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DENSITY FLUCTUATIONS FOR A ZERO-RANGE PROCESS ON THE PERCOLATION CLUSTER

PATRÍCIA GONÇALVES
Centro de Matemática, Universidade do Minho, Campus de Gualtar, 4710-057 Braga, Portugal
email: patg@math.uminho.pt

MILTON JARA
FYMA, Université Catholique de Louvain, Chemin du Cyclotron 2, B-1348, Louvain-la-Neuve, Belgium and CEREMADE, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassygny,Paris CEDEX 75775, France
email: milton.jara@uclouvain.be

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Abstract

We prove that the density fluctuations for a zero-range process evolving on the d-dimensional supercritical percolation cluster, with $d \geq 3$, are given by a generalized Ornstein-Uhlenbeck process in the space of distributions $\mathscr{S}'(\mathbb{R}^d)$.

1 Introduction

Consider the infinite cluster of a supercritical bond percolation model. On this random graph, we define a zero-range process, which can be defined as a system of symmetric, simple random walks on which the hopping time of particles at a given site depends only on the number of particles at that site. It has been recently proved that the scaling limit of a simple random walk on the percolation cluster is given by a Brownian motion with a diffusion coefficient that does not depend on the particular realization of the percolation cluster [3], [10], [11]. With this result in mind, it is natural to raise the question about the collective behavior of a system of random walks evolving on the percolation cluster, possibly with some interaction. More precisely, we want to study the density fluctuations of this model. Scale the lattice by $1/n$ and give a mass $1/n$ to each particle. In this way we obtain a measure in $\mathbb{R}^d$, which we call the *empirical density* of particles. Under a diffusive time scaling, it has been proved for a simple exclusion process [5] that the empirical measure converges in a proper sense, to a solution of the heat equation $\partial_t u = D\Delta u$, where D is the diffusion coefficient of the underlying random walk.

This result can be interpreted as a law of large numbers for the density of particles, and therefore the question about the central limit theorem arises. For the simple exclusion process, duality techniques reduce the law of large numbers and the central limit theorem for the density to suitable problems for simple random walks. In order to treat a more general case, in these notes we discuss the density fluctuations for the zero-range process. Due to the inhomogeneity introduced by the randomness of the percolation cluster, the model turns out to be non-gradient. Following our previous work [6], a functional transformation of the empirical density puts the model back into the setup of gradient systems. The main tool allowing to make this functional transformation is a form of the *compensated compactness lemma* of Tartar [12], which is obtained in [5] using two-scale convergence. Therefore, the proof of a central limit theorem for the density fluctuations is reduced to the so-called *Boltzmann-Gibbs principle*, which roughly states that fluctuations of non-conserved quantities are faster than fluctuations of conserved quantities. In the right scaling limit, only fluctuations of the density of particles survive, allowing to obtain a martingale characterization of the scaling limit. In this way we prove that the limiting density field is given by a generalized Ornstein-Uhlenbeck process, as predicted by the fluctuation-dissipation principle. In dimension $d = 2$, our method requires regularity of solutions of an associated elliptic problem which, up to our knowledge, have not been obtained for this model .

The main technical innovation in this paper is what we call the *connectivity lemma*. In [6], an ellipticity condition makes it possible to compare the relaxation properties of our system with the relaxation properties of a zero-range process in the absence of the random environment. But the percolation cluster does not satisfy this ellipticity condition, since by construction the jump rate through a closed bond is 0. Moreover, the percolation cluster is locally non connected, since for a typical box of fixed side, the part of the percolation cluster lying inside this box is not connected. The connectivity lemma says that enlarging a bit the box we obtain a connected graph with good ergodic properties, allowing to get bounds on the relaxation to equilibrium.

We point out that our results are still true in a more general context. We have chosen the zero-range process on the percolation cluster for simplicity and to capture the essential difficulties. Take for example a random barrier model on $\mathbb{Z}^d$ on which the conductances are not bounded below (see [5] for a more detailed definition), and take an interacting particle system satisfying the gradient condition (before randomizing the lattice). Our results apply as well for these models, leading to the same results.

These notes are organized as follows. In Section 2 we introduce the model and we state the main results. In Section 3 we introduce the corrected density field and we show how to obtain the main results from the compensated compactness lemma and the Boltzmann-Gibbs principle. In Section 4 we state and prove the connectivity lemma and we obtain the Boltzmann-Gibbs principle starting from it.

2 Definitions and main results

2.1 The supercritical percolation cluster

Let $\mathscr{E} = \{e = \langle xy \rangle; x, y \in \mathbb{Z}^d, |x-y| = 1\}$ be the set of nearest-neighbor, non oriented bonds in $\mathbb{Z}^d$. Let $\omega = \{\omega(e); e \in \mathscr{E}\}$ be a sequence of i.i.d. random variables with $P(\omega(e) = 1) = 1 - P(\omega(e) = 0) = p$. Let us fix a realization of the sequence ω. Whenever $\omega(e) = 1$, we say that the bond e is *open*. Otherwise we say that the bond e is closed. We say that two sites x, y in $\mathbb{Z}^d$ are *connected*, which we denote by $x \longleftrightarrow y$, if there is a finite sequence of points $\{x_0 = x, x_1, \dots, x_n = y\}$ such that for any $i \in \{0, \dots, n-1\}$, $|x_{i+1} - x_i| = 1$ and the bond $\langle x_i x_{i+1} \rangle$ is open. For each $x \in \mathbb{Z}^d$, we define $\mathscr{C}(\omega, x) = \{y \in \mathbb{Z}^d; x \longleftrightarrow y\}$. We call $\mathscr{C}(\omega, x)$ the cluster containing x. It is well known [7] that the probability of $\mathscr{C}(\omega, x)$ being infinite is an increasing function of p, and that there exists $p_c \in (0, 1)$ such that this probability is strictly positive for $p > p_c$ and equal to 0 for $p < p_c$.

