression is $\exp[-\phi(g)\log k]$, which exactly cancels out the exponential part of the volume growth and does not depend on n. The lower order terms help give a qualitative description of the long term behavior of the walk. They show that the mass at time n is concentrated on elements with $\phi(g)$ on the order of $\sqrt{n}$. This fact was already known to the extent that for any finitely supported walk for which the lamplighter has no drift, $\limsup |X_n|(n\log\log n)^{-1/2} \in (0,\infty)$ and $\liminf |X_n|(\log\log n)^{1/2}n^{-1/2} \in (0,\infty)$ (see [14]).
The computations used in proving Theorems 1 and 2 also lend themselves to the study of a problem that appears in the mathematical physics literature. In the Rosenstock trapping model a particle performs simple random walk in a lattice that contains absorbing traps (see e.g., [10] for a more detailed description). Physically, the particle is often described as a photon, with the traps being impurities in the crystal that absorb light.
Each point in the crystal is a trap with probability $1-q$, and independently of the other locations. The probability of the photon surviving up to time n is then simply Eq^{R_n}, where R_n is the number of sites (including 0) visited in the first n steps. The probability of the photon having survived and being at a given point x is then $\mathbb{E}(q^{R_n}; S_n = x)$. Even though S_n is distributed fairly evenly on the order of $[-n^{1/2}, n^{1/2}]$, it turns out that a surviving particle tends to be at a distance of no more than $\alpha n^{1/3}$ from the origin, again with a sharp cutoff.

Theorem 3. *For $q \in (0,1)$, as $n \to \infty$ the survival probability is asymptotic to*

$$\mathbb{E}q^{R_n} \sim \frac{8(1-q)^2 n^{1/2}}{-q\log q}\sqrt{\frac{2}{3\pi}}\exp\left[-\frac{3}{2}(\pi\log q)^{2/3}n^{1/3}\right]. \quad (8)$$

Conditioned on survival, the probability of returning to the origin at time $2n$ is given by

$$\mathbb{E}(q^{R_{2n}}; S_{2n} = 0) \sim \frac{2(1-q)^2\pi^{5/6}}{q(\log q)^{2/3}}\sqrt{\frac{2}{3}}(2n)^{1/6}\exp\left[-\frac{3}{2}(\pi\log q)^{2/3}(2n)^{1/3}\right], \quad (9)$$

which is the same expression as $\mathbb{P}(X_{2n} = e)$. Again conditioned on survival, for $a < 1$, the probability of being at $x = 2\lfloor a\alpha n^{1/3}/2\rfloor$ at even times is

$$\mathbb{E}(q^{R_{2n}}; S_{2n} = x) \sim \left[(1-a)\cos(\pi a) + \frac{1}{\pi}\sin(\pi a)\right]\mathbb{E}(q^{R_{2n}}; S_{2n} = 0). \tag{10}$$

For $a > 1$, there is a constant $c(a,q)$ such that for $x = 2\lfloor a\alpha n^{1/3}/2\rfloor$,

$$\mathbb{E}(q^{R_n}; S_n = x) \sim \frac{c(a,q)}{n}\exp\left[-\left(a + \frac{1}{2a^2}\right)(\pi\log q)^{2/3}n^{1/3}\right]. \tag{11}$$

Because $a + (2a^2)^{-1}$ has a minimum when $a = 1$, (11) implies that $\mathbb{P}(S_n = x \mid \text{survival})$ is concentrated on $x \in (-\alpha n^{1/3}, \alpha n^{1/3})$. A nonrigorous argument giving the qualitative behavior of (8) was given by Anlauf [1], but for reasons that we will see later, the constant was incorrect. For the case of $\mathbb{E}(q^{R_n}; S_n = 0)$, (9) was essentially given in [16], with again an error in the constant that is corrected in Hughes [10], eq 6.316. This trapping model was also considered for general dimension d by Antal [2], who found asymptotics for the log of the survival probability. Many of the techniques that Antal used were developed for a continuous analog to this problem studied by Sznitman [15].

