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A TYPE OF GAUSS' DIVERGENCE FORMULA ON WIENER SPACES

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Abstract
We will formulate a type of Gauss' divergence formula on sets of functions which are greater than a specific value of which boundaries are not regular. Such formula was first established by L. Zambotti in 2002 with a profound study of stochastic processes. In this paper we will give a much shorter and simpler proof for his formula in a framework of the Malliavin calculus and give alternate expressions. Our approach also enables to show that such formulae hold in other Gaussian spaces.

1 Introduction

Gauss' divergence formula plays a fundamental role in a wide range of fields in mathematics and other sciences. The formula is simply stated in the case of Euclidian space and Lebesgue measure;

$$\int_D \operatorname{div} f(x)\, dx = \int_{\partial D} \langle f(x), n(x) \rangle \, \sigma_D(dx),$$

where D is a smooth domain, $\operatorname{div} f(x)$ is a divergence of the vector field $f(x)$, $n(x)$ denotes the exterior normal vector at $x \in \partial D$, and $\sigma_D(dx)$ denotes the surface measure on ∂D. In the case that f is a scalar field and the vector field is given by, using a vector h, $f(x)h$, and if the measure has a Gaussian density $g(x) = \frac{1}{Z_Q} \exp\{-\frac{1}{2} \left\langle Q^{-1}x, x \right\rangle\}$, then the divergence formula leads to

$$\int_D \langle \nabla f(x), h \rangle \, g(x)\, dx = \int_D f(x) \left\langle Q^{-1}x, h \right\rangle g(x)\, dx + \int_{\partial D} f(x) \langle n(x), h \rangle \, g(dx \mid \partial D),$$

where $g(dx \mid \partial D)$ is a surface measure on ∂D induced by the Gaussian measure $g(x)\,dx$. It is well known that, in infinite dimensional spaces, there is no Lebesgue measure but we can execute some analysis based on Gaussian measures instead. Hence it is natural to expect that a similar

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divergence formula still holds in the infinite dimensional spaces. To our knowledge, such formulae were first studied by Goodman [6], and, among others, recently by Shigekawa [15] to establish a Hodge–Kodaira vanishing theorem in infinite dimensional spaces. They, however, assume that the domains on which divergence formulae are stated have a regular boundary in the sense of Malliavin calculus.

In this paper we are concerned with a divergence formula for a subset W_D of a Banach space $W = C([0,1],\mathbb{R})$ which consists of $\{f \in W; f(t) \in D, \forall t \in [0,1]\}$, where $D = (-a,\infty)$, $a > 0$. The (topological) boundary ∂W_D is clearly given by the set of functions whose minima are $-a$. We will show that, though $w \in W_D$ may hit $-a$ many times, the metrical boundary consists of the set that $w \in W_D$ hits $-a$ exactly once. We also note that the minimum $m(w)$ of each continuous function is not H-C^1 but H-Lipschitz (see [1, 15] for these notions), and is degenerate in terms of Malliavin calculus.

The divergence formula for W_D was first established by Zambotti [18] and the proof based on profound insights of a theory of stochastic processes. He then applied his formula to analyze solutions to stochastic partial differential equations with reflection of Nualart–Pardoux type [13] through a theory of Dirichlet forms. His formula, which has its own interest, has been extended by Funaki–Ishitani [5] for domains $\{f \in W; h_1 < f < h_2\}$ with some continuous functions h_1 and h_2 using a random walk approximation for Brownian paths, and by Hariya [7] for multidimensional cases applying a concrete representation of the heat kernel. Such approaches rely on a Markovian property of the Brownian motions. Closely related studies to such divergence formulae have been treated by Fukushima, Hino, and Uchida [3, 4, 8, 9].

The aim of the present paper is to give a simple and shorter proof, and an alternate representation for Zambotti's divergence formula in a frame work of Malliavin calculus. After that we will recover his original formula applying simple and well-known probabilistic formulae. Our approach here relies on the Gaussian property of Brownian motions and enables to extend his result to other Gaussian spaces.