It is also well known that for $p > p_c$, with probability one, there exists a unique cluster of infinite cardinality. We denote this cluster by $\mathscr{C}(\omega)$. We call $\mathscr{C}(\omega)$ the *supercritical percolation cluster*. From now on, we fix a number $p \in (p_c, 1]$ and a sequence ω for which $\mathscr{C}(\omega)$ is well defined. We also define $\theta(p) = P(0 \in \mathscr{C}(\omega))$.

2.2 The zero-range process in $\mathscr{C}(\omega)$

For x, y in $\mathbb{Z}^d$, we say that $x \sim y$ if $|x-y| = 1$ and $\langle xy \rangle$ is open. Let $g : \mathbb{N}_0 = \{0, 1, ...\} \to [0, \infty)$ be a function with $g(0) = 0$. The zero-range process in $\mathscr{C}(\omega)$ with interaction rate $g(\cdot)$ is the Markov process ξ_t in $\Omega = \mathbb{N}_0^{\mathscr{C}(\omega)}$ generated by the operator L given by

$$Lf(\xi) = \sum_{\substack{x,y \in \mathbb{Z}^d \\ x \sim y}} g(\xi(x)) \big[f(\xi^{x,y}) - f(\xi) \big],$$

where $f : \Omega \to \mathbb{R}$ is a *local* function, that is, it depends on $\xi(x)$ only for a finite number of sites $x \in \mathscr{C}(\omega)$, and $\xi^{x,y}$ is given by

$$\xi^{x,y}(z) = \begin{cases} \xi(x) - 1, & z = x \\ \xi(y) + 1, & z = y \\ \xi(z), & z \neq x, y. \end{cases}$$

The dynamics of this process is easy to understand. At each time, a particle jumps from a site $x \in \mathscr{C}(\omega)$ to a neighbor $y \sim x$ with exponential rate $g(k)$, where k is the number of particles at site x at that time. When the initial number of particles is finite, the process ξ_t is a continuous-time Markov chain in Ω.
In order to have a well defined family of ergodic, invariant measures for the process ξ_t, we assume that $g(n) > 0$ for any $n > 0$ and that

$$\sup_n |g(n+1) - g(n)| < +\infty. \tag{2.1}$$

Under these conditions, $g(\cdot)$ is bounded by a linear function: there exists c_0 such that $g(n) \leq c_0 n$ for any n. Let us define the product probability measures $\bar{\nu}_\varphi$ in Ω by the relation

$$\bar{\nu}_\varphi\{\xi(x) = k\} = \frac{1}{Z(\varphi)} \frac{\varphi^k}{g(k)!},$$

where $g(k)! = g(1) \cdots g(k)$ for $k \geq 1$, $g(0)! = 1$, $\varphi \in [0, \infty)$ and $Z(\varphi)$ is the normalization constant. Due to (2.1), these measures are well defined up to some critical value $\varphi_c > 0$. Notice that the parameter φ has an interpretation as the strength of the interaction, since $\int g(\xi(x)) \bar{\nu}_\varphi(d\xi) = \varphi$. It has been proved [1] that the measures $\{\bar{\nu}_\varphi; \varphi \in [0, \varphi_c)\}$ are ergodic and invariant under the evolution of ξ_t. Since we are interested in density fluctuations, we will reparametrize this family of measures by its density of particles. For that purpose, define $\rho(\varphi) = \int \xi(x) \bar{\nu}_\varphi(d\xi)$. It is not hard to see that $\varphi \mapsto \rho(\varphi)$ is a diffeomorphism from $[0, \varphi_c)$ to $[0, \rho_c)$, where $\rho_c = \lim_{\varphi \to \varphi_c} \rho(\varphi)$. In particular, the inverse function $\rho \mapsto \varphi(\rho)$ is well defined. We define $\nu_\rho = \bar{\nu}_{\varphi(\rho)}$, measure for which now $\int \xi(x) \nu_\rho(d\xi) = \rho$.

2.3 The density fluctuations

Let $\rho \in (0,\rho_c)$ be fixed and consider the process $\xi_t^n = \xi_{tn^2}$ starting from the measure ν_ρ. We denote by $\mathbb{P}_n$ the distribution of the process $\xi_\cdot^n$ in the Skorohod space $\mathscr{D}([0,\infty),\Omega)$ of càdlàg trajectories in Ω, and by $\mathbb{E}_n$ the expectation with respect to $\mathbb{P}_n$.
Denote by $\mathscr{C}_c(\mathbb{R}^d)$ the set of continuous functions $G : \mathbb{R}^d \to \mathbb{R}$ with compact support. By the ergodic theorem, it is not hard to see that

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x\in\mathscr{C}(\omega)} G(x/n)\xi(x) = \theta(p)\rho \int_{\mathbb{R}^d} G(x)dx,$$

almost surely with respect to the probability measure $P \otimes \nu_\rho$. In the previous result, the ergodic theorem is invoked twice: first to state that the density of points belonging to $\mathscr{C}(\omega)$ when properly rescaled is equal to $\theta(p)$, and then to state that the density of particles is equal to ρ. Since the measure ν_ρ is invariant under the dynamics of ξ_t, the same result is also valid if we replace ξ by ξ_t^n in the previous expression. We are interested on a version of the central limit theorem for this quantity. Let us define, for each test function G, the *density fluctuation field* $\mathscr{Y}_t^n(G)$ by

$$\mathscr{Y}_t^n(G) = \frac{1}{n^{d/2}} \sum_{x\in\mathscr{C}(\omega)} G(x/n)\big(\xi_t^n(x) - \rho\big).$$

For topological reasons, it will be convenient to restrict the previous definition to functions $G \in \mathscr{S}(\mathbb{R}^d)$, the Schwartz space of test functions, although $\mathscr{Y}_t^n(G)$ makes sense for more general test functions. The process $\mathscr{Y}_t^n$ defined in this way corresponds to a process on the Skorohod space $\mathscr{D}([0,\infty),\mathscr{S}'(\mathbb{R}^d))$, where $\mathscr{S}'(\mathbb{R}^d)$ is the space of distributions on $\mathbb{R}^d$. Now we are ready to state our main result:

Theorem 2.1. *Fix a particle density $\rho \in (0,\rho_c)$. In dimension $d > 2$, for almost all ω, the sequence of processes $\{\mathscr{Y}_t^n\}_n$ converges in the sense of finite-dimensional distributions to a Gaussian process $\mathscr{Y}_t$ with mean zero and covariance given by*

$$E[\mathscr{Y}_{t+s}(H)\mathscr{Y}_s(G)] = \chi(\rho)\theta(p) \int_{\mathbb{R}^d} G(y)S_tH(y)dy,$$

where we define $\chi(\rho) = \mathrm{Var}(\xi(0);\nu_\rho)$, S_t denotes the semigroup generated by the operator $\varphi'(\rho)D\Delta$, $s,t \geq 0$ and $G,H \in \mathscr{S}(\mathbb{R}^d)$ and D is the limiting variance of a symmetric random walk in $\mathscr{C}(\omega)$.

We call the process $\mathscr{Y}_t$ the *generalized Ornstein-Uhlenbeck* process of characteristics $\varphi'(\rho)D\Delta$ and $\sqrt{2\theta(p)\varphi(\rho)D}\nabla$. Notice that this theorem implies in particular a central limit theorem for the sequence $\{\mathscr{Y}_t^n(G)\}_n$ for any $G \in \mathscr{S}(\mathbb{R}^d)$.

3 The corrected fluctuation field

3.1 The corrected fluctuation field

The proof of Theorem 2.1 is based on Holley-Stroock's characterization of generalized Ornstein-Uhlenbeck processes [8]. For each $t \geq 0$, let $\mathscr{F}_t$ be the σ-algebra on $\mathscr{D}([0,\infty),\mathscr{S}'(\mathbb{R}^d))$ generated by the projections $\{\mathscr{Y}_s(H); s \leq t, H \in \mathscr{S}(\mathbb{R}^d)\}$. The process $\mathscr{Y}_t$ admits the following characterization:

Proposition 3.1. *There exists a unique process $\mathscr{Y}_t$ in $\mathscr{C}([0,\infty),\mathscr{S}'(\mathbb{R}^d))$ such that:*

i) For every function $G \in \mathscr{S}(\mathbb{R}^d)$,

$$M_t(G) = \mathscr{Y}_t(G) - \mathscr{Y}_0(G) - \varphi'(\rho) D \int_0^t \mathscr{Y}_s(\Delta G) ds$$

and

$$\big(M_t(G)\big)^2 - 2\theta(p)\varphi(\rho) D t \int_{\mathbb{R}^d} \big(\nabla G(x)\big)^2 dx$$

are $\mathscr{F}_t$-martingales.

ii) $\mathscr{Y}_0$ is a Gaussian field of mean zero and covariance given by

$$E\big[\mathscr{Y}_0(G)\mathscr{Y}_0(H)\big] = \theta(p)\chi(\rho)\int_{\mathbb{R}^d} G(x)H(x)dx,$$

where G,H are test functions in $\mathscr{S}(\mathbb{R}^d)$. The process $\mathscr{Y}_t$ is called the generalized Ornstein-Uhlenbeck process of mean zero and characteristics $\varphi'(\rho)D\Delta$ and $\sqrt{2\theta(p)\varphi(\rho)D}\nabla$.

Now the idea behind the proof of Theorem 2.1 is simple. We will prove that the sequence of processes $\{\mathscr{Y}_t^n\}_n$ is tight and that every limit point satisfies the martingale problem stated in Proposition 3.1. With these two elements in hand, we will be able to conclude Theorem 2.1.
By Dynkin's formula, for each $G \in \mathscr{S}(\mathbb{R}^d)$,

$$M_t^n(G) = \mathscr{Y}_t^n(G) - \mathscr{Y}_0^n(G) - \int_0^t n^2 L \mathscr{Y}_s^n(G) ds$$

is a martingale with respect to $\mathscr{F}_t^n = \sigma\{\xi_s^n; s \le t\}$. The quadratic variation of $M_t^n(G)$ is given by

$$\int_0^t n^2\{L\mathscr{Y}_s^n(G)^2 - 2\mathscr{Y}_s^n(G)L\mathscr{Y}_s^n(G)\}ds.$$

Notice that the second step (that is, that the limit points satisfy the martingale problem) requires to replace, in some sense, $n^2 L\mathscr{Y}_s^n(G)$ by $\varphi'(\rho)D\mathscr{Y}_s^n(\Delta G)$, and to replace $L\mathscr{Y}_s^n(G)^2 - 2\mathscr{Y}_s^n(G)L\mathscr{Y}_s^n(G)$ by $\theta(p)\varphi(\rho)D\int_{\mathbb{R}^d}\big(\nabla G(x)\big)^2 dx$. Let us take a more careful look at these two terms. We start with the second one. Simple computations show that

$$n^2\Big\{L\mathscr{Y}_s^n(G)^2 - 2\mathscr{Y}_s^n(G)L\mathscr{Y}_s^n(G)\Big\} = \frac{1}{n^d}\sum_{\substack{x\in\mathscr{C}(\omega)\\ y:x\sim y}} g\big(\xi_s^n(x)\big)n^2\big(G(y/n)-G(x/n)\big)^2.$$

By the ergodic theorem, when integrated in time this quantity should converge to

$$2\theta(p)\varphi(\rho)t\int_{\mathbb{R}^d}\big(\nabla G(x)\big)^2 dx.$$

Notice that we have missed the factor D in the previous computation. We will return to this point later. Now let us take a look at the first term:

$$n^2 L\mathscr{Y}_s^n(G) = \frac{1}{n^{d/2}}\sum_{x\in\mathscr{C}(\omega)}\big(g(\xi_s^n(x)) - \varphi(\rho)\big)\mathscr{L}_n G(x/n),$$

where $g(\cdot)$ is the interaction rate and $\mathcal{L}_n$ is the generator of the associated symmetric random walk in $\mathscr{C}(\omega)$. For a function $F:\mathbb{R}^d \to \mathbb{R}$ and $x \in \mathscr{C}(\omega)$, $\mathcal{L}_n F(x/n)$ is defined by

$$\mathcal{L}_n F(x/n) = n^2 \sum_{\substack{y \in \mathscr{C}(\omega) \\ y \sim x}} F(y/n) - F(x/n).$$