2 Asymptotic Summations

Before analyzing the random walk, we need some preliminary computations. The heart of all of the bounds rely on computing asymptotics for a variety of summations. All of the summations in this section are over the integers–when the limits of a summation are irrational, it is to be understood that the summation is over integers between the limits.

Lemma 1. *For $q < 1$, let $\alpha = (-\pi^2/\log q)^{1/3}$ and $\varphi(\rho) = -n\pi^2/(2\rho^2) + \rho\log q$. Fix a constant $a > 0$, and suppose that $f(\rho)$ is a differentiable function on $[a\alpha n^{1/3}, \infty)$ such that $|\rho f'(\rho)/f(\rho)| \le k$ for some $k \ge 1$. Then if $a < 1$,*

$$\sum_{\rho > a\alpha n^{1/3}} f(\rho)\exp[\varphi(\rho)] \sim f\left(\alpha n^{1/3}\right)\sqrt{\frac{2}{3}}\frac{\pi^{5/6}}{(\log q)^{2/3}}n^{1/6}\exp[\varphi(\alpha n^{1/3})]. \tag{12}$$

If $a > 1$, then

$$\sum_{\rho > a\alpha n^{1/3}} \left(\rho - a\alpha n^{1/3}\right)^3 f(\rho)\exp[\varphi(\rho)] \sim f\left(a\alpha n^{1/3}\right)\exp\left[\varphi\left(a\alpha n^{1/3}\right)\right]C(a,q), \tag{13}$$

where $C(a,q)$ is given by

$$C(a,q) = \frac{q^{(1-a^{-3})}\left(1 + 4q^{(1-a^{-3})} + q^{2(1-a^{-3})}\right)}{\left(1 - q^{(1-a^{-3})}\right)}.$$

Proof. Let $\theta = \alpha n^{1/3}$, so $\varphi(\theta)$ is the maximum value of $\varphi(\rho)$. The idea of the proof is that the mass of $\exp[\varphi(\rho)]$ is centered around θ, and so we can truncate the tails. Near θ, we then use a power series expansion to obtain a tractable summation.

By our assumption on the derivative of f, we have $x^{-k} \le f(x\rho)/f(\rho) \le x^k$ for $x > 1$. Because $\theta > 1$ and $\varphi(\rho)$ is increasing on $(0, \theta)$, this implies that for $\epsilon > 0$,

$$\sum_{1<\rho<\theta(1-\epsilon)} f(\rho)\exp[\varphi(\rho)] \le \theta^{k+1} f(\theta)\exp[\varphi([1-\epsilon]\theta)].$$

But this expression is $o(f(\theta)\exp[\varphi(\theta)])$ and thus does not contribute significantly to the summation. To bound the upper tail, note that

$$\begin{aligned}\sum_{\rho>(1+\epsilon)\theta} f(\rho)\exp[\varphi(\rho)] = &\sum_{\rho=(1+\epsilon)\theta}^{2\theta} f(\rho)\exp[\varphi(\rho)] + \sum_{\rho>2\theta} f(\rho)\exp[\varphi(\rho)] \\ \le &2\theta 2^k f(\theta)\exp[\varphi([1+\epsilon]\theta)] + \sum_{\rho>2\theta} f(\rho) q^\rho.\end{aligned}$$

which is again of a lower order than $f(\theta)\exp[\varphi(\theta)]$. Because both the upper and lower tails are relatively small, the summation is asymptotically equivalent to

$$\sum_{\rho>a\alpha n^{1/3}} f(\rho)\exp[\varphi(\rho)] \sim \sum_{\rho=\theta(1-\epsilon)}^{\theta(1+\epsilon)} f(\rho)\exp[\varphi(\rho)].$$

To estimate this last summation, we will use the power series expansion for $\varphi(\rho)$. Because $\varphi'(\theta) = 0$, the power series expansion around θ for $\varphi(\rho)$ is of the form

$$\varphi(\rho) = \varphi(\theta) - \left[\frac{3n\pi^2}{2\theta^4}(\rho-\theta)^2\right] - \epsilon^3 O(n^{-2/3}).$$

