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DISAGGREGATION OF LONG MEMORY PROCESSES ON $\mathcal{C}^\infty$ CLASS

DIDIER DACUNHA-CASTELLE
Équipe de Probabilités et Statistiques, Université de Paris-Sud, 91405 Orsay CEDEX, France.
email: `didier.dacunha-castelle@math.u-psud.fr`

LISANDRO FERMíN [1]
Équipe de Probabilités et Statistiques, Université de Paris-Sud, 91405 Orsay CEDEX, France.
Departamento de Matemática, Universidad Central de Venezuela, Caracas 1020, Venezuela.
email: `fermin.lisandro@math.u-psud.fr`

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Abstract
We prove that a large set of long memory (LM) processes (including classical LM processes and all processes whose spectral densities have a countable number of singularities controlled by exponential functions) are obtained by an aggregation procedure involving short memory (SM) processes whose spectral densities are infinitely differentiable (C^∞). We show that the $\mathcal{C}^\infty$ class of spectral densities infinitely differentiable is the best class to get a general result for disaggregation of LM processes in SM processes, in the sense that the result given in $\mathcal{C}^\infty$ class cannot be improved by taking for instance analytic functions instead of indefinitely derivable functions.

1 Introduction

Let X be a stochastic second-order stationary process with covariance function γ and density spectral F. We define

$$\|\gamma\| = \sum_{k=0}^{\infty} |\gamma(k)|.$$

If $\|\gamma\| < \infty$, we say that the process is SM and if $\|\gamma\| = \infty$, we say that the process is LM. The long memory is generally associated to the singularities of the spectral density F.

The most important LM processes used in applications are obtained by a model of aggregation, see [2, 3, 5, 6, 8, 9, 11]. This is the case in finance, hydrology or communication networks.

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By aggregations, we mean the following procedure: let $Z^i = \{Z_t^i(Y^i(\omega), \omega'),\ t \in T\}$ be doubly stochastic elementary processes centered second-order stationary, whose structure is driven by $Y = \{Y^i,\ i \in \mathbb{N}\}$, an ergodic process with distribution ν. The aggregation of the elementary processes $\{Z^i\}$ is possible if there exists a summation-normalization procedure which converges in the following way: for every fixed trajectory of Y, let $X^N(Y) = \{X_t^N(Y),\ t \in T\}$ of partial aggregations of elementary processes $\{Z^i\}$, defined by

$$X_t^N(Y) = \frac{1}{\sqrt{B_N}} \sum_{i=1}^{N} Z_t^i(Y^i), \tag{1}$$

where B_N is a sequence of real numbers such that $B_N \to \infty$, and the sequence of partial aggregation $\{X^N(Y)\}$ converges $\nu - a.s.$ in distribution to a process X, independent of Y, which is called the aggregation of the elementary processes $\{Z^i\}$.

By disaggregation, we mean the inverse problem. Let X a given stochastic process. Is there a sequence of elementary process $\{Z^i\}$ driving an ergodic process Y and a sequence $\{B_N\}$ such that the sequence $\{X^N(Y)\}$ converges in distribution to X for almost all trajectory Y. Disaggregation existence can allow to consider X, for instance in a statistic mechanics context, as an observable macro-process resulting of a suitable mean of the $\{Z^i\}$, non necessarily observable micro-processes; and in some cases, disaggregation allows to understand how LM could be generated.

Of course the previous framework is too general. We investigate here only a very particular situations. We consider that X is a given centered stationary process with spectral density F. Let $\mathbb{G}$ be a class of centered second-order stationary processes, we denoted by $\mathcal{G}$ the class of the corresponding spectral densities. Then, there exists a disaggregation of X on $\mathbb{G}$ iff the sequence $\{X^N(Y)\}$ converges in distribution to X for almost all trajectory Y, with $Z^i(Y^i) \in \mathbb{G}$ and Y is an ergodic process with stationary distribution ν. We study in detail the following two cases:

1. The $\{Z^i(Y^i), i \in \mathbb{N}\}$ is a sequence of independent elementary processes. In this case, we prove in [2] that necessarily B_N is equivalent to $\sqrt{N}$.
2. The $\{Z^i(Y^i), i \in \mathbb{N}\}$ is a sequence of linear gaussian elementary processes with the same innovation process $\varepsilon = \{\varepsilon_t, t \in T\}$ independent of Y, then B_N is equivalent to N, [2].

