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BOUNDS FOR DISCONNECTION EXPONENTS

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Abstract
We improve the upper bounds of disconnection exponents for planar Brownian motion that we derived in an earlier paper. We also give a plain proof of the lower bound $1/(2\pi)$ for the disconnection exponent for one path.

1 Introduction

The first purpose of this paper is to improve the upper bounds of Brownian critical exponents derived in Werner [14]. The basic ideas and tools of the proof are similar to those used in [14]. We refer to this paper for a detailed introduction and definitions of disconnection exponents for planar Brownian motion and for more references. Recall that, if $B^1, \dots, B^n$ denote n independent planar Brownian motions started from $(1,0)$, the disconnection exponent η_n (for $n \geq 1$) describes the asymptotical decay of the probability

$$\mathbb{P}\{\cup_{j=1}^{j=n} B^j[0,t] \text{ does not disconnect } 0 \text{ from infinity}\},$$

when $t \to \infty$, which is logarithmically equivalent to $t^{-\eta_n/2}$ (we say that a compact set K disconnects 0 from infinity if it contains a closed loop around 0). We are going to show that:

Theorem 1

$$\eta_n \leq \frac{n}{2} - \frac{1}{8\pi^2 n} \sum_{i \in \mathbb{Z}} 2^i \ln \left(\sum_{k \in \mathbb{Z}} \exp(-k^2 2^i) \right) < \frac{n}{2} - \frac{.03125}{n}$$

In particular $\eta_1 < .469$ and $\eta_2 < .985$ (the upper bound in [14] was $\eta_n < n/2 - .0243/n$). Lawler [10] recently showed that the Hausdorff dimension h of the 'frontier' of planar Brownian motion is exactly $2 - \eta_2$. Combined with our estimate, this implies that $h > 1.0156$ (see also Bishop et al. [2], Burdzy-Lawler [3]). Let us just recall that it has been conjectured that $\eta_1 = 1/4$ and $\eta_2 = 2/3$ (see e.g. Duplantier et al. [5], Puckette-Werner [12]). These conjectures have been confirmed by simulations [12]. One of the motivations of this paper is to understand why the upper bounds in [14] are so far from the conjectured value.

The second result of this paper is the lower bound

Theorem 2

$$\eta_1 \geq \frac{1}{2\pi}.$$

This result has been anounced by Burdzy and Lawler (see e.g. in Lawler [6]) but (to our knowledge) it has never been written up. Proofs of the fact that $\eta_1 \geq 1/\pi^2$ can be found in [3], [6]. We supply a short proof of Theorem 2 to fill in this gap in the literature. This result has consequences for the Hausdorff dimension of the frontier of planar Brownian motion, and also for the Hausdorff dimension of the set of cut-points of planar Brownian motion (see Burdzy-Lawler [3], Lawler [7], Lawler [10]). For random walk counterparts, see e.g. Puckette-Lawler [11], Lawler [8]; see also Lawler [9] and Werner [16] for some other related results on disconnection exponents and non-intersection exponents.

The paper is structured as follows. We first derive Theorem 2 in Section 2; in section 3, we derive some results concerning extremal distance and we finally prove Theorem 1 in Section 4.

2 Lower bound

We will often identify $\mathbb{R}^2$ and $\mathbb{C}$. Let B denote a complex Brownian motion started from 1. If T_R denotes the hitting time of the circle $\{z,\ |z| = R\}$ by B, then the disconnection exponent η_1 is defined by

$$\eta_1 = \lim_{R\to\infty} \frac{-\ln \mathbb{P}\{B[0,T_R] \text{ does not disconnect } 0 \text{ from } \infty\}}{\ln R};$$

see e.g. [14] and the references therein for more details. We want to derive a lower bound for η_1, i.e. upper bounds for $(R > 1)$

$$\mathbb{P}\{B[0,T_R] \text{ does not disconnect } 0 \text{ from } \infty\}.$$

Using the exponential mapping and conformal invariance of planar Brownian motion, one can notice that this is the same as finding an upper bound for the probabilities $(r = \log R > 0)$

$$Q_r = \mathbb{P}\{\forall (s,t) \in [0,\tilde{T}_r]^2,\ Z_s - Z_t \neq 2i\pi \text{ and } Z_s - Z_t \neq -2i\pi\}$$

