Elect. Comm. in Probab. **7** (2002) 37–46

ELECTRONIC COMMUNICATIONS in PROBABILITY

FURTHER EXPONENTIAL GENERALIZATION OF PITMAN'S 2M-X THEOREM

FABRICE BAUDOIN
Laboratoire de Probabilités et Modèles aléatoires,
Universités Paris 6 et Paris 7, Laboratoire de Finance-Assurance CREST
email: baudoin@ensae.fr

submitted November 5, 2001 *Final version accepted* January 11, 2002

AMS 2000 Subject classification: 60J60
Diffusion processes, Exponential analogue of the $2M - X$ Pitman's theorem

Abstract
We present a class of processes which enjoy an exponential analogue of Pitman's 2M-X theorem, improving hence some works of H. Matsumoto and M. Yor.

1 Introduction

In a recent series of papers [6-13], H. Matsumoto and M. Yor have shown the following theorem:

Theorem 1 *Let $e_t^{(\mu)} := \exp(B_t + \mu t)$, $t \geq 0$, a geometric Brownian motion with drift $\mu \in \mathbb{R}_+^*$. The following identity (called Dufresne identity) holds*

$$\Big(\frac{1}{\int_0^t (e_s^{(-\mu)})^2 ds} \ , \ t > 0 \Big) \overset{law}{=} \frac{1}{\int_0^t (e_s^{(\mu)})^2 ds} + \frac{1}{\int_0^{+\infty} (\widetilde{e}_s^{(-\mu)})^2 ds} \ , \ t > 0 \Big)$$

where $\int_0^{+\infty} (\widetilde{e}_s^{(-\mu)})^2 ds$ is a copy of $\int_0^{+\infty} (e_s^{(-\mu)})^2 ds$ independent of the process $\left(\int_0^t (e_s^{(\mu)})^2 ds \right)_{t \geq 0}$. Moreover, the process $\frac{\int_0^t (e_s^{(\mu)})^2 ds}{e_t^{(\mu)}}$, $t \geq 0$, is, in its own filtration which is strictly included in the original Brownian filtration, a diffusion with infinitesimal generator

$$\mathcal{G} = \frac{1}{2} x^2 \frac{d^2}{dx^2} + \left[\big(-\mu + \frac{1}{2} \big) x + \frac{K_{1+\mu}}{K_\mu} \big(\frac{1}{x} \big) \right] \frac{d}{dx}$$

As shown in [9], by an elementary Laplace method argument, this theorem allows to recover the classical $2M - X$ Pitman's theorem.
In this work, we show that theorem 1 can be extended to a large class of processes which are constructed from the geometric Brownian motion by a simple transformation and which satisfy a generalized Dufresne identity (originally discussed in [5] and [10]).
This paper is essentially self-contained and contains a new proof of theorem 1.

2 Diffusions for which $(X_t^{-1}\int_0^t X_s^2 ds)_{t\geq 0}$ is also a diffusion

Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ a filtered probability space on which a standard Brownian motion $(B_t)_{t\geq 0}$ and Z a positive $\mathcal{F}_\infty$ -measurable variable independent of the process $(B_t)_{t\geq 0}$ are defined.
In [6], C. Donati-Martin, H. Matsumoto and M. Yor introduced the following transforms from $C(\mathbb{R}_+, \mathbb{R})$ to itself

$$T_\alpha(\phi)_t = \phi(t) - \ln\left(1 + \alpha\int_0^t \exp(2\phi(s))\, ds\right)$$

In their paper these authors have shown that if the law of Z is equivalent to the Lebesgue measure, then the laws of the processes $T_Z\left(B^{(\mu)}\right)_s$, $s \leq t$, and $B_s^{(\mu)} = B_s + \mu s$, $s \leq t$ are equivalent.
In what follows we study $T_Z\left(B^{(\mu)}\right)$ in its own filtration and we characterize the variables Z such that $T_Z\left(B^{(\mu)}\right)$ is a diffusion. As a consequence of this characterization, we will be able to give a new proof of Matsumoto-Yor's theorem 1. More precisely, in all this paper, we deal with the process $(X_t)_{t\geq 0}$ defined as follows

$$X_t = \frac{e_t^{(\mu)}}{1 + Z\int_0^t (e_s^{(\mu)})^2 ds}, \; t \geq 0$$

where $e_t^{(\mu)} := x_0 \exp(B_t + \mu t)$, $t \geq 0$, is the geometric Brownian motion with drift $\mu \in \mathbb{R}_+^*$ and started at $x_0 > 0$.