Intermediate cases, with interactive elementary processes $\{Z^i\}$ can be also considered, see [2]. We do not detail here this situation.

Let μ be the common distribution of Y^i, i.e. $\nu = \mu^{\bigotimes \mathbb{N}}$. From [2], it can be seen that the existence of a disaggregation for the case 1 is equivalent to the existence of a representation of F as a mixture

$$F(\lambda) = \int g(\lambda, y) d\mu(y), \tag{2}$$

with $g(\lambda, y) \in \mathcal{G}\ \mu - a.s.$ and for the case 2 is equivalent to the existence of a representation of the spectral density as

$$F_1(\lambda) = \left| \int h(\lambda, y) d\mu(y) \right|^2, \tag{3}$$

with $g(\lambda, y) = |h(\lambda, y)|^2 \in \mathcal{G}\ \mu - a.s.$.

Remark 1 *When we consider interactive elementary processes the spectral density of the aggregated process is always a positive convex combination of the form $aF + bF_1$. So the results in these cases can be deduced from the two previous cases.*

Let $\mathcal{G}$ be a class of spectral densities and let $\mathcal{M}(\mathcal{G})$ denotes the set of mixtures, given by (2), of spectral densities $g(\lambda, y)$ belonging to $\mathcal{G}$. $\mathcal{M}_1(\mathcal{G})$ denotes the set of mixture, given by (3), of transfer functions $h(\lambda, y)$ such that $|h|^2 \in \mathcal{G}$. We say that a process with spectral density $F(\lambda)$ can be disaggregated into elementary processes with spectral densities in $\mathcal{G}$ iff $F \in \mathcal{M}(\mathcal{G}) \bigcup \mathcal{M}_1(\mathcal{G})$; equivalently by disaggregation in $\mathbb{G}$, we understand the existence of a representation of a given process as an aggregation of elementary processes with spectral densities belonging to $\mathcal{G}$. Under previous considerations, the disaggregation problem is then equivalent to the following question: when do we have $F \in \mathcal{M}(\mathcal{G})$ or $F \in \mathcal{M}_1(\mathcal{G})$ for F and $\mathcal{G}$ given?

What are the "interesting" choices of $\mathcal{G}$? Small classes $\mathcal{G}$ with exponentially decreasing covariances as $\mathcal{AR}(p)$ class of spectral densities of $AR(p)$ processes (p-order autoregressive processes), $\mathcal{OU}(p)$ class of spectral densities of $OU(p)$ processes, (p-order Ornstein-Uhlenbeck processes), or the class $\mathcal{A}$ of analytical spectral densities have nice analytical and practical properties (for instance for simulation). We show that disaggregation in these classes needs very specific properties of F, properties which are not related with singularities and LM; for instance the conditions for the existence of a disaggregation on $\mathbb{AR}(1)$ class, the set of $AR(1)$ processes, are related to specific algebraic properties of F, for instance to be a Mellin transform. A known example is the disaggregation of the $FARIMA(d)$ process on $\mathbb{AR}(1)$, see [9]. A more general development of the disaggregation on $\mathbb{AR}(p)$, the set of $AR(p)$ processes is given in [3] where we also show results of disaggregation, for continuous time processes on $\mathbb{OU}(p)$, the set of $OU(p)$ processes.

We address the disaggregation procedure in SM processes and on subclasses $\mathbb{G}$ for which the covariances decrease as fast as possible. We prove that a very large set of stationary processes, whose spectral densities have singularities of different kinds, can be disaggregated by involving processes whose spectral densities are in $\mathcal{C}^\infty$ (or in $\mathcal{C}^H$, with $H \in \mathbb{N}$). We get general results for $F \in C^H$, $H \leq \infty$, except in a countable set of singularities; we conjecture that we are close to a kind of necessary and sufficient condition for a density to be in $\mathcal{M}(\mathcal{C}^\infty)$. Classical LM processes are included in this set, for instance processes whose spectral densities F have a single singularity where F and its derivatives are explicitly controlled by functions with exponential growth.