Since our assumptions on f imply that $f[\rho(1+\epsilon)]/f(\rho) \le (1+\epsilon)^k$, there is a function $g(\epsilon)$ satisfying $\lim_{\epsilon\to 0} g(\epsilon) = 1$ and

$$\begin{aligned}g(\epsilon)f(\theta)\exp[\varphi(\theta)]\exp\left[-\frac{1}{2}\varphi''(\theta)(\theta-\rho)^2\right] &\le f(\rho)\exp[\varphi(\rho)] \\ &\le \frac{1}{g(\epsilon)} f(\theta)\exp[\varphi(\theta)]\exp\left[-\frac{1}{2}\varphi''(\theta)(\theta-\rho)^2\right].\end{aligned}$$

Thus, up to a factor of $g(\epsilon)$, we have

$$\sum_{\rho=\theta(1-\epsilon)}^{\theta(1+\epsilon)} f(\rho)\exp[\varphi(\rho)] \simeq f(\theta)\exp[\varphi(\theta)] \sum_{\rho=\theta(1-\epsilon)}^{\theta(1+\epsilon)} \exp\left[-\frac{3\pi^2(\rho-\theta)^2}{2\alpha^4 n^{1/3}}\right].$$

Recognizing the summation as a Riemann sum, note that for $\beta > 0$ and $\epsilon >> n^{-1/6}$,

$$\begin{aligned}\sum_{\rho=\theta(1-\epsilon)}^{\theta(1+\epsilon)} \exp\left[-\frac{\beta(\rho-\theta)^2}{n^{1/3}}\right] &\sim \frac{n^{1/6}}{\sqrt{\beta}} \int_{-\epsilon\theta\sqrt{\beta}/n^{1/6}}^{\epsilon\theta\sqrt{\beta}/n^{1/6}} e^{-x^2} dx \\ &\sim n^{1/6}\sqrt{\frac{\pi}{\beta}}.\end{aligned}$$

To complete the proof, take $\epsilon = n^{-1/12}$, so $g(\epsilon) \to 1$. Taking $\beta = 3\pi^2/(2\alpha^4)$ and $\alpha = (-\pi^2/\log q)^{1/3}$ gives $(\pi/\beta)^{1/2} = (\log q)^{-2/3}\pi^{5/6}(2/3)^{1/2}$, which is the claimed constant in (12).

The ideas of the proof of (13) are similar, but when $a > 1$, $\varphi(\rho)$ is maximized at the left endpoint $\theta = a\alpha n^{1/3}$ instead of in the middle. Again, most of the mass of the summation is near this critical θ. For ρ close to θ, the linear expansion $\varphi(\rho) \simeq \varphi(\theta) + (\rho - \theta)\varphi'(\theta)$ is reasonably accurate, and so truncating the tails as before, it follows that

$$\begin{aligned}
\sum_{\rho>\theta}(\rho-\theta)^3 f(\rho)\exp[\varphi(\rho)] &\sim f(\theta)\exp[\varphi(\theta)]\sum_{\rho=\theta}^{\infty}(\rho-\theta)^3\exp[(\rho-\theta)\varphi'(\theta)] \\
&= f(\theta)\exp[\varphi(\theta)]\frac{\exp[\varphi'(\theta)]\left(\exp[2\varphi'(\theta)] + 4\exp[\varphi'(\theta)] + 1\right)}{\left(1-\exp[\varphi'(\theta)]\right)^4} \\
&= f(\theta)\exp[\varphi(\theta)]\frac{q^{(1-a^{-3})}\left(1+4q^{(1-a^{-3})}+q^{2(1-a^{-3})}\right)}{\left(1-q^{(1-a^{-3})}\right)}.
\end{aligned}$$

Substituting $a\alpha n^{1/3}$ back in for θ completes the proof. □

3 Random Walks

With Lemma 1 in hand, we turn now to our main results. We begin by examining the most general case, and then later do the simplifications to the special cases that are of particular interest.

Let S_n be a simple random walk on $\mathbb{Z}$ starting at $S_0 = 0$, $R_n = \#\{0 \le i \le n : S_i\}$ denote the number of visited sites (including 0) up to time n, $M_n = \max\{S_k : k \le n\}$ the rightmost visited site, and $m_n = -\min\{S_k : k \le n\}$ the leftmost visited site. We will ultimately be computing a variety of joint probability estimates of these quantities.