Proposition 2 *The process $(\int_0^t X_s^2 ds)_{t\geq 0}$ converges $\mathbb{P}$ a.s. when $t \to +\infty$ to $\frac{1}{Z}$ and the following generalized Dufresne identity holds:*

$$\left(\frac{1}{\int_0^t X_s^2 ds}, \; t > 0\right) \overset{a.s.}{=} \left(\frac{1}{\int_0^t (e_s^{(\mu)})^2 ds} + Z, \; t > 0\right)$$

Moreover,

$$\left(\frac{X_t}{\int_0^t X_s^2 ds}, \; t > 0\right) \overset{a.s.}{=} \left(\frac{e_t^{(\mu)}}{\int_0^t (e_s^{(\mu)})^2 ds}, \; t > 0\right),$$

thus $(\frac{X_t}{\int_0^t X_s^2 ds}, \; t > 0)$ is independent of $\int_0^{+\infty} X_s^2 ds$.

Theorem 3 *Assume that the law of Z admits a bounded C^2 density with respect to the law $\frac{2}{x_0^2}\gamma_\mu$. Then $(X_t)_{t\geq 0}$ is a diffusion in its own filtration if and only if there exists $\delta \geq 0$ such that:*

$$\mathbb{P}(Z \in dx) = \frac{x_0^\mu}{2\delta^\mu K_\mu(\delta x_0)}\, x^{\mu-1} e^{-\frac{\delta^2}{2x} - \frac{x_0^2}{2}x} dx, \; x > 0 \tag{2.1}$$

Moreover, in this case there exists a standard Brownian motion $(\beta_t)_{t\geq 0}$ adapted to the natural filtration of $(X_t)_{t\geq 0}$ such that

$$dX_t = X_t\left[\left(\mu + \frac{1}{2} - \delta X_t \frac{K_{1+\mu}(\delta X_t)}{K_\mu(\delta X_t)}\right) dt + d\beta_t\right], \; t \geq 0 \tag{2.2}$$

Remark 4 *For* $\delta = 0$, $Z \overset{law}{=} \frac{2}{x_0^2}\gamma_\mu$ *and* (2.2) *becomes*

$$dX_t = X_t\Big[\big(-\mu + \frac{1}{2}\big)dt + d\beta_t\Big] \;,\; t \geq 0$$

Corollary 5 *The process* $Z_t := \frac{\int_0^t X_s^2 ds}{X_t}$, $t \geq 0$, *is, in its own filtration, a diffusion independent of* $\int_0^{+\infty} X_s^2 ds$ *with infinitesimal generator*

$$\mathcal{G}^{x_0} = \frac{1}{2}x^2\frac{d^2}{dx^2} + \Big[\big(-\mu + \frac{1}{2}\big)x + x_0\frac{K_{1+\mu}}{K_\mu}\big(\frac{x_0}{x}\big)\Big]\frac{d}{dx}$$

Moreover, under the assumption (2.1) *, for* $t > 0$ *and* $x, z > 0$

$$\mathbb{P}\left(X_t \in dx \mid \mathcal{Z}_t, Z_t = z\right) = \Big(\frac{x}{x_0}\Big)^{2\mu}\frac{K_\mu(\delta x)}{K_\mu(\delta x_0)}e^{-\frac{\delta^2}{2}zx - \frac{1}{2z}\left(\frac{x}{x_0} + \frac{x_0}{x}\right)}\frac{dx}{2xK_\mu\big(\frac{1}{z}\big)}$$

where $(\mathcal{Z}_t)_{t\geq 0}$ *is the natural filtration of* $(Z_t)_{t\geq 0} = \big(\frac{\int_0^t X_s^2 ds}{X_t}\big)_{t\geq 0}$.

Proposition 6 *Under the assumption* (2.1), *for all* $t \geq 0$

$$\mathbb{E}\Big(\frac{X_t^2}{\int_t^{+\infty} X_s^2 ds} - \delta^2\int_t^{+\infty} X_s^2 ds \,\Big|\, \mathcal{X}_t\Big) = 2\mu \tag{2.3}$$

where $(\mathcal{X}_t)_{t\geq 0}$ *is the natural filtration of* $(X_t)_{t\geq 0}$.