In section 2, we study disaggregation using multiplicative kernels and elementary densities roughly of the form $g(\lambda, y) = F(\lambda)K(y\phi(\lambda))$, where $K \circ \phi$ is a kernel such that at each singularity λ_0 of F we have $F(\lambda_0)K(y\phi(\lambda_0)) = 0$ as well as for all its derivatives. The C^∞ behavior of $K \circ \phi$ drives the C^∞ behavior of $g(\lambda, y)$. For instance, if we consider the $\mathcal{AR}(1)$ class then $K(y) = \frac{1}{1+y}$ and $\phi(\lambda) = 1 - cos\lambda$.

In section 3, we study the following representation

$$F(\lambda) = \int F(\lambda)\phi(\lambda)K(y\phi(\lambda))dy,$$

where F is singular at $\lambda_0 = 0$ and $g(\lambda, y) = F(\lambda)\phi(\lambda)K(y\phi(\lambda)) \in \mathcal{C}^\infty$. Thus, under suitable conditions, we get $F \in \mathcal{M}(\mathcal{C}^\infty)$ for a large class of densities with singularities. The main theorem extends the previous ideas to F with a countable set Λ_s of singularities. If these singularities are controlled (by very weak conditions) then for $F \in C^H$ on Λ_s^c we have that $F \in \mathcal{M}(\mathcal{C}^H)$.

In section 4, we study extreme cases. We prove that $F \in \mathcal{M}(\mathcal{C})$ iff it is lower semi continuous (l.s.c.). We give then an example of LM process which cannot be disaggregated in SM processes. We should call such a situation as a "hard" long memory but we are in almost pathological situations. Then we detail the disaggregation on $\mathcal{A}$ class of analytical functions to show how $\mathcal{A}$ and $\mathcal{C}^\infty$ play a different role. Disaggregation on $\mathcal{C}^\infty$ is linked to very weak analytical properties much easier to be satisfied by F than the algebraic properties required by disaggregation on $\mathcal{A}$. We give examples of $\mathcal{C}^\infty$ functions that can not be disaggregated on the class $\mathcal{A}$. The boundary between $\mathcal{C}^\infty$ and $\mathcal{A}$ is very clear, a function in $\mathcal{C}^\infty$ which is zero at some frequency λ can be disaggregated in $\mathcal{A}$ iff it belongs to $\mathcal{A}$.

Finally, in section 5 we consider the disaggregation procedure for the case of non-independent elementary processes and present some other considerations.

2 Disaggregation using Multiplicative Kernels

Let $K(y)$ be a positive kernel, $y \in \mathbb{R}$, μ a bounded measure such that $K(\tau y) \in L^1(\mu)$ for every $\tau \in \mathbb{R}$. Let $\mathcal{G}$ be a given class of spectral densities, we suppose there exists a function $\phi(\lambda)$ defined on Λ, where $\Lambda = (-\pi, \pi]$ in the case of discrete process and $\Lambda = \mathbb{R}$ in the continuous case, such that $K(\phi(\lambda)y) \in \mathcal{G}$ for every $y \in \mathbb{R}$.

We say that F is disaggregated on $\mathcal{G}$ by means of the multiplicative kernel $K(\phi(\lambda)y)$ if

$$F(\lambda) = \int K(\phi(\lambda)y)d\mu(y),$$

with $K(\phi(\lambda)y) \in \mathcal{G}$, or if $\hat{\mu}$ is the K-transform, i.e. $\hat{\mu}(\tau) = \int K(\tau y)d\mu(y)$ then $F(\lambda) = \hat{\mu}(\phi(\lambda))$.