where $Z = (X,Y)$ is a two-dimensional Brownian motion started from 0 and

$$\tilde{T}_r = \inf\{t > 0;\ X_t = r\}.$$

More precisely,

$$\eta_1 = \lim_{r\to\infty} \frac{-\ln Q_r}{r}. \tag{1}$$

We now put down some notation: For all $r > 0$, define

$$\tilde{T}_r^+ = \inf\{t > 0,\ X_t > r\}$$

and

$$h_r = \tilde{T}_r^+ - \tilde{T}_r.$$

h_r's are the lengths of the excursions of X below its maximae. For $u \in [0, h_r]$, we define

$$E_r(u) = E_r^1(u) + iE_r^2(u) = Z_{\tilde{T}_r+u}.$$

Lévy's identity (see e.g. Revuz-Yor [13], Chapter VI, Theorem (2.3)) shows that $(r - E_r^1(.), r \geq 0)$ is identical to the excursion process of reflected linear Brownian motion. Put also $H_r = \sup_{u\in[0,h_r]}(r - E_r^1(u))$.
Note that

$$F_r(.) = E_r^2(.) - E_r^2(0)$$

is a linear Brownian motion started from 0, which is independent from X, E^1 and also from $F_{r'}$, $r' \neq r$.

It is easy to check that:

$$Q_r \leq \mathbb{P}\{\forall v \in [0,r] \text{ such that } H_v < v \text{ and } \forall u \in [0,h_v],\ \forall t < \tilde{T}_u,\ E_v(u) \neq Z_t \pm 2i\pi\}.$$

Proposition 2 in Werner [15] (which is in some sense a slightly improved version of Beurling's Theorem), readily implies that for all $v \in [0, r]$, conditional on $\{X_t, t \geq 0\}$ and $\{Y_t, t \leq \tilde{T}_v\}$, such that $0 < H_v < v$ (this depends only on X), one has:

$$\mathbb{P}^{F_v}\{\forall u \in [0,h_v],\ \forall t < \tilde{T}_v,\ E_v(u) \neq Z_t \pm 2i\pi\} \leq \mathbb{P}^{F_v}\{\forall u \in [0,h_v], |F_v(u)| < 2\pi\},$$

where $\mathbb{P}^{F_v}$ denotes the probability measure corresponding to $\{F_v(u), 0 \leq u \leq h_v\}$. Let us put

$$A_v = \{\forall u \in [0,h_v], |F_v(u)| < 2\pi\}.$$

For $v \neq v'$, the strong Markov property shows that A_v and $A_{v'}$ are independent. Hence,

$$Q_r \leq \mathbb{P}\{\forall v \in [0,r] \text{ such that } H_v < v, A_v\}.$$

It is well-known (see e.g. Chung [4], page 206) that:

$$\mathbb{P}^{F_v}\{A_v\} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left[\frac{-(2k+1)^2 h_v}{32}\right].$$

We define:

$$a(h_v) = 1 - \mathbb{P}^{F_v}\{A_v\} = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(1 - \exp\left[\frac{-(2k+1)^2 h_v}{32}\right]\right).$$

Let $\{e(u), 0 \leq u \leq h\}$ denote an Brownian excursion under the Itô measure n. Let $H = \sup\{e(u),\ 0 \leq u \leq h\}$ and

$$A = \{\forall u \in [0,h], |F(u)| < 2\pi\},$$

where F is an independent linear Brownian motion started from 0 under the probability measure $\mathbb{P}^F$. Fix $\varepsilon > 0$ and put $r_0 = 1/\varepsilon$. so that $n(H > r_0) = \varepsilon$ (see Revuz-Yor [13], Chapter XII, Exercice (2.10)). For all $r > r_0$, one has

$$Q_r \leq \mathbb{P}\{\forall v \in [r_0, r],\ A_v \text{ or } \{H_v > r_0\}\}$$

and as the Excursion process of Brownian motion is a Poisson point process,

$$\begin{aligned} Q_r &\le \exp\left[-\int_{r_0}^{r} dv\ \mathbb{E}^F\{n((A\cup\{H>r_0\})^c)\}\right] \\ &\le \exp\left[-(r-r_0)\mathbb{E}^F\{n(A^c))-n(\{H>r_0\}\}\right] \\ &\le \exp\left[-(r-r_0)\left(\left(\int_0^\infty \frac{dh}{h^{3/2}(2\pi)^{1/2}}a(h)\right)-\varepsilon\right)\right]. \end{aligned}$$