In the following remarks, we assume that (2.1) is satisfied for some $\delta \geq 0$.

Remark 7

1. *We recall the definition of the Mac-Donald function*

$$K_\mu(x) = \frac{1}{2}\Big(\frac{x}{2}\Big)^\mu\int_0^{+\infty}\frac{e^{-t-\frac{x^2}{4t}}}{t^{1+\mu}}dt$$

but for further details, we refer to [7].

2. *The law* (2.1) *is called a generalized inverse Gaussian distribution. These laws have been widely discussed by Barndorff-Nielsen [1], Letac-Wesolowski [8], Matsumoto-Yor [12], and Vallois [19] among others.*

3. *We get the theorem 1.1. by taking* $\delta = 0$, *because in this case*

$$(X_t \;,\; t \geq 0) \overset{law}{=} (e_t^{(-\mu)} \;,\; t \geq 0)$$

4. *The stochastic differential equation* (2.2) *solved by* $(X_t)_{t\geq 0}$, *enjoys the pathwise uniqueness property. Indeed, a simple computation shows that a scale function is given by*

$$s(x) = \frac{I_\mu}{K_\mu}(\delta x)$$

where I_μ is the classical modified Bessel function with index μ. Using Wronskian relation for Bessel functions, we get then the speed measure

$$m(dx) = -\frac{2}{x} K_\mu (\delta x)^2 \, dx$$

And finally, Feller's test for explosions (see for example [17] pp.386) gives

$$\mathbb{P}(\mathbf{e} = +\infty) = 1$$

where $\mathbf{e}$ *is the lifetime of a solution.*

5. *If we apply Itô's formula conditionally to Z, we see that in the natural filtration of $(X_t)_{t\geq 0}$ enlarged by $\int_0^{+\infty} X_s^2 ds$ the process $(X_t)_{t\geq 0}$ is a semimartingale whose decomposition is*

$$dX_t = X_t \left[\left(\mu + \frac{1}{2} - \frac{X_t^2}{\int_t^{+\infty} X_s^2 ds} \right) dt + dB_t \right] \ , t \geq 0$$

6. *For $\mu = \frac{1}{2}$, (2.2) becomes*

$$dX_t = -\delta X_t^2 dt + X_t d\beta_t$$

and this equation is solved by

$$X_t = \frac{x_0 e^{-\frac{1}{2}t + \beta_t}}{1 + \delta x_0 \int_0^t e^{-\frac{1}{2}s + \beta_s} ds} \ , t \geq 0$$

We have then the following surprising identity in law

$$\left(\frac{x_0 e^{-\frac{1}{2}t + \beta_t}}{1 + \delta x_0 \int_0^t e^{-\frac{1}{2}s + \beta_s} ds}, \ t \geq 0 \right) \overset{law}{=} \left(\frac{x_0 e^{\frac{1}{2}t + B_t}}{1 + Z x_0^2 \int_0^t e^{s + 2B_s} ds} \ , t \geq 0 \right)$$

for which it would be interesting to have a direct proof.
For $\mu \neq \frac{1}{2}$, it seems difficult to solve explicitly the equation (2.2), even in the case $\mu = n + \frac{1}{2}$, with $n \in \mathbb{N}$. Let us just mention that

$$X_t = \frac{x_0 e^{-\mu t + \beta_t}}{1 + \delta x_0 \int_0^t e^{-\mu s + \beta_s} ds}$$

solves

$$dX_t = \left[\left(-\mu + \frac{1}{2} \right) X_t - \delta X_t^2 \right] dt + X_t d\beta_t$$

which coincides with (2.2) if and only if $\mu = \frac{1}{2}$.