Example 1 *We consider the $\mathcal{AR}(1)$ class, then the K-transform is the Mellin transform*

$$\hat{\mu}(\tau) = \int \frac{d\mu(y)}{1+\tau y}, \qquad \textit{and} \qquad \phi(\lambda) = 1 - cos\lambda.$$

The density of the $AR(1)$ process with parameter ρ is $g(\lambda,\rho) = \frac{1}{1-2cos\lambda+\rho^2}$. Let $y = \frac{2\rho}{(1-\rho)^2}$, from where $g(\lambda,\rho(y)) = \frac{1}{(1-\rho(y))^2(1+y(1-cos\lambda))}$. Then $F \in \mathcal{AR}(1)$ is equivalent to

$$F(\lambda) = \int_{-1}^{1} \frac{d\nu(\rho)}{1-2\rho cos\lambda+\rho^2},$$

for some bounded measure ν. Let $\bar{\nu}$ be the image of ν by the application $y = \frac{2\rho}{(1-\rho)^2}$ and $d\mu(y) = \frac{d\bar{\nu}(y)}{(1-\rho(y))^2}$ so that we can rewrite F as

$$F(\lambda) = \int \frac{d\mu(y)}{1+y(1-cos\lambda)}.$$

Details on the integrability condition for $d\mu(y)$ are given in [3, 5] with the characterization of F, for instance the with Toeplitz and Hankel matrix properties. The disaggregation on $\mathbb{AR}(1)$ is unique when it exists.

Example 2 *Let $\mathcal{G}_{exp} = \{e^{-|\lambda|y}, y \in \mathbb{R}^+\}$, so if there exists $\phi(\lambda)$ such that $F(\lambda) = \hat{\mu}(\phi(\lambda))$, where $\hat{\mu}$ is the Laplace transform of the bounded measure μ, then $F \in \mathcal{M}(\mathcal{G}_{exp})$. In particular, if there exists ϕ a monotone positive function such that $F(\phi(\lambda))$ is a Laplace transform, then $F(\lambda)$ can be written as $\int e^{-\phi(\lambda)y} d\mu(y)$.*

Example 3 *This example will introduce the main theorem of this paper. We take $F \in C^\infty$ for every λ except at $\lambda = 0$. Let K a kernel infinitely differentiable such that $K^{(j)}(0) = 0$ for every $j \in \mathbb{N}^*$, $\int K(y)dy = 1$ and $K(y) = K(-y)$.*
Let $g^*(\lambda, y) = \phi(\lambda)K(\phi(\lambda)y)F(\lambda)$, then

$$F(\lambda) = \int g^*(\lambda, y)dy = \int \frac{g^*(\lambda, y)}{\sigma(y)}\sigma(y)dy,$$

for any $\sigma(y) \in L^1(dy)$ and $\sigma > 0$. We can extend the definition of mixtures of spectral densities, given in (2), taking an unbounded measure, as the Lebesgue measure dy. In this case we can consider that we take a strictly positive density of probability $\sigma(y)$ and $\mu(y)$ the respective probability distributions, i.e. $d\mu(y) = \sigma(y)dy$. Then we can rewrite F as the mixture of the spectral densities $g^*(\lambda, y)\sigma^{-1}(y)$ by the measure mixture μ. Thus, if $\phi(\lambda)K(\phi(\lambda)y)F(\lambda) \in \mathcal{C}^\infty$ then by taking $g(\lambda, y) = g^*(\lambda, y)\sigma^{-1}(y)$ we have that $F \in \mathcal{M}(\mathcal{C}^\infty)$.

Lemma 1 *$g(\lambda, y) \in \mathcal{C}^\infty$ is implied by the following conditions:*

1. *For every $j \in \mathbb{N}$, $F^{(j)}(\lambda) \leq C_j e^{\frac{1}{|\lambda|^q}}$ for $|\lambda| < \varepsilon$, $\lambda \neq 0$ and for some $0 < q < 1$. In this case we say that the singularity is exponentially controlled.*
2. *$\phi(\lambda) = \frac{1}{|\lambda|^p}$, with $0 < q < p < 1$.*

These conditions imply that $\lim_{\lambda \to 0} F^{(k)}(\lambda)\phi^{(l)}(\lambda)K^{(m)}(\phi(\lambda)y) = 0$, for every $(k, l, m) \in \mathbb{N}^3$ and this implies $g \in \mathcal{C}^\infty$.