We now state the following technical lemma:

Lemma 1

$$\frac{2\sqrt{2}}{\pi^{3/2}}\int_0^\infty \frac{dh}{h^{3/2}}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\left(1-\exp[-(2k+1)^2h/32]\right)=\frac{1}{2\pi}.$$

This lemma yields immediately that

$$Q_r \le \exp\left[-(r-r_0)(1/(2\pi)-\varepsilon)\right]$$

and completes the proof of Theorem 2. ∎

Proof of Lemma 1: There are various ways of deriving this identity. Let c denote the integral on the left-hand side of the identity in Lemma 1. It is very easy, using a reflection argument, to check that for all $n>0$,

$$\lim_{h\to 0} h^{-n}a(h) \le \lim_{h\to 0} 4h^{-n}\mathbb{P}\{F_h > 2\pi\} = 0.$$

Hence, integrating by parts yields:

$$c=\frac{4\sqrt{2}}{32\pi^{3/2}}\int_0^\infty \frac{dh}{\sqrt{h}}\sum_{k=0}^{\infty}(-1)^k(2k+1)\exp[-(2k+1)^2h/32]. \tag{2}$$

Define $g(x)=x\exp(-x^2/32)$. Note that:

$$\sum_{k\ge 0}\left\{g(4k\sqrt{h})-g((4k+4)\sqrt{h})\right\}=0.$$

We put

$$d(k,h)=\left(g((4k+1)\sqrt{h})-g((4k+3)\sqrt{h})-\frac{g(4k\sqrt{h})-g((4k+4)\sqrt{h})}{2}\right).$$

We can rewrite (2) as follows:

$$c=\frac{1}{4\sqrt{2}\pi^{3/2}}\int_0^\infty \frac{dh}{h}\sum_{k=0}^{\infty} d(k,h).$$

It is an easy exercice that we safely leave to the reader to check that for some fixed constants c', c'' and for all $k>0$,

$$\int_0^\infty \frac{dh}{h}d(k,h) \le \int_0^\infty \frac{dh}{h}c'h \sup_{[4k\sqrt{h},(4k+4)\sqrt{h}]}|g''| \le \frac{c''}{k^2}.$$

Hence, by dominated convergence,

$$c = \frac{1}{4\sqrt{2}\pi^{3/2}} \sum_{k=0}^{\infty} \int_0^\infty \frac{dh}{h} d(k,h).$$

For all $n > 0$,

$$\int_0^\infty \frac{dh}{h} g(n\sqrt{h}) = \int_0^\infty \frac{dx}{x} g(\sqrt{x}) = \sqrt{32} \int_0^\infty \frac{dx}{\sqrt{x}} e^{-x} = 4\sqrt{2\pi}.$$

Hence, for all $k \geq 1$,

$$\int_0^\infty dh h^{-1} d(k,h) = 0,$$

and as $g(0) = 0$,

$$c = \frac{1}{4\pi^{3/2}\sqrt{2}} \int_0^\infty \frac{dh}{h} d(0,h) = \frac{1}{4\pi^{3/2}\sqrt{2}} \frac{4\sqrt{2\pi}}{2} = \frac{1}{2\pi}$$

which concludes the proof. ∎

3 Extremal distance estimates

Let us fix $a > 0$ and an integer $N > 0$. Put $r = Na$. Define $(\varphi_1, \ldots, \varphi_N) \in (-\pi/2, \pi/2)^N$ and the continuous odd function f on $[-r, r]$ charaterized by

$$f'(x) = \tan \varphi_j \quad \text{for} \quad x \in [(j-1)a, ja],$$

for all $j \in \{1, \ldots, N\}$. We consider the strip $S := S(f)$ of the complex plane

$$S = \{(x,y);\ y \in (f(x) - \pi, f(x) + \pi),\ x \in (-r, r)\} \tag{3}$$

and we put:

$$U = \{(x, f(x) + \pi);\ x \in (-r, r)\} \subset \partial S \tag{4}$$

$$L = \{(x, f(x) - \pi);\ x \in (-r, r)\} \subset \partial S. \tag{5}$$

We are going to evaluate the extremal distance $d_S(U, L)$ between U and L in S. Let us just recall that $d_S(U, L)$ is the only value d, such that there exists a conformal mapping ϕ of S onto the rectangle $(-1,1)\times(-d,d)$, which maps the four corners $(-r+i(f(-r)-\pi))$, $(-r+i(f(-r)+\pi))$, $(r+i(f(r)+\pi))$, $(r+i(f(r)-\pi))$ onto the four corners $(-1-id)$, $(-1+id)$, $(1+id)$, $(1-id)$ respectively. Note that by symmetry and uniqueness of ϕ, $\phi(0) = 0$.
Recall also briefly the following alternative definition of $d_S(U, L)$: Let Γ denote the set of all rectifiable arcs γ in S, which join U to L. For all Borel measurable function ρ in S, we define the ρ-length of γ in Γ by $l_\rho(\gamma) = \int_\gamma \rho |dz|$. Then,

$$d = \sup_\rho \inf_{\gamma \in \Gamma} (l_\rho(\gamma))^2, \tag{6}$$

where the supremum is taken over all positive measurable functions ρ such that $\int_S \rho^2 dx dy = 1$. (see e.g. Ahlfors [1], chapter IV for definition, properties and more details on extremal distance).

Proposition 1

$$\frac{1}{d_S(U,L)} \leq \frac{aN}{\pi} + \frac{a}{\pi}\sum_{j=1}^{N}(\tan\varphi_j)^2. \tag{7}$$

Proof: For all x, we define $j(x) \in \mathbb{N}$ such that $|x| \in [(j(x)-1)a, j(x)a)$. Consider the function ρ in S defined by:

$$\rho(x,y) = \frac{1}{\cos\varphi_{j(x)}}.$$

We now show that the ρ-distance of any continuous path joing L to U in S is greater or equal to 2π. Take a C^1 path $(x(s), y(s))_{s\in[0,l]}$ in S, joining L to U (s is the euclidean arclength parameter and l the euclidean length of the path), that is such that $y(0) = f(x(0)) - \pi$ and $y(l) = f(x(l)) + \pi$. Define $Y_s = y(s) - f(x(s))$. It is easy to notice that

$$dY_s \cos\varphi_{j(x(s))} \leq ds.$$

As $\int_0^l dY_s = 2\pi$, one has:

$$\int_0^l ds\rho(x(s), y(s)) = \int_0^l ds/\cos\varphi_{j(x(s))} \geq \int_0^l dY_s = 2\pi.$$

The ρ-area $\mathcal{A}$ of S is

$$\mathcal{A} = \int\int_S \rho(x,y)^2 dx\ dy = 2\pi 2a\sum_{j=1}^{N}\frac{1}{(\cos\varphi_j)^2} = 4\pi a(N + \sum_{j=1}^{N}(\tan\varphi_j)^2).$$

Hence, (6) yields (considering the function $\rho/\sqrt{\mathcal{A}}$)

$$d_S(U,L) \geq \frac{(2\pi)^2}{\mathcal{A}}$$

and the proposition follows. ■

Note that for a C^1 odd function f on $(-r,r)$ and S,U, L defined as in (3), (4) and (5), the same method shows that (in fact, this can also be viewed as a corollary of Proposition 1, using approximations of f by piecewise linear functions),

$$\frac{1}{d_S(U,L)} \leq \frac{1}{\pi}\int_0^r (1 + f'(x)^2)dx, \tag{8}$$

which generalizes (7) in [14] and Proposition 1.

Let us now just recall the following observation from [14]. If B denotes a planar Brownian motion started from 0 and τ its exit time from the domain S, then:

$$\begin{aligned} \mathbb{P}\{\Re(B_\tau) = aN\} &= \frac{1}{2}\mathbb{P}\{B_\tau \notin U \cup L\} \\ &= \frac{1}{2}\mathbb{P}\{|\Re(\phi(B_\tau))| = 1\} \\ &= \frac{1}{2}\mathbb{P}\{|\Re(B)| \text{ hits } 1 \text{ before } |\Im(B)| \text{ hits } d_S(U,L)\} \\ &\geq \frac{1}{\pi}\exp\left[\frac{-\pi}{2d_S(U,L)}\right], \end{aligned} \tag{9}$$

where the first equality is a consequence of the symmetry of S, the third follows from conformal invariance of B under ϕ, and the last inequality is a consequence of properties of hitting times by reflected linear Brownian motion. Hence, with the same notation than in (8),

$$\mathbb{P}\{\Re(B_\tau) = r\} \geq \frac{1}{\pi} \exp\left[\frac{-r}{2} - \frac{1}{2}\int_0^r f'(x)^2 dx\right]. \qquad (10)$$