The so-called *Boltzmann-Gibbs principle* guarantees the replacement, when integrated in time, of the expression $g(\xi_s^n(x)) - \varphi(\rho)$ by $\varphi'(\rho)\big(\xi_s^n(x) - \rho\big)$ in the sum above, allowing us to write $n^2 L\mathscr{Y}_t^n(G)$ as $\varphi'(\rho)\mathscr{Y}_t(\mathcal{L}_n G)$ plus a term that is negligible as $n \to \infty$. Now we can see what the problem is. Assume that $\{\mathscr{Y}_t^n\}_n$ is tight. Take a limit point $\mathscr{Y}_t^\infty$ of $\{\mathscr{Y}_t^n\}_n$. We want to say that $\mathscr{Y}_t^n(\mathcal{L}_n G)$ converges to $\mathscr{Y}_t^\infty(D\Delta G)$ along the corresponding subsequence. But $\mathscr{Y}_t^n$ converges to $\mathscr{Y}_t^\infty$ only in a weak sense. Therefore, we should need strong convergence of $\mathcal{L}_n G$, which is easily checked not to hold.

The way to overcome this problem is to use the *compensated compactness lemma*. The idea is to choose, for each n, a test function G_n in such a way that $\mathcal{L}_n G_n$ converges strongly to $D\Delta G$ as $n \to \infty$. We therefore define the *corrected fluctuation field* as in [6]. Fix some $\lambda > 0$. For each $G \in \mathscr{S}(\mathbb{R}^d)$, define $G_n : \mathscr{C}(\omega) \to \mathbb{R}^d$ as the solution of the *resolvent* equation

$$\lambda G_n(x) - \mathcal{L}_n G_n(x) = \lambda G(x/n) - D\Delta G(x/n). \tag{3.1}$$

Then the corrected fluctuation field $\mathscr{Y}_t^{n,\lambda}$ is given by

$$\mathscr{Y}_t^{n,\lambda}(G) = \frac{1}{n^{d/2}} \sum_{x \in \mathscr{C}(\omega)} \big(\xi_t^n(x) - \rho\big) G_n(x).$$

Notice that $\mathcal{L}_n G_n = \lambda(G_n - G) + D\Delta G$. In particular, strong convergence of $\mathcal{L}_n G_n$ to $D\Delta G$ follows from strong convergence of G_n to G. The following proposition tells us that this is, indeed, the case.

Proposition 3.2 (Faggionato [5]). *There is a set of P-total probability such that for any $G \in \mathscr{S}(\mathbb{R}^d)$, the sequence $\{G_n\}_n$ converges to G in the following strong sense:*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{x \in \mathscr{C}(\omega)} \Big| G_n(x) - G(x/n) \Big|^2 = 0.$$

In particular, since the invariant measure ν_ρ is of product form, we conclude that $\mathscr{Y}_t^{n,\lambda}(G_n) - \mathscr{Y}_t^n(G)$ vanishes in $\mathscr{L}^2(\mathbb{P}_n)$ as $n \to \infty$. We will see that the standard scheme, tightness plus uniqueness of limit points via martingale characterization, can be accomplished for $\mathscr{Y}_t^{n,\lambda}$.

3.2 The martingale problem for the corrected fluctuation field

When considering the corrected fluctuation field $\mathscr{Y}_t^{n,\lambda}$ instead of $\mathscr{Y}_t^n$, the martingale representation gives

$$M_t^{n,\lambda}(G) = \mathscr{Y}_t^{n,\lambda}(G) - \mathscr{Y}_0^{n,\lambda}(G) - \int_0^t \Theta_s^n(\lambda(G_n - G) + D\Delta G) ds.$$

In this formula, we have defined

$$\Theta_t^n(F) = \frac{1}{n^{d/2}} \sum_{x \in \mathscr{C}(\omega)} \big(g(\xi_t^n(x)) - \varphi(\rho)\big) F(x/n).$$

The quadratic variation of $M_t^{n,\lambda}$ is given by

$$\langle M_t^{n,\lambda} \rangle = \int_0^t \frac{1}{n^d} \sum_{\substack{x \in \mathscr{C}(\omega) \\ y: x \sim y}} n^2 g\big(\xi_s^n(x)\big)\big(G_n(y) - G_n(x)\big)^2 ds.$$

Multiplying the resolvent equation (3.1) by $\frac{1}{n^d} G_n(x)$ and summing over $x \in \mathscr{C}(\omega)$ we see that

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{\substack{x \in \mathscr{C}(\omega) \\ y: x \sim y}} \frac{n^2}{2}\big(G_n(y) - G_n(x)\big)^2 = \theta(p) D \int_{\mathbb{R}^d} \big(\nabla G(x)\big)^2 dx. \tag{3.2}$$

Let us denote by Q_n^λ the distribution of the process $\mathscr{Y}_t^{n,\lambda}$ in $\mathscr{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$. Following [6], it is not hard to prove that the sequence $\{Q_n^\lambda\}_n$ is tight. In the same way, for $d \geq 3$, we can prove that

$$\lim_{n\to\infty} \langle M_t^{n,\lambda} \rangle = 2\varphi(\rho)\theta(p) D \int_{\mathbb{R}^d} \big(\nabla G(x)\big)^2 dx. \tag{3.3}$$

The arguments in [6], and only in dimension $d = 2$, require some regularity of G_n that is missing in our situation. This is the only point in this article where we require $d \geq 3$, otherwise the proofs do not depend on the dimension. Now we will state in a more precise way the Boltzmann-Gibbs principle.