7. *The process $(\rho_t)_{t\geq 0}$ related to $(X_t)_{t\geq 0}$ by the Lamperti's relation $X_t = \rho_{\int_0^t X_s^2 ds}$, $t \geq 0$, is a well-known diffusion discussed in [11], [15], [16], and [20] the $BES(\mu, \delta \downarrow)$ process, i.e. the diffusion process with the infinitesimal generator*

$$\frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{\mu + \frac{1}{2}}{x} - \delta \frac{K_{1+\mu}(\delta x)}{K_\mu (\delta x)} \right) \frac{d}{dx}$$

From [6] (see also [3]), it implies then that there exists a Bessel process Q with index μ, starting at x_0 and independent of the first hitting T_0 of 0 for ρ such that

$$(\rho_t, \ t < T_0) \overset{law}{=} \left(\left(1 - \frac{t}{T_0} \right) Q_{\frac{T_0 t}{T_0 - t}}, \ t < T_0 \right)$$

3 Proofs

Proof of proposition 2
This proof is very simple and inspired from [6].
As

$$X_t = \frac{e_t^{(\mu)}}{1+Z\ \int_0^t (e_s^{(\mu)})^2 ds}\ ,\ t\geq 0 \tag{3.1}$$

we have

$$X_t^2 = \Big(\frac{e_t^{(\mu)}}{1+Z\ \int_0^t (e_s^{(\mu)})^2 ds}\Big)^2\ ,\ t\geq 0$$

and hence

$$\int_0^t X_s^2 ds = \frac{\int_0^t (e_s^{(\mu)})^2 ds}{1+Z\int_0^t (e_s^{(\mu)})^2 ds}\ ,\ t\geq 0 \tag{3.2}$$

This equality implies immediately the $\mathbb{P}-a.s.$ convergence of the process $\int_0^t X_s^2 ds$, $t\geq 0$, to $\frac{1}{Z}$, it also implies

$$\frac{1}{\int_0^t X_s^2 ds} = \frac{1}{\int_0^t (e_s^{(\mu)})^2 ds} + Z\ ,\ t>0$$

Finally, by dividing (3.2) by (3.1), we deduce

$$\frac{\int_0^t X_s^2 ds}{X_t} = \frac{\int_0^t (e_s^{(\mu)})^2 ds}{e_t^{(\mu)}}\ ,\ t\geq 0$$

Proof of theorem 3
Let us set in all the proof

$$Y = \frac{1}{Z} = \int_0^{+\infty} X_s^2 ds$$

Let now $y>0$ and denote $\mathbb{Q}^y$ the law of the process $(X_t)_{t\geq 0}$ conditioned with $Y=y$.
From theorem 1.5. of [6], we see that the following absolute continuity relation takes place

$$d\mathbb{Q}^y_{/\Xi_t} = \big(\frac{\chi_t}{x_0}\big)^{2\mu}\Big(\frac{y}{y-\int_0^t \chi_s^2 ds}\Big)^{1+\mu} e^{\frac{x_0^2}{2y}-\frac{\chi_t^2}{2(y-\int_0^t \chi_s^2 ds)}} 1_{\int_0^t \chi_s^2 ds<y}\ d\mathbb{P}^{-\mu}_{/\Xi_t}\ ,\ t\geq 0$$

where $(\chi_t)_{t\geq 0}$ is the coordinate process, $(\Xi_t)_{t\geq 0}$ its natural filtration, and $\mathbb{P}^{-\mu}$ the law of the process $(x_0\exp(-\mu t+B_t))_{t\geq 0}$. By integrating this absolute continuity relation with respect to the law of Y, i.e.

$$\mathbb{P}(Y\in dx) = \frac{x_0^{2\mu}\xi(x)}{2^\mu\Gamma(\mu)}\ \frac{e^{-\frac{x_0^2}{2x}}}{x^{1+\mu}}dx\ ,\ x>0$$

where ξ is the bounded density of Y with respect to $\frac{2}{x_0^2\gamma_\mu}$, we deduce after some elementary computation that the law $\mathbb{Q}$ of our process $(X_t)_{t\geq 0}$ satisfies the following equivalence relation

$$d\mathbb{Q}_{/\Xi_t} = \Big(\int_0^{+\infty}\frac{e^{-u}u^{\mu-1}}{\Gamma(\mu)}\xi\Big(\int_0^t \chi_s^2 ds+\frac{\chi_t^2}{2u}\Big)du\Big)\ d\mathbb{P}^{-\mu}_{/\Xi_t}\ ,\ t\geq 0 \tag{3.3}$$