4 Upper bound

We very briefly recall some notation and results from [14]. We want to derive an upper bound for η_1. We define for $r > 0$,

$$q_r = \mathbb{P}\{\exists f : (-\infty, r] \to \mathbb{R}, \text{ continuous }, \forall t \in [0, \tilde{T}_r), |Y_t - f(X_t)| < \pi\}$$

where (as in the previous section) X and Y are two independent linear Brownian motions started from 0 and

$$\tilde{T}_r = \inf\{t > 0;\ X_t = r\}.$$

For all $r > 0$, it is easy to see that $q_r \leq Q_r$ (with Q_r defined as at the beginning of the previous section). Combining this with (1) shows that

$$\eta_1 \leq \liminf_{r\to\infty} \frac{-\ln q_r}{r}. \qquad (11)$$

As in [14], we are going to consider a family of functions f such that the events:

$$A_f^r = \{\forall t \in [0, \tilde{T}_r),\ |Y_t - f(X_t)| < \pi\}$$

are disjoint. We will use (10) to evaluate each probability $\mathbb{P}\{A_f^r\}$ and then sum over all functions f in this family.

We define the sets:

$$I = \{(i,j) \in \mathbb{N}^2;\ i \geq 1 \text{ and } j \in \{1, \ldots, 2^{i-1}\}\},$$
$$I' = I \cup \{(0,1)\},$$
$$J = \mathbb{Z}^I \text{ and } J' = \mathbb{Z}^{I'}.$$

We also define:

$$J'' = \{K = (k_{i,j})_{(i,j)\in I'} \in J';\ \text{for all but finitely many } (i,j) \in I',\ k_{i,j} = 0\}.$$

For $K = (k_{i,j})_{(i,j)\in I'} \in J''$ with

$$i_0 = i_0(K) = \sup\{i,\ \exists j \in \{1, \ldots, 2^{i-1}\},\ k_{(i,j)} \neq 0\}$$

(we put $i_0(0) = 0$), we define the function f_K on $[-r, r]$ as follows:

1. f_K is odd and continuous
2. For all $1 \leq j \leq 2^{i_0}$, f_K is linear on the interval $[r(j-1)2^{-i_0}, rj2^{-i_0}]$.
3. $f_K(r) = 2k_{0,1}\pi$

4. If $i_0(K) \neq 0$: For all $1 \leq i \leq i_0$ and $1 \leq j \leq 2^{i-1}$,

$$f_K\left(j\frac{r}{2^{i-1}} - \frac{r}{2^i}\right) = 2k_{i,j}\pi + \frac{1}{2}\left[f\left(\frac{jr}{2^{i-1}}\right) + f\left(\frac{(j-1)r}{2^{i-1}}\right)\right].$$

Note that Condition 2 implies that Condition 4 holds for all $i \geq 1$ and $j \in \{1, \ldots, 2^{i-1}\}$. Also if $K = (k_{i,j}) \neq K' = (k'_{i,j})$ in J'', then for

$$i_1 = \inf\{i \geq 0;\ \exists j,\ k_{i,j} \neq k'_{i,j}\}$$

and

$$j_1 = \inf\{j \geq 1;\ k_{i_1,j} \neq k'_{i_1,j}\},$$

the definition of f_K yields

$$|f_K((2j_1 - 1)r/2^{i_1}) - f_{K'}((2j_1 - 1)r/2^{i_1})| \geq 2\pi$$

and consequently $A^r_{f_K} \cap A^r_{f_{K'}} = \emptyset$. Hence,

$$q_r \geq \sum_{K \in J''} \mathbb{P}\{A^r_{f_K}\}. \tag{12}$$

We now evaluate $\int_0^r (f'_K(x))^2 dx$. An easy induction (over i_0) shows that

$$\int_0^r (f'_K(x))^2 dx = \frac{4\pi^2 k_{0,1}^2}{r} + \frac{8\pi^2}{r} \sum_{(i,j) \in I} 2^i (k_{i,j})^2. \tag{13}$$

Hence, using (10),

$$\mathbb{P}\{A^r_{f_k}\} \geq \frac{1}{\pi} \exp\left[\frac{-r}{2} - \frac{2\pi^2 k_{0,1}^2}{r} - \frac{4\pi^2}{r} \sum_{(i,j) \in I} 2^i (k_{i,j})^2\right].$$