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2 Framework and main result

Our main subjects in the present paper are divergence formulae for two types of Wiener spaces consisting of a Brownian motion starting from zero and a pinned Brownian motion starting from and ending at zero on time interval $[0,1]$. We will, however, begin with a slight general setting and introduce a Hilbert space $E = L_2(0,1)$. Let Q be an operator $E \to E$ given by a symmetric, nonnegative, and factorizable integral kernel $\rho(s,t)$, namely, $Qf(t) := \int_0^1 \rho(s,t)f(s)\,ds$ and $\rho(s,t) := \int_0^1 r(s,u)r(t,u)\,du$. We will assume that r is in $L_2([0,1]\times[0,1])$ so that Q is nuclear. We will, in addition, assume $\operatorname{Ker} Q = \{0\}$. Typical examples of $\rho(s,t)$ are $s \wedge t$ (Brownian motion) and $s \wedge t - st$ (pinned Brownian motion). The corresponding factors of ρ are, respectively, $r(s,u) = 1_{[0,s]}(u)$ and $r(s,u) = 1_{[0,s]}(u) - s$. If we define Hilbert–Schmidt operators on E by $Rf(t) := \int_0^1 r(s,t)f(s)\,ds$ and $R^\dagger f(t) := \int_0^1 r(t,s)f(s)\,ds$, it is easy to see that $\langle Qf, g\rangle_E = \langle Rf, Rg\rangle_E$ and $\left\langle Q^{-1}f, g\right\rangle_E = \left\langle (R^\dagger)^{-1}f, (R^\dagger)^{-1}g\right\rangle_E$, though R and $R^\dagger$ are not symmetric nor nonnegative.

It is well-known that there exists a Gaussian measure on E with covariance operator Q if and only if Q is symmetric nonnegative nuclear operator. With above settings, let us introduce a mean zero Gaussian measure μ on E with covariance operator Q. Then $H := Q^{1/2}E$ is a reproducing kernel Hilbert space with an inner product $\langle f, g\rangle_H := \left\langle Q^{-1/2}f, Q^{-1/2}g\right\rangle_E$. We will always identify H and

its dual H^* hereafter. We shall denote by $W \subset C([0,1],\mathbb{R})$ a Banach space which actually supports μ. We note that the above setting also includes the canonical probability space induced by the fractional Brownian motion [10].
We introduce here several notions which are used later before proceeding. The space $\mathbb{D}_p^s$ is the completion of the set of stochastic polynomials with respect to the norm $\|F\|_{\mathbb{D}_p^s} := \|(I-L)^{s/2}F\|_p$, where L is the Ornstein–Uhlenbeck operator and $\|\cdot\|_p$ denotes the usual L_p-norm on W with respect to μ for $p>1$ and $s \in \mathbb{R}$. And the space $\mathbb{D}_{\infty-}^{\infty}$ is defined by $\bigcap_{p>1,s\in\mathbb{R}} \mathbb{D}_p^s$. For every $F \in \mathbb{D}_p^1$ we can define the Gross–Shigekawa derivative $D : \mathbb{D}_p^1 \to L_p(W,H)$ which is the set of H-valued L_p-functions on W as usual, see, e.g., [12, 17] for detailed discussions and analysis on such spaces.
Before stating our result, we will introduce a concept of locally nondegenerate Wiener functional following Florit–Nualart [2].

Definition 2.1. A Wiener functional $F : W \to \mathbb{R}^n$, $F \in \mathbb{D}_2^1$ is called locally nondegenerate on an open set $A \subset \mathbb{R}^n$ if there exist $R_A : W \to H^n$, ${R_A}^j \in \mathbb{D}_{\infty-}^{\infty}(H)$ for $j = 1,\ldots,n$ and an $n \times n$ matrix σ_A whose components are all in $\mathbb{D}_{\infty-}^{\infty}$ such that $\rho_A(w) := \det \sigma_A(w)$ satisfies $1/\rho_A \in L_p(W)$ for every $1 < p < \infty$ and ${\sigma_A}^i_j(w) = \left\langle DF^i(w), {R_A}^j(w)\right\rangle_H$ holds on $W_A := \{w \in W; F(w) \in A\}$.

It is known that the minimum $m(w)$ is locally nondegenerate on $(-\infty,-\alpha)$ for every $\alpha > 0$ for the Brownian motion and the fractional Brownian motion [10] starting from zero.