Hence, by Girsanov theorem, if X is a diffusion in its own filtration, then the infinitesimal generator of this diffusion can be written

$$\mathcal{L} = \frac{1}{2}x^2 \frac{d}{dx^2} + x\left(-\mu + \frac{1}{2} + xb(x)\right)\frac{d}{dx}$$

where $b : \mathbb{R}_+^* \to \mathbb{R}$ is related to

$$\varphi(t,x) = \int_0^{+\infty} \frac{e^{-u}u^{\mu-1}}{\Gamma(\mu)} \xi\left(t + \frac{x^2}{2u}\right) du$$

by the Hopf-Cole transformation

$$b = \frac{\partial}{\partial x} \ln \varphi \tag{3.4}$$

From (3.4) the homogeneity of b implies that there exist two functions f and g such that

$$\varphi(t,x) = f(x)\ \ g(t)\ \ ,\ t,x > 0$$

but φ is a solution of the following partial differential equation

$$\frac{\partial \varphi}{\partial t} + \frac{1}{2}\frac{\partial^2 \varphi}{\partial x^2} + \frac{-\mu + \frac{1}{2}}{x}\frac{\partial \varphi}{\partial x} = 0$$

it immediately implies that there exists a constant C such that

$$g(t) = e^{Ct}$$

We note now that the constant C is negative, because ξ is bounded and we have the limit condition

$$\lim_{x \to 0^+} \varphi(t,x) = \xi(t) \tag{3.5}$$

Denote the constant C by $-\frac{\delta^2}{2}$ with $\delta \in \mathbb{R}_+$ and conclude that

$$\mathbb{P}(Y \in dx) = \frac{x_0^{\mu}}{2\delta^{\mu} K_{\mu}(\delta x_0)} \frac{e^{-\frac{\delta^2}{2}x - \frac{x_0^2}{2x}}}{x^{1+\mu}} dx\ ,\ x > 0$$

It proves the first part of our theorem.
On the other hand, if (2.1) is satisfied the following equivalence relation takes place (it suffices to use the equivalence relation (3.3))

$$d\mathbb{Q}_{/\Xi_t} = e^{-\frac{\delta^2}{2}\int_0^t \chi_s^2 ds} \left(\frac{\chi_t}{x_0}\right)^{\mu} \frac{K_{\mu}(\delta \chi_t)}{K_{\mu}(\delta x_0)}\, d\mathbb{P}^{-\mu}_{/\Xi_t}\ ,\ t \geq 0 \tag{3.6}$$

which implies, by Girsanov theorem

$$dX_t = X_t\left[\left(\mu + \frac{1}{2} - \delta X_t \frac{K_{1+\mu}(\delta X_t)}{K_{\mu}(\delta X_t)}\right) dt + d\beta_t\right]\ ,\ t \geq 0$$

where $(\beta_t)_{t\geq 0}$ is a $\mathbb{P}$ standard Brownian motion adapted to the natural filtration of $(X_t)_{t\geq 0}$.

Proof of corollary 5

Let $\delta > 0$.
Let us consider a sequence $(Z_n)_{n>0}$ of random variables independent of $(B_t)_{t\geq 0}$ and such that

$$\mathbb{P}\left(Z_n \in dx\right) = \frac{n^\mu}{2\delta^\mu K_\mu\left(\delta n\right)}\ x^{\mu-1}e^{-\frac{\delta^2}{2x}-\frac{n^2}{2}x}dx\ ,\ x>0$$

We associate with this sequence $(Z_n)_{n>0}$ the sequence of processes $\left(X_t^{(n)}\right)_{t\geq 0}$ defined by

$$X_t^{(n)} = \frac{n\exp\left(B_t+\mu t\right)}{1+Z_n n^2\int_0^t \exp\left(2B_s+2\mu s\right)ds}\ ,\ t\geq 0$$

We have

$$\frac{1}{X_t^{(n)}} = \frac{1}{n}\exp\left(-B_t-\mu t\right)+Z_n n\ \frac{\int_0^t \exp\left(2B_s+2\mu s\right)ds}{\exp\left(B_t+\mu t\right)}$$

But, one can show that

$$nZ_n \to^{\text{law}}_{n\to+\infty} \delta$$

hence

$$\left(\frac{1}{X_t^{(n)}}\ ,\ t\geq 0\right) \to^{law}_{n\to+\infty} \left(\delta\ \frac{\int_0^t \exp\left(2B_s+2\mu s\right)ds}{\exp\left(B_t+\mu t\right)}\ ,\ t\geq 0\right)$$