The reason for the following computations is that if g is a group element such that $F_R(g) = F_R$, $F_L(g) = F_L$, and $\pi(g) = x$, then $\mathbb{P}(X_n = g) = \sum_\rho q^\rho P_n(\rho, x, F_L, F_R)$, where $q = 1/|G|$ and

$$P_n(\rho, x, F_L, F_R) = \mathbb{P}\{R_n = \rho, S_n = x, m_n \ge F_L, M_n \ge F_R\}.$$

This relationship is because, as mentioned earlier, the lamps visited by the lamplighter have been independently randomized. In order for X_n to be equal to g, the lamplighter must visit both $F_L(g)$ and $F_R(g)$, all lamps visited must be in the correct configuration, and the lamplighter needs to end at the correct place.

For $l, r > 0$, let $U_n(x, l, r) = \mathbb{P}\{S_n = x, S_k \in (-l, r), k \le n\}$. Using reflection arguments, $U_n(x, l, r) = \sum_{i=-\infty}^{\infty} \mathbb{P}\{S_n = x + 2i(l+r)\} - \mathbb{P}\{S_n = x + 2i(l+r) + 2l\}$. When $l + r = o\left(n^{1/2}\right)$, Fourier analysis (see e.g. [10] eq. 3.291) turns this into the more useful expression

$$\begin{aligned}
U_n(x, l, r) &= \frac{1}{l+r}\sum_{|k|<l+r}\cos^n\left(\frac{\pi k}{l+r}\right)\sin\left(\frac{\pi k l}{l+r}\right)\sin\left(\frac{\pi k(x+l)}{l+r}\right) && (14) \\
&= \frac{1}{2(l+r)}\sum_{|k|<l+r}\cos^n\left(\frac{\pi k}{l+r}\right)\left[\cos\left(\frac{\pi k x}{l+r}\right) - \cos\left(\frac{\pi k(x+2l)}{l+r}\right)\right]. && (15)
\end{aligned}$$

By viewing cosine as the real part of the exponential, summing a geometric series gives the identity

$$\sum_{l=a}^{b} \cos\left(\frac{\pi k(x+2l)}{\rho}\right) = \sin\left(\frac{\pi k(b-a+1)}{\rho}\right)\cos\left(\frac{\pi k(b+a+x)}{\rho}\right)\csc\left(\frac{\pi k}{\rho}\right).$$

Let $V_n(x,l,r) = \mathbb{P}\{S_n = x, M_n = r, m_n = l\}$. Then by a simple inclusion-exclusion argument, $V_n(x,l,r) = U_n(x,l+1,r+1) - U_n(x,l,r+1) - U_n(x,l+1,r) + U_n(x,l,r)$. Because

$$P_n(\rho+1,x,F_L,F_R) = \sum_{l=F_L}^{\rho-F_R} V_n(x,l,\rho-l),$$

we also have

$$\begin{aligned}P_n(\rho+1,x,F_L,F_R) = &\sum_{l=F_L+1}^{\rho-F_R+1} U_n(x,l,\rho-l+2) - \sum_{l=F_L+1}^{\rho-F_R+1} U_n(x,l,\rho-l+1)\\ &- \sum_{l=F_L}^{\rho-F_R} U_n(x,l,\rho-l+1) + \sum_{l=F_L}^{\rho-F_R} U_n(x,l,\rho-l).\end{aligned}$$

Using (15), we can combine many of the above terms into the form

$$\begin{aligned}G(F_L,F_R,x,\rho) = &\sum_{l=F_L}^{\rho-F_R} U_n(x,l,\rho-l)\\ =&\frac{1}{2\rho}\sum_{|k|<\rho}\cos^n\left(\frac{\pi k}{\rho}\right)\Bigg[(\rho+1-F_L-F_R)\cos\left(\frac{\pi kx}{\rho}\right)\\ &- \frac{\sin\left(\frac{\pi k(\rho-F_R-F_L+1)}{\rho}\right)\cos\left(\frac{\pi k(\rho-F_R+F_L+x)}{\rho}\right)}{\sin\left(\frac{\pi k}{\rho}\right)}\Bigg].\end{aligned}$$