Theorem 2.1. *Assume $m(w)$ is locally nondegenerate on $(-\infty,-\alpha)$ for every $\alpha > 0$. Then we have, for every $F \in \mathbb{D}_p^2$ with $p \geq 2$ and $h \in Q(E)$,*

$$\int_{W_D} \langle DF(w), h\rangle_H \,\mu(dw) = \int_{W_D} F(w) \left\langle w, Q^{-1}h\right\rangle_E \mu(dw) - \left\langle F, (h \circ \tau_m)\delta_{-a}(m)\right\rangle, \tag{1}$$

where $\tau_m(w)$ is an almost surely unique $t \in [0,1]$ such that $w(t)$ attains its minimum $m(w)$ $(w(\tau_m(w)) = m(w))$, $\delta_{-a}(m)$ is a Watanabe composition of Dirac measure δ_{-a} at $-a < 0$ and locally nondegenerate functional m, and $\langle\cdot,\cdot\rangle$ is the canonical bilinear form.

We note that the fact that $w(t)$ attains its minimum at a unique $t \in [0,1]$ can be also proved in the framework of Malliavin calculus (see [1]) and $\langle Dm(w), h\rangle_H = h(\tau_m(w))$.
Since $\delta_{-a}(m(w))$ is a positive generalized Wiener functional, it determines a positive measure on W by Sugita's theorem [16]. Moreover, by virtue of Theorem 6.1 in [16], we obtain the following integral representation.

Corollary 2.2. *With the settings of the above theorem, we have*

$$\int_{W_D} \langle DF(w), h\rangle_H \,\mu(dw) = \int_{W_D} F(w) \left\langle w, Q^{-1}h\right\rangle_E \mu(dw) - \int_{\partial W_D} \tilde{F}(w) h(\tau(w)) \sigma_{-a}(dw),$$

where $\tilde{F}$ is $(p,1)$-quasi-continuous version of F and $\sigma_{-a}(dw) := \delta_{-a}(m(w))\mu(dw)$ in the sense of Sugita.

3 Watanabe compositions for locally nondegenerate functionals

Florit and Nualart [2] showed that the maximum of Brownian sheet has a C^∞ density by introducing a concept of locally nondegenerate functionals. To show the result, they used a standard method in the Malliavin calculus, namely, they showed Malliavin's integration by parts formula still holds for locally nondegenerate functionals [12, Proposition 2.1.4]. Another well-known approach to show the smoothness of the law of nondegenerate functionals is the Watanabe composition method [17], which is also based on Malliavin's integration by parts.

We will show that Watanabe composition still holds for locally nondegenerate functionals with a little restricted situations that are not essential for our purpose. Before stating the result, let us recall that $\mathbb{D}_p^{-2}$ is the completion of the set of stochastic polynomials with respect to $\|F\|_{\mathbb{D}_p^{-2}} := \|(I-L)^{-1}F\|_p$. It is also known that $\mathbb{D}_p^{-2}$ is the dual space of $\mathbb{D}_q^2$ with $1/p + 1/q = 1$ under identification of $L_2(E,\mu)$ with its dual.

Proposition 3.1. *Let $F : W \to \mathbb{R}^n$ be locally nondegenerate on an open set $A \subset \mathbb{R}^n$ satisfying $F \in L_{\infty-}(W,\mu)$. Then for every $1 < p < \infty$ and $\varphi \in C_0^\infty(A)$ which is the set of smooth functions supported on A, we have*

$$\|\varphi \circ F\|_{\mathbb{D}_p^{-2}} \leq C_p \|\varphi\|_{\mathscr{T}_{-2}},$$

where $\|\varphi\|_{\mathscr{T}_{-2}} := \|(1+|x|^2-\Delta)^{-1}\varphi(x)\|_{L_\infty}$