On the other hand, from theorem 3, $\left(\frac{1}{X_t^{(n)}}\ ,\ t\geq 0\right)$ is a diffusion with infinitesimal generator

$$\mathcal{G}^\delta = \frac{1}{2}x^2\frac{d^2}{dx^2}+\left[\left(-\mu+\frac{1}{2}\right)x+\delta\frac{K_{1+\mu}}{K_\mu}\left(\frac{\delta}{x}\right)\right]\frac{d}{dx}$$

Now, to conclude the proof of the first part of the corollary, we use proposition 2 which asserts that

$$\frac{\int_0^t X_s^2 ds}{X_t} = \frac{\int_0^t \left(e_s^{(\mu)}\right)^2 ds}{e_t^{(\mu)}}\ ,\ t\geq 0$$

The computation of the conditional probabilities is easily deduced from (3.6) and proposition 6 of [9], by a change of probability, namely the absolute continuity (3.6).

Proof of proposition 6
A simple computation shows that for $t\geq 0$

$$\mathbf{D}_t\int_0^{+\infty} e^{2\chi_s-2\mu s}ds = 2\int_t^{+\infty} e^{2\chi_s-2\mu s}ds$$

where $\mathbf{D}$ is the Malliavin's differential and $(\chi_t)_{t\geq 0}$ the coordinate process on the Wiener space. Clark-Ocone's formula (see [4]) applied to $\exp\{-\frac{\delta^2}{2}\int_0^{+\infty} e^{2\chi_s-2\mu s}ds\}$ implies then that for all $t\geq 0$

$$\mathbb{E}\left(\int_t^{+\infty} X_s^2 ds \mid \mathcal{X}_t\right) = \frac{X_t}{\delta}\frac{K_{1+\mu}\left(\delta X_t\right)}{K_\mu\left(\delta X_t\right)}-\frac{2\mu}{\delta^2} \tag{3.7}$$

As noticed in the remarks of section 2, in the enlarged filtration $\left(\mathcal{X}_t \vee \sigma\left(\int_0^{+\infty} X_s^2 ds\right)\right)_{t\geq 0}$, our process $(X_t)_{t\geq 0}$ is a semimartingale whose decomposition is

$$dX_t = X_t \left[\left(\mu + \frac{1}{2} - \frac{X_t^2}{\int_t^{+\infty} X_s^2 ds} \right) dt + dB_t \right] \ , t \geq 0$$

By projecting this decomposition on $(\mathcal{X}_t)_{t\geq 0}$, using the classical filtering formulas, we get

$$\mathbb{E}\Big(\frac{1}{\int_t^{+\infty} X_s^2 ds} \,\Big|\, \mathcal{X}_t \Big) = \frac{\delta}{X_t} \frac{K_{1+\mu}(\delta X_t)}{K_\mu(\delta X_t)} \tag{3.8}$$

And finally, (3.7) and (3.8) imply (2.3)

4 Opening

To conclude the paper, let us relate our result to another simple transformation of the Brownian motion (first considered by T. Jeulin and M. Yor and then generalized by P.A. Meyer, see [14]). It is easily seen that for a standard Brownian motion $(B_t)_{0\leq t<1}$, the process

$$\left(B_t - \int_0^t \frac{B_s}{s} ds, \ 0 \leq t < 1 \right)$$

is well defined and is a Brownian motion in its own filtration. Now, let us consider a random variable Z independent of $(B_s)_{0\leq s<1}$. Consider the process

$$X_t = B_t + tZ \ , 0 \leq t < 1$$

We have the following proposition (for further details on it, we refer to [2]).

Proposition 8 *The process*

$$\left(X_t - \int_0^t \frac{X_s}{s} ds, \ 0 \leq t < 1 \right)$$

is well defined and is a Brownian motion in its own filtration which is independent of X_1. Moreover, $(X_t)_{0\leq t<1}$ is a (homogeneous) diffusion in its own filtration if and only if, there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\mathbb{P}(Y \in dx) = C \ \cosh(\alpha x + \beta)\, e^{-\frac{x^2}{2}} \ dx$$

where $C > 0$ is a normalization constant. In this case

$$dX_t = \alpha \tanh(\alpha X_t + \beta)\, dt + dW_t \ , 0 \leq t < 1$$

where $(W_t)_{0\leq t<1}$ is a standard Brownian motion adapted to the natural filtration of $(X_t)_{0\leq t<1}$.