3 Disaggregation of LM Processes on $\mathcal{C}^\infty$ Class

Long memory of a process with spectral density F is in general associated to singularities of F or of some of its derivatives at a frequency λ_0. Singularities are often classified as a first order when a one-side limit exists and second order singularity when the function has no limit at λ_0 or not limit at λ_0 exists (the function being bounded or not, with bounded variation or not, etc). We try to take into account most of these situations. Our main purpose is to obtain, for a class as broad as possible, including all classical examples but not limited to more or less explicit densities, a disaggregation on elementary processes with the best possible decay of correlations.

We are lead to work mainly with $\mathcal{G} = \mathcal{C}^\infty$ (resp. $\mathcal{C}^H$) the class of all spectral densities belonging to C^∞ (resp. C^H, for $H \in \mathbb{N}$). An equivalent property is that the covariance function γ, of $F \in \mathcal{C}^\infty$, is rapidly decreasing in the sense that $n^j|\gamma(n)|$ tend to 0 as $n \to \infty$, for every $j \in \mathbb{N}$, see ([4], p. 34). If $F \in \mathcal{C}^H$ then $n^j|\gamma(n)|$ tend to 0 as $n \to \infty$, for every $j \leq H$, in this case the reciprocal is false, nevertheless we can prove that $F \in \mathcal{C}^{H-2}$. For $1 \leq H \leq \infty$ given and for a function $F \in C^H$ except for a finite or countable set of frequencies, we get a disaggregation on $\mathcal{C}^H$ class. The disaggregation on the class $\mathcal{C}^\infty$ is easier to reach than a disaggregation on the SM class.

The next definition extends the Example 3 given in the section 2.

Definition 1 *Let $\Lambda_s \subset \Lambda$ be a finite set of frequencies and $1 \leq H \leq \infty$, we define $\mathcal{G}^H_{\Lambda_s}$ as the set of spectral densities in $\mathcal{C}^H$ which are 0 on Λ_s as are all their H derivatives. If $\Lambda_s = \{0\}$ we denote $\mathcal{G}^H_{\Lambda_s}$ by $\mathcal{G}^H_0$.*

Let K a kernel infinitely differentiable such that $\int K(y)dy = 1$ and $K(y) = K(-y)$ and let ϕ be a positive function on Λ. Let $g(\lambda, y) = \phi(\lambda)K(\phi(\lambda)y)F(\lambda)\sigma^{-1}(y)$, where $\sigma(y)$ is a strictly positive density of probability and $d\mu(y) = \sigma(y)dy$, then $F(\lambda) = \int \frac{g(\lambda,y)}{d}\mu(y)$. The pair $[K;\phi]$ should be called a killer kernel, as it annihilates the singularities of f. The generic situation is given by the formula $F(\lambda_0)\phi(\lambda_0)K(\phi(\lambda_0)y)\sigma^{-1}(y) = \lim_{\lambda\to\lambda_0} F(\lambda)\phi(\lambda)K(\phi(\lambda)y)\sigma^{-1}(y) = 0$ for every $y \in \mathbb{R}$, even if $F(\lambda_0) = \infty$. Therefore, F is the μ-mixture of $\{g(\lambda, y)\} \subset \mathcal{G}^H_{\Lambda_s}$.

Let us give some examples of disaggregation of a function F with a single singularity at λ_0. We consider the standard situation where F and its derivatives are explicitly controlled by functions with exponential growth.