Proof. Define $\psi(x) := (1+|x|^2-\Delta)^{-1}\varphi(x) \in \mathscr{S}$. Let $G : W \to \mathbb{R}$ be $G \in \mathbb{D}_{\infty-}^\infty$ and $G(w) = 0$ on $w \in W_A^c := \{w \in W; G(w) \notin A\}$. Then it is easy to see that (cf. [2, 17]) there exists $\Phi_2(F,G;w)$ such that

$$\int_W \left((1+|x|^2-\Delta)\psi\right)(F(w))\cdot G(w)\,\mu(dw) = \int_W \psi(F(w))\Phi_2(F,G;w)\,\mu(dw) \tag{2}$$

with $\|\Phi(F,G)\|_{L_1} \leq C_{n,p,F}\|G\|_{\mathbb{D}_q^2}$ and $\frac{1}{p}+\frac{1}{q}=1$. Hence we have, from (2),

$$\begin{aligned}
&\int_W \varphi\circ F(w)G(w)\,\mu(dw)\\
&\qquad \leq \int_W \left|(1+|x|^2-\Delta)^{-1}\varphi\circ F(w)\right|\cdot\left|\Phi_2(F,G;w)\right|\,\mu(dw)\\
&\qquad \leq \|\varphi\|_{\mathscr{T}_{-2}}\|\Phi_2(F,G)\|_{L_1}.
\end{aligned}$$

Note that, since $\varphi \in C_0^\infty(A)$, $\varphi(F(w)) \neq 0$ only for $\{w \in W; F(w) \in \operatorname{supp}\varphi \subset A\}$. Hence if we introduce an effective domain $\mathscr{D}(G) := \{w \in W; G(w) \neq 0\}$,

$$\begin{aligned}
\|\varphi\circ F\|_{\mathbb{D}_p^{-2}} &= \sup_{\|G\|_{\mathbb{D}_q^2}\leq 1}\left|\langle\varphi\circ F, G\rangle\right|\\
&= \sup_{\|G\|_{\mathbb{D}_q^2}\leq 1, \mathscr{D}(G)\subset W_A}\left|\int_W \varphi\circ F(w)G(w)\,\mu(dw)\right|\\
&\leq K_{q,F}\|\varphi\|_{\mathscr{T}_{-2}},
\end{aligned}$$

where $K_{q,F} := \sup_{\|G\|_{\mathbb{D}_q^2}\leq 1,\mathscr{D}(G)\subset W_A}\|\Phi_2(F,G)\|_{L_1} < \infty$. □

Corollary 3.2. *For each locally nondegenerate functional $F \in L_{\infty-}(W)$ on A and a Schwartz distribution $T \in \mathscr{T}_{-2}$ with $\operatorname{supp} T \subset A$, we can uniquely define a composition $T \circ F \in \mathbb{D}_p^{-2}$ for every $1 < p < \infty$.*

Since $\delta_x \in \mathscr{T}_{-2}$ if $n = 1$, we have

Corollary 3.3. *Let $m(w)$ be the minimum of w. If $m(w)$ is locally nondegenerate, then $\delta_x(m(w))$ has a rigorous meaning for every $x < 0$ as an element of $\mathbb{D}_p^{-2}$ for every $1 < p < \infty$.*

Hence we can derive the regularity of the density for the maximum/minimum of some Gaussian processes in this framework (cf. [17]). We do not describe them since they are entirely standard.

4 Proof of the theorem and other representations

Once the Watanabe composition (Corollary 3.3) is established, the proof goes easily. We first note that $\int_{W_D} \langle DF(w), h\rangle_H \, \mu(dw) = \int_W 1_{W_D}(w) \langle DF(w), h\rangle_H \, \mu(dw)$. Since $D = (-a, \infty)$ and $W_D = \{w \in C([0,1], \mathbb{R}); w(t) \in D, \forall t \in [0,1]\}$, we have $1_{W_D}(w) = 1_{(-a,\infty)}(m(w))$.
Note that, for every $\varphi \in C_b^1(\mathbb{R})$, $G \in \mathbb{D}_q^1$ and $F \in \mathbb{D}_p^1$ with $1/p + 1/q = 1$, we have the integration by parts formula:

$$\begin{aligned}\int_W &\varphi(G(w)) \langle DF(w), h\rangle_H \, \mu(dw) \\ &= \int_W F(w)\varphi(G(w)) \langle w, h\rangle_H \, \mu(dw) - \int_W \varphi'(G(w)) \langle DG(w), h\rangle_H F(w) \mu(dw),\end{aligned}$$

and $\langle f, g\rangle_H$ is defined by $\left\langle Q^{-1/2} f, Q^{-1/2} g\right\rangle_E$. Therefore we can approximate $1_{(-a,\infty)}$ by smooth functions and obtain