Theorem 3.3 (Boltzmann-Gibbs principle). *For any $t > 0$ and any $F \in \mathscr{C}_c(\mathbb{R}^d)$,*

$$\lim_{n\to\infty} \mathbb{E}_n \Big[\Big(\int_0^t \{\Theta_s^n(F) - \varphi'(\rho)\mathscr{Y}_s^n(F)\} ds \Big)^2 \Big] = 0.$$

As in [6], tightness of $\{Q_n^\lambda\}_n$, Theorem 3.3 and expression (3.2) permit to conclude the following result:

Theorem 3.4. *Fix a particle density $\rho \in (0, \rho_c)$. In dimension $d > 2$, for almost all ω, the sequence of processes $\{\mathscr{Y}_t^{n,\lambda}\}_n$ converges in distribution with respect to the Skorohod topology in $\mathscr{D}([0,\infty), \mathscr{S}'(\mathbb{R}^d))$ to a generalized Ornstein-Uhlenbeck process $\mathscr{Y}_t$ of mean zero and characteristics $\varphi'(\rho) D \Delta$ and $\sqrt{2\theta(p)\varphi(\rho) D}\nabla$.*

Theorem 2.1 is an immediate consequence of this Theorem and Proposition 3.2.

4 The Boltzmann-Gibbs principle

In this section we prove Theorem 3.3. The main point that makes the proof of this theorem different from the proof in [6] is the lack of ellipticity: by construction, the jump rate between two neighboring sites x, y is equal to 0 when the bond $\langle xy \rangle$ is not open. In [6], due to the ellipticity condition we could compare the generator of the process with the generator of a zero-range process without the random environment, but with a slower jump rate. Since the Boltzmann-Gibbs principle holds for the latter process, it should hold for the former, since the dynamics is faster. In our case, we do not have a proper slower process to compare with.

Let us explain better the intuition behind Theorem 3.3. The idea is that non-conserved quantities fluctuate faster than conserved ones. Since the only conserved quantity for the zero-range process is the number of particles, it is reasonable that at the right scale, the only part of the fluctuation field Θ_t^n that is seen at a macroscopic level is its projection over the conservative field $\mathscr{Y}_t^n$. We can think that non-conserved quantities equilibrate locally, while conserved quantities need to be transported in order to equilibrate. In particular, we will see that the form of the graph is not really important in Theorem 3.3. What is really important is the connectivity of the graph: if a graph has more than one connected component, then there is more than one conserved quantity: the number of particles on each connected component.

For $n \in \mathbb{N}_0$, define $\mathscr{V}(n) = g(n) - \varphi(\rho) - \varphi'(\rho)(n-\rho)$. The strategy of the proof of Theorem 3.3 is the following. Fix a positive integer k. Divide the support of the test function F into small boxes of side k/n. Since F is continuous, we can average the function $\mathscr{V}(\xi(x))$ over the corresponding boxes on the lattice. Then we use some sort of ergodic theorem to reduce the sum over many blocks integrated in time, into a sum over a single block. This last problem is a static one. A new ergodicity argument, now with respect to the invariant measure ν_ρ is enough to conclude.

An important property of the lattice is that it is *locally connected*, in the sense that the restriction of the lattice to any box is still a connected graph. This is no longer true for the percolation cluster: the restriction of the percolation cluster to a box is, in general, not connected. Although most of it belongs to a big single component, the rest is spread over many small connected components. Of course, all these connected components are connected by paths that pass outside the initial box. The point is that these paths can be chosen is such a way that they do not go too far from the original box. We will develop these ideas in the next paragraphs.

4.1 A connectivity lemma

In this section we will state a result that we call the *connectivity lemma* and we will prove Theorem 3.3 starting from it. The proof of this connectivity lemma is postponed to the next section.

Let k, l be fixed positive integers. We will send k and l to infinity after n. We introduce two intermediate scales in our problem as follows. For simplicity we assume $n/k \in \mathbb{N}$, and pasting a sufficient number of cubes, we can assume that the support of F is contained in the cube $(\delta, 1-\delta)^d$ for some $\delta > 0$. We can split the cube $\Lambda_n = \{1, 2, \dots, n\}^d$ into $(n/k)^d$ non-overlapping cubes of side k. Let $\{\bar{B}_j^0, j = 1, \dots, (n/k)^d\}$ be an enumeration of those cubes. Define $\bar{B}_j$ as the box of side $(2l+1)k$, centered at $\bar{B}_j^0$. In particular, the cubes $\bar{B}_j$ are the union of $(2l+1)^d$ cubes in $\{\bar{B}_i^0\}_i$, except for the ones near the border of Λ_n. Now define

$$B_j^0 = \bar{B}_j^0 \cap \mathscr{C}(\omega),$$

$$B_j = \{x \in \bar{B}_j; x \longleftrightarrow B_j^0 \ \ inside \ \ \bar{B}_j\}.$$

In other words, B_j^0 is the portion of the cluster $\mathscr{C}(\omega)$ inside the box $\bar{B}_j^0$, and B_j is the portion of the cluster $\mathscr{C}(\omega)$ inside $\bar{B}_j$ and connected to the box $\bar{B}_j^0$. We say that a cube $\bar{B}_j^0$ is *good* if B_j is connected. In other words, $\bar{B}_j^0$ is good if the connected components of B_j^0 are connected between them by paths that lie entirely in $\bar{B}_j$. We will denote by $\bar{\mathscr{B}}_n$ the union of good cubes, and by $\mathscr{B}_n$ the set $\bar{\mathscr{B}}_n \cap \mathscr{C}(\omega)$. The numbers k and l will be fixed most of the time. Therefore, in order to keep notation simple we do not make explicit the dependence of $\mathscr{B}_n$ in k and l. A cube that is not in $\mathscr{B}_n$ will be called *bad*. We will call B_j indistinctely the set of points already defined, and the subgraph of $\mathscr{C}(\omega)$ corresponding to these points.