Taking into account the varying end points of the summation yields

$$\begin{aligned}P_n(\rho+1,x,F_L,F_R) = &G(F_L,F_R,x,\rho+2) - 2G(F_L,F_R,x,\rho+1) + G(F_L,F_R,x,\rho)\\ &- U_n(x,F_L,\rho+2-F_L) - U_n(x,\rho+2-F_R,F_R)\\ &+ U_n(x,F_L,\rho+1-F_L) + U_n(x,\rho+1-F_R,F_R).\end{aligned}$$

Two applications of Abel's summation formula followed by some algebra gives

$$\begin{aligned}\sum q^\rho P_n(\rho,x,F_L,F_R) = &\frac{(1-q)^2}{q}\sum_\rho \frac{q^\rho}{2\rho}\sum_{|k|<\rho}\cos^n\left(\frac{\pi k}{\rho}\right)\times\\ &\left[(\rho-F_R-F_L)\cos\left(\frac{\pi kx}{\rho}\right) + \frac{\sin\left(\frac{\pi k(F_R+F_L-1)}{\rho}\right)\cos\left(\frac{\pi k(F_L-F_R+x)}{\rho}\right)}{\sin\left(\frac{\pi k}{\rho}\right)} + O(1)\right].\end{aligned} \tag{16}$$

In computing asymptotics for these summations, note that $\cos^n \alpha = \exp(-n\alpha^2/2)\exp[nO(\alpha^4)]$. For $\rho \geq n^{7/24}$, we can thus approximate $\cos^n\left(\frac{\pi}{\rho}\right)$ by $\exp[-n\pi^2/2\rho^2]$. The terms up to $\rho = n^{7/24}$ can be crudely bounded by $C\rho\exp[-\pi^2 n^{5/12}/2]$, which is of a lower order than the claimed asymptotic and so can be dropped from the summation. Since the terms in the summation when $\rho - 1 > |k| > 2$ contribute at most $[C\rho + O(1)]\rho\cos^n\left(\frac{\pi 2}{\rho}\right)$, they are also of a much lower order than the claimed result and so do not affect the leading term of the asymptotic. Dropping those terms and combining the terms corresponding to $k = 1, -1, \rho - 1$ and $-\rho + 1$ reduces things to

$$\sum q^\rho P_n(\rho, x, F_L, F_R) \sim \frac{(1-q)^2}{q}\sum_\rho \frac{2q^\rho}{\rho}\exp\left(\frac{-\pi^2 n}{2\rho^2}\right)\times$$
$$\left[(\rho - F_R - F_L)\cos\left(\frac{\pi x}{\rho}\right) + \sin\left(\frac{\pi(F_R + F_L - 1)}{\rho}\right)\cos\left(\frac{\pi(F_L - F_R + x)}{\rho}\right)\csc\left(\frac{\pi}{\rho}\right)\right]. \quad (17)$$

Note that when $X_{2n} = e$ we have $F_L = F_R = x = 0$. This reduces (17) to

$$\mathbb{P}(X_{2n} = e) \sim \frac{(1-q)^2}{q}\sum_\rho 2q^\rho \exp\left(-\frac{\pi^2(2n)}{2\rho^2}\right). \quad (18)$$

Applying Lemma 1 to evaluate (18) yields (4). For $g \in U_{2n}$,

$$\mathbb{P}(X_{2n} = g) \geq \sum_{\rho > (F_R + F_L)/(1-\epsilon/2)} q^\rho P_{2n}(\rho, x, F_L, F_R).$$

But some calculus shows that if $F_R + F_L < (1 - \epsilon/2)\rho$, then because $0 \leq F_R, F_L$ and $x \in [-F_L, F_R]$, there is a $\delta = \delta(\epsilon)$ such that

$$\left[(\rho - F_R - F_L)\cos\left(\frac{\pi x}{\rho}\right) + \sin\left(\frac{\pi(F_R + F_L - 1)}{\rho}\right)\cos\left(\frac{\pi(F_L - F_R + x)}{\rho}\right)\csc\left(\frac{\pi}{\rho}\right)\right] > \delta\rho.$$