$$\begin{aligned}\int_W 1_{W_D}(w) \langle DF(w), h\rangle_H \, \mu(dw) = \int_W &F(w) 1_{W_D}(w) \left\langle w, Q^{-1}h\right\rangle_E \mu(dw) \\ &- \left\langle F, \delta_{-a} \circ m \langle Dm(w), h\rangle_H \right\rangle,\end{aligned}$$

where $\langle \cdot, \cdot \rangle$ of the last term is a natural coupling between $\mathbb{D}_p^2$ and $\mathbb{D}_q^{-2}$ with $1/p + 1/q = 1$. However it is known that $\langle Dm(w), h\rangle_H = h(\tau_m(w))$, we have the conclusion. $\square$
We have now proved the theorem which was stated in a framework of infinite dimensional analysis. We here reformulate the formula using some concrete expressions for several random variables related to Brownian motions and pinned Brownian motions.
The following representation corresponds to those obtained by Hariya [7].

Proposition 4.1. *Let $h \in C_0^2((0,1))$. Then we have:*

(A) The case of Brownian motion starting from zero:

$$\begin{aligned}\int_{W_D} \langle DF(w), h\rangle_H \, \mu(dw) = -\int_{W_D} &\left\langle w, h''\right\rangle_E \mu(dw) \\ &- \frac{2}{\sqrt{\pi}} e^{-\frac{a^2}{2}} E\left[F \cdot (h \circ \tau_m) \mid m(w) = -a\right].\end{aligned}$$

(B) The case of pinned Brownian motion starting from and ending at zero:

$$\begin{aligned}\int_{W_D} \langle DF(w), h\rangle_H \, \mu(dw) = -\int_{W_D} &\left\langle w, h''\right\rangle_E \mu(dw) \\ &- 4a e^{-2a^2} E\left[F \cdot (h \circ \tau_m) \mid m(w) = -a\right].\end{aligned}$$

Proof. Since Watanabe composition is a dual operator of the conditional expectation, we have

$$\begin{aligned}&\langle F, (h\circ\tau_m)\delta_{-a}(m)\rangle\\ &\qquad= \int_{\mathbb{R}} E[F\cdot(h\circ\tau_m) \mid m(w)=x]\delta_{-a}(x)\mu\circ m^{-1}(dx)\\ &\qquad= E[F\cdot(h\circ\tau_m)\mid m(w)=-a]p_m(-a),\end{aligned}$$

where $p_m(x)=\mu\circ m^{-1}(dx)/dx$. Then it is well known that $p_m(-a)=\frac{2}{\sqrt{\pi}}e^{-a^2/2}$ for Brownian motion and $p_m(-a)=4ae^{-2a^2}$ for pinned Brownian motion. □

We can derive one more expression which corresponds to Zambotti's representation.

Proposition 4.2. *Let $h\in C_0^2((0,1))$. Then we have:*

(A) The case of Brownian motion starting from zero:

$$\begin{aligned}\int_{W_D}\langle DF(w),h\rangle_H\,\mu(dw) &= -\int_{W_D}\langle w,h''\rangle_E\,\mu(dw)\\ &\quad -\int_0^1 du\,h(u)\frac{a}{\pi\sqrt{u^3(1-u)}}\exp\left\{-\frac{a^2}{2u}\right\}E[F\mid m(w)=-a,\tau_m(w)=u].\end{aligned}$$

(B) The case of Brownian motion starting from and ending at zero:

$$\begin{aligned}\int_{W_D}\langle DF(w),h\rangle_H\,\mu(dw) &= -\int_{W_D}\langle w,h''\rangle_E\,\mu(dw)\\ &\quad -\int_0^1 du\,h(u)\frac{\sqrt{2}a^2}{\sqrt{\pi u^3(1-u)^3}}\exp\left\{-\frac{a^2}{2u}-\frac{a^2}{2(1-u)}\right\}\\ &\qquad\times E[F\mid m(w)=-a,\tau_m(w)=u].\end{aligned}$$