Define $\mathscr{C}_n(\omega) = \mathscr{C}(\omega) \cap \Lambda_n$. Theorem 3.3 is a consequence of the following lemma:

Lemma 4.1 (Connectivity lemma)**.** *For each $\epsilon > 0$ there exists $l > 0$ such that*

$$i)\lim_{k\to\infty}\limsup_{n\to\infty}\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{x\in\mathscr{B}_n}\mathscr{V}(\xi_s^n(x))F(x/n)ds\Big)^2\Big]=0 \tag{4.1}$$

$$ii)\limsup_{k\to\infty}\limsup_{n\to\infty}\frac{|\mathscr{C}_n(\omega)\setminus\mathscr{B}_n|}{n^d}\le\epsilon, \tag{4.2}$$

where $|\mathscr{C}_n(\omega)\setminus\mathscr{B}_n|$ denotes the cardinality of the set $\mathscr{C}_n(\omega)\setminus\mathscr{B}_n$.

The first part of the lemma says, roughly speaking, that Theorem 3.3 is true if the considered graph is locally connected. The second part says that $\mathscr{C}(\omega)$ is not locally connected only on a small portion of the lattice.

Now let us prove Theorem 3.3 assuming Lemma 4.1. The expectation appearing in the statement of Theorem 3.3 is bounded by

$$2\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{x\in\mathscr{B}_n}\mathscr{V}(\xi_s^n(x))F(x/n)ds\Big)^2\Big] + 2\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{x\in\mathscr{C}_n(\omega)\setminus\mathscr{B}_n}\mathscr{V}(\xi_s^n(x))F(x/n)ds\Big)^2\Big].$$

The first expectation goes to 0 as $n\to\infty$ and then $k\to\infty$ by Lemma 4.1, part i). By Schwartz inequality, the second expectation is bounded by

$$\frac{t^2}{n^d}\sum_{x\in\mathscr{C}_n(\omega)\setminus\mathscr{B}_n}F(x/n)^2E_\rho[\mathscr{V}(\xi(x))^2],$$

which turns out to be bounded by $C(t,F,\rho)|\mathscr{C}_n(\omega)\setminus\mathscr{B}_n|/n^d$. This last expression vanishes as $n\to\infty$ and then $k\to\infty$ by Lemma 4.1, part ii), which proves Theorem 3.3.

4.2 Proof of the connectivity lemma, part i)

In order to simplify the notation, we also denote by $\mathscr{B}_n$ the set of indices j in $\{1,\dots,(n/k)^d\}$ for which $\bar{B}_j^0$ is a good cube. Let us start with part i). We will manipulate the term

$$\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{x\in\mathscr{B}_n}\mathscr{V}(\xi_s^n(x))F(x/n)ds\Big)^2\Big] \tag{4.3}$$

until we arrive to an expression that does not depend on t. Taking the positive and negative parts of F, we can assume, without loss of generality, that F is non-negative. For each j, take a point y_j in $\bar{B}_j^0$. Since the function F is uniformly continuous, we can rewrite the integrand in (4.3) as

$$\frac{1}{n^{d/2}}\sum_{j\in\mathscr{B}_n}F(y_j/n)\sum_{x\in B_j^0}\mathscr{V}(\xi_s^n(x)) \tag{4.4}$$

plus a rest that vanishes in $\mathscr{L}^2(\mathbb{P}_n)$ as $n\to\infty$ and then $k\to\infty$. Notice that now we have introduced an averaging of the function $\mathscr{V}(\xi(x))$ over boxes of side k.

Now we explain the point where we make use of the time average in (4.3). For each j, denote by L_{B_j} the restriction of the generator L to the set B_j. Observe that the zero-range process restricted to $\mathbb{N}_0^{B_j}$ is ergodic on the set of configurations with a fixed number of particles, exactly due to the fact that $\bar{B}_j^0$ is a good cube. For any two functions $f, h : \Omega \to \mathbb{R}$, denote by $\langle f, h\rangle_\rho$ the inner product with respect to ν_ρ. We define by

$$||f||_1^2 = \langle f, -Lf\rangle_\rho,$$
$$||f||_{-1}^2 = \sup_h \{2\langle f, h\rangle_\rho - ||h||_1^2\}$$

the Sobolev norms associated to L. Here the supremum is taken over functions $h \in \mathscr{L}^2(\nu_\rho)$. Take an arbitrary (by now) family of functions $\{f_j; j \in \mathscr{B}_n\}$ with $f_j : \mathbb{N}_0^{B_j} \to \mathbb{R}$. We have the following *Sobolev inequality* (Prop. A1.6.1 of [9]):

$$\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{j\in\mathscr{B}_n} F(y_j/n)L_{B_j}f_j(\xi_s^n)ds\Big)^2\Big] \le \frac{20t}{n^2}\Big|\Big|\frac{1}{n^{d/2}}\sum_{j\in\mathscr{B}_n} F(y_j/n)L_{B_j}f_j\Big|\Big|_{-1}^2.$$

Let us call $\mathscr{W}_n$ the term inside the norm. We will bound $||\mathscr{W}_n||_{-1}^2$ using the variational formula of $||\cdot||_{-1}^2$ introduced above. Then, we need to estimate $\langle h, \mathscr{W}_n\rangle_\rho$ for h given. By the weighted Schwartz inequality,

$$\int hL_{B_j}f_j d\nu_\rho \le \frac{1}{2\gamma_j}\langle f_j, -L_{B_j}f_j\rangle_\rho + \frac{\gamma_j}{2}\langle h, -L_{B_j}h\rangle_\rho.$$

Choose $\gamma_j = n^{d/2}/F(y_j/n)$ and observe that $\sum_j \langle h, -L_{B_j}h\rangle_\rho \le \langle h, -Lh\rangle_\rho$. Plug this estimate into the variational formula for $||\mathscr{W}_n||_{-1}^2$ to discover that

$$\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{j\in\mathscr{B}_n} F(y_j/n)L_{B_j}f_j(\xi_s^n)ds\Big)^2\Big] \le \frac{20t}{n^{d+2}}\sum_{j\in\mathscr{B}_n} F(y_j/n)^2\langle f_j, -L_{B_j}f_j\rangle_\rho.$$