Hence, a somewhat vague question arise: Are there other *natural* transformations of the Brownian motion for which the same phenonemon of loss of information takes place ?

Acknowledgments. I would like to thank Marc Yor for his encouragements.

References

[1] 0. Barndorff-Nielsen: Hyperbolic distributions and distributions on hyperbolae, Scand. J. Stat., 5 (1978), 151-157

[2] F. Baudoin : Conditioned stochastic differential equations: Theory and applications, preprint (2001) to appear in Stochastic Processes and their Applications available on http://www.proba.jussieu.fr/mathdoc/preprints/index.html

[3] F. Baudoin: Bessel process with random drift and the $\frac{\langle X \rangle}{X}$ transformation, preprint (2001) available on http://www.proba.jussieu.fr/mathdoc/preprints/index.html

[4] J.M.C Clark, The representation of functionals of Brownian motion by stochastics integrals, Ann. Math. Stat. 41 (1970), 1282-1295, 42 (1971), 1778

[5] D. Dufresne : An affine property of the reciprocal Asian option process, Osaka J. Math. 38, 2001, 379-381

[6] C. Donati-Martin, H. Matsumoto, M. Yor: Some absolute continuity relationships for certain anticipative transformations of geometric Brownian motions, Publ. RIMS Kyoto, 2001, Vol. 37, 295-326

[7] N. N. Lebedev : Special Functions and their Applications, Dover, New-York, 1972

[8] G. Letac and J. Wesolowski : An independence property for the product of GIG and gamma laws, Ann. Prob. 2000, Vol. 28, 3, 1371-1383

[9] H. Matsumoto, M. Yor : A version of Pitman's $2M - X$ theorem for geometric Brownian motions, C.R. Acad. Sci. Paris, t. 328, Série I, p. 1067-1074, 1999

[10] H. Matsumoto, M. Yor : A Relationship between Brownian Motions with Opposite Drifts via Certain Enlargements of the Brownian Filtration, Osaka J. Math. 38, 2001, 383-398

[11] H. Matsumoto, M. Yor : An Analogue of Pitman's $2M - X$ Theorem for Exponential Wiener Functionals, Part I: A time-inversion approach, Nagoya Math. J., Vol. 159 125-166, 2000

[12] H. Matsumoto, M. Yor : An Analogue of Pitman's $2M - X$ Theorem for Exponential Wiener Functionals, Part II: The Role of the Generalized Inverse Gaussian laws, Nagoya Math. J., Vol. 162 65-86, 2001

[13] H. Matsumoto, M. Yor : Some changes of probabilities related to a geometric Brownian motion version of Pitman's $2M - X$ theorem, Elect. Comm. in Probab. 4 (1999) 15-23

[14] P.A. Meyer : Sur une transformation du mouvement brownien due à Jeulin et Yor. Sém. Prob. XXVIII, 98-101, LNM 1583, Springer 1994

[15] J. W. Pitman and M. Yor : Processus de Bessel, et mouvement brownien, avec ≪drift≫, C.R. Acad. Sc. Paris Série I, 291 (1980), 151-153

[16] J. W. Pitman and M. Yor : Bessel processes and infinitely divisible laws, Stochastic Integrals, ed. by D. Williams, Lecture Notes Maths. 851, 285-370, Springer-Verlag, Berlin, 1981

[17] D. Revuz and M. Yor : Continuous Martingales and Brownian Motion, 3rd. Ed., Springer-Verlag, Berlin, 1999

[18] L.C.G. Rogers, Characterizing all diffusions with the $2M - X$ property, Ann. Prob., 9 (1981), 561-572

[19] P. Vallois : La loi gaussienne inverse généralisée comme premier ou dernier temps de passage de diffusions, Bull. Sc. math., $2^{\text{è}}$ série, 115 (1991), 301-368

[20] S. Watanabe : On time inversion of one-dimensional diffusion process, Zeitschrift für Wahr., 31, 1975, p. 115-124