Proof. The proof can be done similarly to the above proposition. Therefore it is sufficient to recall that, for a Brownian motion starting from $a>0$,

$$P_a\left(H(t)\in dv, \inf_{0\le s\le t}W(s)\in dy\right)=\frac{1_{[0,t]}(v)(a-y)}{\pi\sqrt{v^3(t-v)}}\exp\left\{-\frac{(a-y)^2}{2v}\right\}dy\,dv,$$

where $H(t):=\inf\{s<t; W(s)=m(t)\}$. Hence we have

$$P_a\left(m(w)=0,\tau_m(w)=u\right)\,du=\frac{a}{\pi\sqrt{u^3(1-u)}}\exp\left\{-\frac{a^2}{2u}\right\}du.$$

Moreover it is also known that

$$\begin{aligned}&P_a\left(H(t)\in dv, \inf_{0\le s\le t}W(s)\in dy, W(t)\in dz\right)\\ &\qquad=\frac{(a-y)(z-y)}{\pi\sqrt{v^3(t-v)^3}}\exp\left\{-\frac{(a-y)^2}{2v}-\frac{(z-y)^2}{2(t-v)}\right\}dy\,dv\,dz,\end{aligned}$$

so that the pinned case is also derived. □

References

[1] O. Enchev and D. W. Stroock, *Rademacher's theorem for Wiener functionals*, Ann. Probab. **21** (1993), 25–33. MR1207214

[2] C. Florit and D. Nualart, *A local criterion for smoothness of densities and applications to the supremum of the Brownian sheet*, Statistics and Probability Letters **22** (1995), 25–31. MR1327725

[3] M. Fukushima, *BV functions and distorted Ornstein Uhlenbeck processes over the abstract Wiener space*, J. Funct. Anal. **174** (2000), 227–249. MR1761369

[4] M. Fukushima and M. Hino, *On the space of BV functions and a related stochastic calculus in infinite dimensions*, J. Funct. Anal. **183** (2001), 245–268. MR1837539

[5] T. Funaki and K. Ishitani, *Integration by parts formulae for Wiener measures on a path space between two curves*, Probab. Theory Relat. Fields **137** (2007), 289–321. MR2278459

[6] V. Goodman, *A divergence theorem for Hilbert space*, Trans. Amer. Math. Soc. **164** (1972), 411–426. MR0298417

[7] Y. Hariya, *Integration by parts formulae for the Wiener measure restricted to subsets in $\mathbb{R}^d$*, J. Funct. Anal. **239** (2006), 594–610. MR2261339

[8] M. Hino, *On Dirichlet spaces over convex sets in infinite dimensions*, Contemp. Math. **317** (2003), 143–156. MR1966893

[9] M. Hino and H. Uchida, *Reflecting Ornstein-Uhlenbeck processes on pinned path spaces*, RIMS Kôkyûroku Bessatsu **B6** (2008), 111–128. MR2407558

[10] N. Lanjri Zaïdi and D. Nualart, *Smoothness of the law of the supremum of the fractional Brownian motion*, Elect. Comm. Probab. **8** (2003), 102–111. MR2042749

[11] P. Malliavin, *Stochastic analysis*, Springer, 1997. MR1450093

[12] D. Nualart, *The Malliavin calculus and related topics*, 2nd ed., Springer–Verlag, 2006. MR2200233

[13] D. Nualart and E. Pardoux, *White noise driven quasilinear SPDEs with reflection*, Probab. Theory Relat. Fields **93** (1992), 77–89. MR1172940

[14] D. Nualart and J. Vives, *Continuité absolue de la loi du maximum d'un processus continu*, C. R. Acad. Sci. Paris **307** (1988), 349–354. MR0958796

[15] I. Shigekawa, *Vanishing theorem of the Hodge–Kodaira operator for differential forms on a convex domain of the Wiener space*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **6** (2003), 53–63. MR2074767

[16] H. Sugita, *Positive generalized functions and potential theory over an abstract Wiener space*, Osaka J. Math. **25** (1988), 665–696. MR0969026

[17] S. Watanabe, *Lectures on stochastic differential equations and Malliavin calculus*, 1984, Tata Institute of Fundamental Research. MR0742628

[18] L. Zambotti, *Integration by parts formulae on convex sets of paths and applications to SPDEs with reflection*, Probab. Theory Relat. Fields **123** (2002), 579–600. MR1921014