Notice that, for f_j fixed, the right-hand side of this inequality is of order $1/n^2$. Notice also that we can not take the same function f_j for different j, since the clusters B_j are different for each j. However, k and l are fixed. In particular, the possibilities for the cluster B_j are many, but finite. Let Ξ be the set of connected graphs contained in a box of side $(2l+1)k$. Let us denote by η a generic element (that is, a graph) in Ξ. Take now a family of functions $\{f^\eta; \eta \in \Xi\}$ and define $f_j = f^\eta$ if $B_j = \eta$. With this notation, we can bound the right-hand side of the previous inequality by

$$\frac{20t||F||_\infty^2}{n^{d+2}}\sum_{\eta\in\Xi} q^n(\eta)\langle f^\eta, -L^\eta f^\eta\rangle_\rho,$$

where $q^n(\eta)$ is the number of boxes in $\mathscr{B}_n$ with graph η, and L^η is the generator L restricted to the graph η. Since $q^n(\eta)/n^d$ is bounded, the functions $\{f^\eta\}_\eta$ are fixed and also Ξ is fixed, this last quantity goes to 0 as $n \to \infty$. In particular, this means that we can discount a sum of the form $n^{-d/2}\sum_j F(y_j/n)L_{B_j}f_j$ in (4.4). We can also take the infimum over the families $\{f^\eta\}_\eta$, although only after $n \to \infty$. In other words, we have reduced (4.3) to the verification of

$$\lim_{k\to\infty}\sup_{\eta\in\Xi}\inf_{\{f^\eta\}_\eta}\limsup_{n\to\infty}\mathbb{E}_n\Big[\Big(\int_0^t \frac{1}{n^{d/2}}\sum_{j\in\mathscr{B}_n} F(y_j/n)\sum_{x\in B_j^0}\{\mathscr{V}(\xi_s^n(x)) - L_{B_j}f_j(\xi_s^n)\}ds\Big)^2\Big].$$

The good point is that in this sum the time integral has already played its part, allowing the introduction of the functions f_j. Bound the expectation using Schwarz inequality twice, once to get rid of the time integral, and once more to make use of the product form of the measure ν_ρ. Notice that the boxes B_j at a distance greater than $(2l+1)k$ are independent. Then, the expectation in the previous limit is bounded by

$$\frac{t^2(2l+1)^d\|F\|_\infty^2}{n^d}\sum_{\eta\in\Xi}q^n(\eta)E_\rho\Big[\Big(\sum_{x\in\eta^*}\mathscr{V}(\xi(x))-L^\eta f^\eta\Big)^2\Big],$$

where η^* is the set of points in the intersection between η and the central box of side k. Remark that the sum above depends on n only through the number $q^n(\eta)/n^d$. By the ergodic theorem, $q^n(\eta)/(n/k)^d$ converges, as $n\to\infty$, to $q(\eta)$, which is the probability of a given subgraph B_j of $\mathscr{C}(\omega)$ be (good and) equal to η. Therefore, we are left to prove that

$$\lim_{k\to\infty}\sup_{\eta\in\Xi}\inf_{\{f^\eta\}_\eta}\frac{t^2(2l+1)^d\|F\|_\infty^2}{k^d}\sum_{\eta\in\Xi}q(\eta)E_\rho\Big[\Big(\sum_{x\in\eta^*}\mathscr{V}(\xi(x))-L^\eta f^\eta\Big)^2\Big]=0. \tag{4.5}$$

Here we remark that the whole construction we have done has as purpose to get a *connected* graph in the sum above. For each graph η fixed, and due to the ergodicity of the zero-range process on sets with fixed number of particles,

$$\inf_{\{f^\eta\}_\eta}E_\rho\Big[\Big(\sum_{x\in\eta}\mathscr{V}(\xi(x))-L^\eta f^\eta\Big)^2\Big]=E_\rho\Big[E_\rho\Big(\sum_{x\in\eta}\mathscr{V}(\xi(x))\Big|\xi^\eta\Big)^2\Big],$$

where ξ^η denotes the number of particles of the configuration ξ on the graph η. Recall the definition of $V(\xi(x))$ and note that by the equivalence of ensembles (see Sect. 11.1 of [9]) it holds that

$$\sup E_\rho\Big[E_\rho\Big(\sum_{x\in\eta}\mathscr{V}(\xi(x))\Big|\xi^\eta\Big)\Big]^2<+\infty,$$

where the supremum is over $k>0$ *and* $\eta\in\Xi$. Therefore, we have proved that the supremum on the left-hand side of expression (4.5) is bounded by

$$\frac{C(g)t^2(2l+1)^d\|F\|_\infty^2}{k^d}\sum_{\eta\in\Xi}q(\eta),$$

which goes to 0 as $k\to\infty$, since the sum in η is bounded by 1 (remember that $q(\eta)$ is a probability).

4.3 Proof of the connectivity lemma, part $ii)$

We will use a result of [2], which roughly states that the percolation cluster has good connectivity properties.

For any two points $x,y\in\mathscr{C}(\omega)$, we define the distance $D(x,y)$ as the length of the minimal path connecting x and y:

$$D(x,y)=\inf\{n;\exists x=x_0,\dots,x_n=y \text{ with } \langle x_{i-1}x_i\rangle \text{ open}\}.$$

We have the following result:

Proposition 4.2 (Antal-Pisztora [2]). *There exists a constant $\gamma = \gamma(p,d) \in [1,\infty)$ such that*

$$\limsup_{|z|\to\infty} \frac{1}{|z|} \log P\big(0 \longleftrightarrow z, D(0,z) > \gamma|z|\big) < 0,$$

where $|z| = \sum_{i=1}^{d} |z_i|$.