This means that

$$\mathbb{P}(X_{2n} = g) \geq \frac{(1-q)^2}{q}\sum_{\rho > (F_R + F_L)/(1-\epsilon)} 2\delta q^\rho \exp\left(\frac{-\pi^2(2n)}{2\rho^2}\right).$$

But for $g \in U_{2n}(\epsilon)$, $(F_R + F_L)/(1 - \epsilon/2) < a\alpha n^{1/3}$ for $a = (1-\epsilon)/(1-\epsilon/2) < 1$. The case of $a < 1$ of Lemma 1 thus applies, which proves (5).
When $g \in V_{2n}(\epsilon)$, note that $\mathbb{P}(X_{2n} = g) \geq \mathbb{P}(X_{2n} = g')$ if $F_R(g) \leq F_R(g'), F_L(g) \leq F_R(g')$, and $\pi(g) = \pi(g')$. This is simply because the paths for the lamplighter that might result in g include all paths that might result in g'. It thus suffices to check the claim for g such that $0 = F_L(g)$ and $\pi(g) = F_R(g) > (1+\epsilon)\alpha(2n)^{1/3}$. The probability of this is just $\sum_\rho q^\rho \mathbb{P}(R_n = \rho, S_n = x)$, which is also the probability of being at x conditioned on survival.
In this case, equation (16) simplifies to

$$\sum q^\rho \mathbb{P}(R_n = \rho, S_n = x) = \frac{(1-q)^2}{q}\sum_\rho \frac{q^\rho}{2\rho}\sum_{|k|<\rho}\cos^n\left(\frac{\pi k}{\rho}\right)\times$$

$$\left[(\rho - x)\cos\left(\frac{\pi k x}{\rho}\right) + \sin\left(\frac{\pi k(x-1)}{\rho}\right)\csc\left(\frac{\pi k}{\rho}\right)\right]. \quad (19)$$

A power series expansion about $\rho = x$ shows that

$$(\rho - x)\cos\left(\frac{\pi x}{\rho}\right) + \frac{\rho}{\pi}\sin\left(\frac{\pi x}{\rho}\right) = \frac{\pi^2(\rho - x)^3}{3\rho^2} + O\left(\frac{(\rho - x)^4}{\rho^3}\right).$$

In other words, $(\rho - x)\cos\left(\frac{\pi x}{\rho}\right) + \frac{\rho}{\pi}\sin\left(\frac{\pi x}{\rho}\right) = (\rho - x)^3 h(\rho)$, where $h(x) = \frac{\pi^2}{3x^2}$ and $h(\rho)$ satisfies the requirements of Lemma 1.

$$\begin{aligned} E(q^{R_n}; S_n = x) &\sim \frac{(1-q)^2}{q}\sum_{\rho=x}^{\infty}\frac{2q^\rho}{\rho}(\rho - x)^3 h(\rho)\exp\left[-\frac{\pi^2 n}{2\rho^2}\right] \\ &\sim \frac{(1-q)^2}{q}\frac{2\pi^2}{3a^3\alpha^3 n}\exp\left[\varphi(a\alpha n^{1/3})\right]C_1(a,q) \\ &= \frac{c_2(a,q)}{n}\exp\left[-\left(a + \frac{1}{2a^2}\right)(\pi\log q)^{2/3}n^{1/3}\right], \end{aligned}$$

which proves (11).

For $a < 1$, equation (17) and Lemma 1 show that for n even and $x = 2\lfloor a\alpha n^{1/3}/2\rfloor$,

$$\begin{aligned} E\left(q^{R_n}; S_n = x\right) &\sim \frac{(1-q)^2}{q}\sum_\rho 2q^\rho\exp\left[-\frac{\pi^2 n}{2\rho^2}\right]\left[\left(1 - \frac{x}{\rho}\right)\cos\left(\frac{\pi x}{\rho}\right) + \frac{1}{\pi}\sin\left(\frac{\pi x}{\alpha n^{1/3}}\right)\right] \\ &\sim \frac{(1-q)^2}{q}2\left[(1-a)\cos(\pi a) + \frac{1}{\pi}\sin(\pi a)\right]\sqrt{\frac{2}{3}}\frac{\pi^{5/6}}{(\log q)^{2/3}}n^{1/6}\exp\left[\varphi\left(\alpha n^{1/3}\right)\right] \\ &= E(q^{R_n}; S_n = 0)\left[(1-a)\cos(\pi a) + \frac{1}{\pi}\sin(\pi a)\right], \end{aligned}$$

which completes the proof of (10).