Example 4 *Let $F(\lambda) = 1_{(-\lambda_0,\lambda_0)}(\lambda)$ for $\lambda_0 \in \Lambda$, $\phi(\lambda) = 1/|\lambda^2 - \lambda_0^2|^p$ with $0 < p < 1$, and $K(y) = e^{-y}$. In this case it is straight forward to check that $g(\lambda, y) \in \mathcal{G}^\infty_{\{-\lambda_0,\lambda_0\}}$, since all derivatives of g are 0 for $|\lambda| = \lambda_0$.*

Example 5 *Let $F(\lambda) = |1-\cos(\lambda-\lambda_0)|^{-d}$, $0 < d < 1$, $\lambda \in \Lambda = (-\pi, \pi]$. We keep $K(y) = e^{-y}$ and $\phi(\lambda) = 1/|\lambda - \lambda_0|^p$ with $0 < p < 1$. All derivatives of F at $\lambda = \lambda_0$ are controlled by a negative power of $|\lambda - \lambda_0|$ and so $g(\lambda, y) \in \mathcal{G}^H_{\lambda_0}$. The same properties can be easily checked for a strongly oscillating function F as $\cos(\pi(\lambda - \lambda_0))/|\lambda - \lambda_0|^q$, for $0 < q < 1$. thus for these kinds of controlled singularities, we show that $F \in \mathcal{M}(\mathcal{G}^H_{\lambda_0}) \subset \mathcal{M}(\mathcal{C}^\infty)$.*

Definition 2 *Let F be a spectral density. We say that $F \in S^H$, $1 \leq H \leq \infty$, if F has a continuous H derivative at every frequency except for a finite set $\Lambda_s = \{\lambda_j, j \in J\}$ and if there exists q, $0 < q < 1$, and a, $0 < a < 1$, such that for all $j \in J$ and for all $l \leq H$*

$$\lim_{\lambda\to\lambda_j} \exp\left(-\frac{a}{|\lambda - \lambda_j|^q}\right) |F^{(l)}(\lambda)| = 0.$$

If Λ_s is a countable infinite set instead of finite and has only a finite number of accumulation points, then we say that $F \in T^H$.

We state now a theorem for a general situation.

Theorem 1 *Let $F \in S^H$, $1 \leq H \leq \infty$, then $F \in \mathcal{M}(\mathcal{G}^H_{\Lambda_s}) \subset \mathcal{M}(\mathcal{C}^H)$.*

Proof. Let $\phi(\lambda) = \prod_{j\in J} |\lambda - \lambda_j|^{-p}$, $K(y) = e^{-y}$ and $g(\lambda, y) = F(\lambda)\phi(\lambda)K(\phi(\lambda)y)\sigma^{-1}(y)$, where $\sigma(y)$ is a strictly positive density of probability. Then $F(\lambda) = \int g(\lambda, y)d\mu(y)$ with $d\mu(y) = \sigma(y)dy$. We choose p such that $0 < q < p < 1$. If $\Psi(\lambda, y) = \phi(\lambda)K(\phi(\lambda)y)$ then for all the l-derivatives of g, $l \leq H$, we show that there exist constants b_l, C_l and m_l such that

$$\left|g^{(l)}(\lambda, y)\right| = \left|\sigma^{-1}(y) \sum_{k=0}^{l} C_{k,l}F^{(k)}(\lambda)\Psi^{(l-k)}(\lambda, y)\right| \leq C_l|\sigma^{-1}(y)||\Psi(\lambda, y)|^{m_l} e^{\left(-\sum_{j\in J} \frac{ab_l}{|\lambda-\lambda_j|^q}\right)}.$$

So $g(\lambda, y) \in \mathcal{G}^H_{\Lambda_s}$ and $F \in \mathcal{M}(\mathcal{G}^H_{\Lambda_s}) \subset \mathcal{M}(\mathcal{C}^H)$. □

Example 6 *If $F(\lambda) = |\lambda|^{-d}$, $\lambda \in \Lambda = \mathbb{R}$, $-1 < d < 1$, F is the spectral density of continuous fractional Gaussian noise. Then $|F^{(l)}(\lambda)| = O(1/|\lambda|^{d+l})$ and the conditions of Theorem 1 are satisfied.*

The next theorem is an extension of the previous one. Its proof follows the same steps of Theorem 1 but uses a localization required when Λ_s is a countable infinite set.