Combining with (12) yields:

$$q_r \geq \frac{e^{-r/2}}{\pi} \sum_{K \in J''} \exp\left[-\frac{2\pi^2 k_{0,1}^2}{r} - \frac{4\pi^2}{r} \sum_{(i,j) \in I} 2^i (k_{i,j})^2\right].$$

For $K \in J' \setminus J''$,

$$\exp[-\sum_{(i,j) \in I} 2^i (k_{i,j})^2] = 0.$$

Hence,

$$\begin{aligned} q_r &\geq \frac{e^{-r/2}}{\pi} \sum_{K \in J'} \exp\left[-\frac{2\pi^2 k_{0,1}^2}{r} - \frac{4\pi^2}{r} \sum_{(i,j) \in I} 2^i (k_{i,j})^2\right] \\ &= \frac{e^{-r/2}}{\pi} \left(\sum_{k \in \mathbb{Z}} e^{-2\pi^2 k^2 / r}\right) \prod_{(i,j) \in I} \left(\sum_{k \in \mathbb{Z}} \exp[-4\pi^2 k^2 2^i / r]\right) \end{aligned}$$

and eventually

$$q_r \geq \frac{e^{-r/2}}{\pi}\theta\left(\frac{2\pi^2}{r}\right)\prod_{i=1}^{\infty}\theta\left(\frac{4\pi^2 2^i}{r}\right)^{(2^{i-1})}, \tag{14}$$

where $\theta(x) = \sum_{k\in\mathbb{Z}} \exp(-k^2 x)$ is the usual Theta function. We now put $b = 8\pi^2/r$ and define the function:

$$\begin{aligned} g(b) &= \frac{b}{8\pi^2}\left[\ln\theta(b/4) + \sum_{i=0}^{\infty} 2^i \ln(\theta(b2^i))\right] \\ &= \frac{1}{r}\left[\ln(\theta(2\pi^2/r)) + \sum_{i=0}^{\infty} 2^i \ln(8\pi^2 2^i/r)\right]. \end{aligned}$$

We rewrite (14) as follows

$$q_r \geq \frac{1}{\pi}(\exp[-1/2 + g(b)])^r.$$

It remains to study the behaviour of $g(b)$ when $b \to 0+$. It actually turns out that the maximum M of g is obtained at the limit $b \to 0+$, which is not surprising. Considering the sequence $b_n = 2^{-n}$, one can express $M = g(0+)$ as follows:

$$M = \lim_{n\to\infty} \frac{2^{-n}}{8\pi^2}\sum_{i=0}^{\infty} 2^i \ln\theta(2^{i-n}) = \frac{1}{8\pi^2}\sum_{i\in\mathbb{Z}} 2^i \ln(\theta(2^i)).$$

Finally,

$$\eta_1 \leq \frac{1}{2} - M = \frac{1}{2} - \frac{1}{8\pi^2}\sum_{i\in\mathbb{Z}} 2^i \ln(\theta(2^i)).$$

Numerically, $M > .03125$, which completes the proof of Theorem 1 for one walk.

As in [14], exactly the same technique provides an upper bound for the disconnection exponent for $n > 1$ Brownian motions. One just has to consider the sum $\sum \mathbb{P}\{A_f^r\}^n$. The upshot is Theorem 1. Exactly as in [14], this result has some consequences for non-intersection exponents that we leave to the reader. ∎

Remarks.

In Werner [14], the estimates obtained have at least three reasons for being far from the conjectured values. We try to remove one in the present paper (allowing Brownian motion to wind quickly from time to time).

One would expect to obtain better estimates for instance considering a family of functions $\mathcal{F}$ such that for some $f \neq g$ in $\mathcal{F}$, the events A_f^r and A_g^r are not disjoint, and then estimating the sum

$$\sum_{f\in\mathcal{F}}(\mathbb{P}\{A_f^r\} - \frac{1}{2}\sum_{g\neq f}\mathbb{P}\{A_f^r \cap A_g^r\}).$$

But to do this, we would need more precise estimates of $\mathbb{P}\{A_f^r\}$ and $\mathbb{P}\{A_f^r \cap A_g^r\}$ (the latter is more difficult) than those derived in this paper.

The other estimation loss occurs while restricting ourselves to study the asymptotics of q_r. It is in fact likely that q_r and Q_r do have different asymptotic behaviours. This gap seems even more difficult to lift using our type of approach.

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