To prove equation (8), we need to compute Eq^{R_n} without restricting to a final location. The ideas are very similar to the previous cases, so we only sketch the proof.

Let $U_n(l,r) = \mathbb{P}\{S_k \in (-l,r), k \le n\}$. To evaluate this, note that $U_n(l,r) = \sum_x U_n(x,l,r)$, which yields

$$U_n(l,r) = \frac{1}{l+r}\sum_{|2j+1|<l+r}\cos^n\left(\frac{\pi(2j+1)}{l+r}\right)\sin\left(\frac{\pi(2j+1)}{l+r}\right)\cot\left(\frac{(2j+1)\pi}{2(l+r)}\right).$$

As before, if $V_n(l,r) = \mathbb{P}\{m_n = l, M_n = r\}$, then $V_n(l,r) = U_n(l+1,r+1) - U_n(l,r+1) - U_n(l+1,r) + U_n(l,r)$. But then

$$\begin{aligned} \mathbb{P}(R_n = \rho) &= \sum_{l=0}^{\rho-1} V_n(l, \rho - l - 1) \\ &= \sum_{l=0}^{\rho-1}\left[U_n(l+1,\rho-l) - U_n(l,\rho-l-1) - U_n(l+1,\rho-l) + U_n(l+1,\rho-l-1)\right]. \end{aligned}$$

Since $U_n(0,z) = U_n(z,0) = 0$, this reduces to $\mathbb{P}(R_n = \rho) = G(\rho-1) - 2G(\rho) + G(\rho+1)$ where

$$\begin{aligned} G(\rho) &= \sum_{l=0}^{\rho} U(l, \rho - l) \\ &= \frac{1}{\rho} \sum_{|2j+1|<\rho} \cos^n \left(\frac{\pi(2j+1)}{\rho} \right) \cot^2 \left(\frac{(2j+1)\pi}{2\rho} \right). \end{aligned}$$

In the physical literature, $\mathbb{P}(R_n = \rho)$ is often approximated by $d^2G/d\rho^2$, but this approximation requires $n^{1/2}/\rho = o(n^{1/6})$. Since the bulk of the mass of our summation occurs when $\rho = \alpha n^{1/3}$, this approximation is not valid in our case. This is the reason why the constants computed in [1] and [16] were incorrect. Again using Abel's summation formula and arguing much as before, we see that

$$\begin{aligned} Eq^\rho &= \frac{(1-q)^2}{q} \sum_\rho q^\rho G(\rho) \\ &= \frac{(1-q)^2}{q} \sum_\rho \frac{q^\rho}{\rho} \sum_{|2j+1|<\rho} \cos^n \left(\frac{\pi(2j+1)}{\rho} \right) \cot^2 \left(\frac{(2j+1)\pi}{2\rho} \right) \\ &\sim \frac{(1-q)^2}{q} \sum_\rho \frac{2q^\rho}{\rho} \exp\left[-\frac{\pi^2 n}{2\rho^2} \right] \frac{4\rho^2}{\pi^2} \\ &= \frac{8(1-q)^2}{\pi^2 q} \sum_\rho \rho \exp\left[-\frac{\pi^2 n}{2\rho^2} + \rho \log q \right] \\ &\sim \frac{8(1-q)^2}{q} \frac{n^{1/2}}{-\log q} \sqrt{\frac{2}{3\pi}} \exp\left[-\frac{3}{2} (\pi \log q)^{2/3} n^{1/3} \right]. \end{aligned}$$

Finally, in order to prove the upper bound of (7), we need to control $V_n(x,l,r)$.