Theorem 2 *Theorem 1 remains valid if $F \in T^H$.*

Proof. Suppose, in order to simplify notations, that Λ_s is an infinite countable set with only one accumulation point. The general case can be easily obtained by re-indexing Λ_s points using the partition of Λ defined by the points of accumulation of Λ_s and then applying the same proof. We can thus suppose $\Lambda_s = \{\lambda_j, j \geq 1\}$, with $\lambda_j < \lambda_{j+1}$ for every $j \in \mathbb{N}$.

We build in the same way as previously a family of functions $g^{\Lambda_s}(\lambda, y)$, multiplying F by a killer kernel $[K; \phi^{\Lambda_s}](\lambda, y)$ that annihilates the points of discontinuity of F.

Let us note $a = \inf \Lambda_s$, $b = \sup \Lambda_s$ and $\lambda_\infty = \lim_{j\to\infty} \lambda_j = \sup_j \lambda_j$. Let be p such that $0 < q < p < 1$, and we consider

$$\begin{aligned}
[K, \phi^0](\lambda, y) &= \frac{1}{|\lambda - \lambda_1|^p} \exp\left(-\frac{y}{|\lambda - \lambda_1|^p}\right) 1_{(a,\lambda_1)}(\lambda).\\
[K, \phi^j](\lambda, y) &= \frac{1}{|\lambda - \lambda_j|^p |\lambda - \lambda_{j+1}|^p} \exp\left(-\frac{y}{|\lambda - \lambda_j|^p |\lambda - \lambda_{j+1}|^p}\right) 1_{(\lambda_j,\lambda_{j+1})}(\lambda).\\
[K, \phi^\infty](\lambda, y) &= \frac{1}{|\lambda - \lambda_\infty|^p} \exp\left(-\frac{y}{|\lambda - \lambda_\infty|^p}\right) 1_{(\lambda_\infty,b)}(\lambda).
\end{aligned}$$

Then we define

$$[K; \phi^{\Lambda_s}](\lambda, y) = \sum_{j=0}^{\infty} [K; \phi^j](\lambda, y),$$

and

$$g^{\Lambda_s}(\lambda, y) = F(\lambda)[K; \phi^{\Lambda_s}](\lambda, y)\sigma^{-1}(y).$$

We have that

$$\int [K; \phi^{\Lambda_s}](\lambda, y) dy = \int e^{-z} dz \left(1_{(a,\lambda_1)} + \sum_{j\geq 1} 1_{(\lambda_j,\lambda_{j+1})} + 1_{(\lambda_\infty,b)} \right) = 1.$$

So $V(y) = \int_\Lambda g^{\Lambda_s}(\lambda, y) d\lambda$ and by applying Fubini's theorem $\int V(y) d\mu(y) = \int_\Lambda F(\lambda) d\lambda < \infty$. We can prove, by using the same proof as for Theorem 1, that the H derivatives with respect to λ of $g^{\Lambda_s}(\lambda, y)$ converge to 0 when $\lambda \to \lambda_j$, since $q < p$. So $g^{\Lambda_s}(\lambda, y) \in \mathcal{G}^H_{\Lambda_s}$. $\square$

Remark 2 *Killer kernels $[K; \phi]$ selected to build only the mixtures are never the best ones for covariances decay. For instance, we can take* $\exp(-\exp(y))$ *instead of* $\exp(-y)$ *getting covariances decreasing to* 0 *slightly faster and so on.*

4 Condition for the Existence of Disaggregation

4.1 Continuous Densities Class

Let us begin with some remarks about mixtures and LM property giving rough necessary conditions for the existence of a disaggregation on SM densities. We use only the fact that every SM density is continuous.

Lemma 2 *[Characterization of $\mathcal{M}(\mathcal{C})$] A spectral density is a mixture of continuous spectral densities iff it is lower semicontinuous (l.s.c.).*

Proof. If $F(\lambda) = \int g(\lambda, y) d\mu(y)$ where $\{g(\lambda, y)\}$ is a family of continuous spectral densities, then applying Fatou's lemma we see that F is l.s.c.