This Proposition is telling us that, with high probability, two points that belong to $\mathscr{C}(\omega)$, both inside a box of side k, are connected through a path contained on the box of side $(\gamma+\delta)k$, centered on the original box. This means that for $l > \gamma$, (4.2) should hold. Now we will make this point more precise.

Since all the norms in $\mathbb{R}^d$ are equivalent, changing the constant $\gamma(p,d)$ if needed we can assume that Proposition 4.2 also holds for the supremum norm. For simplicity, we assume k odd and we use the supremum norm instead of the sum norm. Fix $\epsilon > 0$. Let Γ_k^0 be the box of side k, centered at the origin. Define $\Gamma_k = \Gamma_k^0 \cap \mathscr{C}(\omega)$, the intersection of Γ_k^0 with the infinite cluster. By the ergodic theorem, $|\Gamma_k|/k^d \to \theta(p)$ $a.s.$ and therefore in probability. Fix k_0 such that $\Gamma_{k_0} \neq \emptyset$ with probability bigger that $1-\epsilon/2$. By Proposition 4.2, there is $\delta > 0$ such that

$$P(0 \longleftrightarrow z, D(0,z) > \gamma|z|) \leq \exp\{-\delta|z|\} \tag{4.6}$$

if $|z|$ is big enough. Taking a bigger k_0 if necessary, we can assume that this inequality holds for any z such that $|z| > k_0$. Fix $x \in \Gamma_{k_0}^0$ and consider for a moment the measure $P(\cdot|x \in \mathscr{C}(\omega))$, which is well defined since $P(x \in \mathscr{C}(\omega)) = \theta(p) > 0$. Fix $l > 0$. In analogy with our previous definitions, we say that the box Γ_k is good if each connected component of Γ_k is connected to x through a path contained in $\Gamma_{(2l+1)k}^0$. This is the case if and only if each point of $\mathscr{C}(\omega) \cap \partial\Gamma_k^0$ is connected to x by a path contained in $\Gamma_{(2l+1)k}^0$. When we do not fix a point $x \in \Gamma_{k_0}^0$, we say that Γ_k is good if any two connected components of Γ_k are connected by a path contained in $\Gamma_{(2l+1)k}^0$. For $k > k_0$, each one of the points in $\mathscr{C}(\omega) \cap \partial\Gamma_k^0$ are at distance at least $(k-k_0)/2$ and at most $(k+k_0)/2$ from x. By (4.6), the probability (with respect to $P(\cdot|x \in \mathscr{C}(\omega))$) of all these points being connected to x by a path contained in $\Gamma_{(2l+1)k}^0$ is bigger than

$$1 - 2dk^{d-1}\exp\{-\delta(k-k_0)/2\} \tag{4.7}$$

as soon as $l \geq \gamma(k+k_0)/(4k)$ and $(k-k_0)/2 > k_0$. This is the case if, for example, $l \geq \gamma/4$. The quantity in (4.7) goes to 1 as $k \to \infty$. Since this estimate is independent of the choice of site $x \in \Gamma_{k_0}^0$, we conclude that

$$\limsup_{k\to\infty} P(\Gamma_k \text{ is not good }) \leq \epsilon/2.$$

Notice that the definition of *goodness* of a given box is translation invariant. In particular, with the B_j's notation,

$$\limsup_{k\to\infty} P(B_j \text{ is not good }) \leq \epsilon/2.$$

By the ergodic theorem,

$$\lim_{n\to\infty} \frac{|\mathscr{C}_n(\omega) \setminus \mathscr{B}_n|}{n^d} = P(B_j \text{ is not good})$$

in probability and $a.s.$ We conclude that for $l \geq \gamma/4$,

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \frac{|\mathscr{C}_n(\omega) \setminus \mathscr{B}_n|}{n^d} \leq \frac{\epsilon}{2},$$

which ends the proof of (4.2).

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References

[1] E. Andjel. Invariant measures for the zero range processes. *Ann. Probab.*, 10(3): 525–547, 1982. MR0659526

[2] Peter Antal and Agoston Pisztora. On the chemical distance for supercritical Bernoulli percolation. *Ann. Probab.*, 24(2): 1036–1048, 1996. MR1404543

[3] Noam Berger and Marek Biskup. Quenched invariance principle for simple random walk on percolation clusters. *Probab. Theory Related Fields*, 137(1-2): 83–120, 2007. MR2278453

[4] Chih Chung Chang. Equilibrium fluctuations of gradient reversible particle systems. *Probab. Theory Related Fields*, 100(3): 269–283, 1994. MR1305583

[5] A. Faggionato. Random walks and exclusion processes among random conductances on random infinite clusters: homogenization and hydrodynamic limit . *Electronic Journal of Probability*, 12: 2217–2247, 2008. MR2469609

[6] P. Gonçalves and M. Jara. Scaling limit of gradient systems in random environment. *J. Stat. Phys.*, 131(4): 691–716, 2008. MR2398949

[7] Geoffrey Grimmett. *Percolation*, volume 321 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, second edition, 1999. MR1707339

[8] Richard A. Holley and Daniel W. Stroock. Generalized Ornstein-Uhlenbeck processes and infinite particle branching Brownian motions. *Publ. Res. Inst. Math. Sci.*, 14(3): 741–788, 1978. MR0527199

[9] Claude Kipnis and Claudio Landim. *Scaling limits of interacting particle systems*, volume 320 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. MR1707314

[10] P. Mathieu and A. Piatnitski. Quenched invariance principles for random walks on percolation clusters. *Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci.*, 463(2085): 2287–2307, 2007. MR2345229

[11] V. Sidoravicius and A. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. *Probab. Theory Related Fields*, 129: 219–244, 2004. MR2063376

[12] L. Tartar. Homogénéisation et compacité par compensation. In *Séminaire Goulaouic-Schwartz (1978/1979)*, pages Exp. No. 9, 9. École Polytech., Palaiseau, 1979. MR0557520