Lemma 2. *There exists a constant C such that*

$$V_n(x,l,r) \le C \left(\frac{(l+r)^2}{n^{-5/2}} + \frac{n^2}{(l+r)^7} \right) \exp\left[-\frac{(l+r)^2}{Cn} - \frac{n}{C(l+r)^2} \right]. \tag{20}$$

For $l + r \le n^{1/3}$, the claim follows from (15). For $l + r$ much larger than $n^{1/2}$, this follows because $V_n(x,l,r) \le \mathbb{P}(S_n = x)$, which has a Gaussian upper bound. We will thus assume that $n^{1/2+\epsilon} > l + r > n^{1/3}$ in what follows.
Let $P_n(x)$ denote $\mathbb{P}(S_n = x)$, $f_n(x) = (2/\pi n)^{1/2} \exp(-x^2/2n)$ the Gaussian approximation for $P_n(x)$, and let $\rho = l + r + 1$. As before,

$$\begin{aligned} V_n(x,l,r) = \sum_{k=-\infty}^{\infty} & P_n[x + 2k(\rho-1)] - 2P_n[x + 2k\rho] + P_n[x + 2k(\rho+1)] \\ - P_n[x + 2k(\rho-1) - 2l] &+ P_n[x + 2k\rho - 2l] + P_n[x + 2k\rho - 2l - 2] - P_n[x + 2k(\rho+1) - 2l - 2]. \end{aligned} \tag{21}$$

Note that when $k = 0$, this expression is 0, so the summation is really over $k \ne 0$. To bound this quantity, we want to replace $P_n(x)$ by $f_n(x)$. Using the local central limit theorem (see e.g. [11], Theorem 1.2.1), the second gradient of the error introduced by replacing $P_n(y)$ by

$f_n(y)$ is bounded by $n^{-5/2}$. Since $\rho > n^{1/3}$ and $P_n(y) = 0$ for $|y| > n$, there are at most $n^{2/3}$ non-zero terms in the summation, and so the error introduced by using Gaussian estimates is of a lower order than the claimed bound.
We will now break the estimate into two further cases: the first when $\rho > 2n^{1/2}$, and the second for $\rho \leq 2n^{1/2}$. When $\rho > 2n^{1/2}$, a second order Taylor expansion yields

$$f_n[x+2k(\rho-1)] - 2f_n[x+2k\rho] + f_n[x+2k(\rho+1)] \\ = \frac{[(x+2k\rho)^2-n]}{n^2} f_n[x+2k\rho] + k^3 O\left(n^{-5/2}\right) f_n[x+2k\rho]. \quad (22)$$

The second term is of a lower order, and for $k \neq 0$, the fact that $\rho > 2n^{1/2}$ and $|x| \leq \rho$ means that $(x+2k\rho)^2 \leq |k|\rho^2$. This means that $\exp\left[-(x+2k\rho)^2/2n\right] < \exp(-2|k|\rho^2/n)$, so summing over all $k \neq 0$ yields the claimed bound.
When $\rho \leq 2n^{1/2}$, Poisson's summation formula implies that

$$\sum_k f_n(x+2k\rho) = \frac{1}{\rho} \sum_k \exp\left[-\frac{\pi^2 k^2 n}{2\rho^2} + i\frac{\pi x k}{\rho}\right]. \quad (23)$$

Since we are interested in a difference of terms of the form of (23), the $k = 0$ terms again cancel out. Taking the analogous second order approximation used in the $\rho > 2n^{1/2}$ case and the fact that the $k = \pm 1$ terms again dominate the sum gives the bound

$$Cn^2 \rho^{-7} \exp[-n/(C\rho^2)].$$

To prove (7), equation (20) implies that

$$\sum_\rho q^\rho P_n(\rho, x, F_L, F_R) \leq C(\rho - F_L + F_R)\left(\frac{\rho^2}{n^{-5/2}} + \frac{n^2}{\rho^7}\right) \exp\left(-\frac{\rho^2}{Cn} - \frac{n}{C\rho^2}\right).$$

Using the asymptotic summation methods as before, there is a $\beta > 0$ such that if $\phi(g) \geq \beta n^{1/3}$ the summation is bounded by a constant times the first term, which is the claimed bound of (7). By taking a larger constant C in (7), the result can be extended to $\phi(g) > an^{1/3}$ for $a < \beta$.

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