Conversely, if F is l.s.c. and positive, there exists a sequence f_n of continuous functions such that $F = \sup_n f_n$. Taking $F_n = \sup_{1 \le k \le n} f_k$, we can choose an increasing sequence of continuous functions $\{F_n\}$ such that $F = \sup_n F_n$. So $F = \sum_{n \ge 1} (F_{n+1} - F_n))$. If we take $\mu(y) = \sum_{n \ge 1} \frac{1}{2^n} \delta(y - n)$ and $g(\lambda, n) = 2^n \left(F_{n+1}(\lambda) - F_n(\lambda) \right)$ then we can rewrite F as $F(\lambda) = \int g(\lambda, y) d\mu(y)$ where $\{g(\lambda, y)\}$ is a family of continuous spectral densities. □

From the lemma, we see that every non l.s.c. spectral density has LM property and cannot be disaggregated on SM class. For instance, if F is the function equal to one on a perfect set, [7], of strictly positive Lebesgue measure, then F is upper semicontinuous (u.s.c.) and not l.s.c. and it is the density of an absolutely continuous probability with respect to the Lebesgue measure. This provides an example of a situation that we can call "hard" LM process which cannot be disaggregated by SM processes. In fact, we are very close to the case of a non absolutely continuous spectral measure.

4.2 Analytic Spectral Densities Class

Let us prove that the previous results cannot be improved by taking analytic densities instead of infinitely differentiable densities.

Disaggregation is a hierarchical procedure: if $F \in \mathcal{M}(\mathcal{G})$ and $\mathcal{G} \subset \mathcal{M}(\mathcal{H})$ then $F \in \mathcal{M}(\mathcal{H})$, in fact if $g(\lambda, y) = \int h(\lambda, z) d\nu(y, z)$ then

$$F(\lambda) = \int g(\lambda, y) d\mu(y) = \int \int [h(\lambda, y, z) d\nu(y, z)] d\mu(y).$$

In general we have $\mathcal{G} \subset \mathcal{M}(\mathcal{G}) \subset \overline{\mathcal{M}(\mathcal{G})}$ with strict inclusion, the closure being taken in $L^1(d\lambda)$. The obvious exception is $\mathcal{G} = \mathcal{MA}(q)$, the set of densities of q-moving average processes, for which $\mathcal{MA}(q) = \mathcal{M}(\mathcal{MA}(q)) = \overline{\mathcal{M}(\mathcal{MA}(q))}$.

We use this hierarchical procedure in order to show that our result cannot be improved in the following sense: we cannot take analytic functions instead of $\mathcal{C}^\infty$. Therefore, we have to check that the functions we have used in $\mathcal{C}^\infty$, as $F(\lambda) = \frac{1}{|\lambda|^p} \exp(-\frac{y}{|\lambda|^p})$, do not belong to $\mathcal{M}(\mathcal{A})$, in order to show that our result cannot be improved.

Proposition 1 *If $F \in \mathcal{C}^\infty_{0,+} = \left\{ F \in \mathcal{C}^\infty : F \ge 0, \exists\, \lambda_0 \text{ such that } F^{(j)}(\lambda_0) = 0 \,\forall\, j \in \mathbb{N} \right\}$, then $F \notin \mathcal{M}(\mathcal{A})$.*

PROOF. Suppose $F(\lambda) = \int_{\mathbb{R}^s} g(\lambda, y)d\mu(y)$ with $g \in \mathcal{A}$ $\mu - a.s.$ The Fatou's Lemma implies

$$F^{(j)}(\lambda_0) \geq \int_{\mathbb{R}^s} g^{(j)}(\lambda_0, y)d\mu(y),$$

and $F^{(j)}(\lambda_0) = 0$ implies that, if $g^{(j)}(\lambda_0, y) \geq 0$ $\mu - a.s.$, then $g^{(j)}(\lambda_0, y) = 0$ $\mu - a.s.$ From $g(\lambda, y) \geq 0$ $\mu - a.s$ we get $g(\lambda_0, y) = 0$ $\mu - a.s.$ and $g^{(1)}(\lambda_0, y) \geq 0$ $\mu - a.s.$, and so $g^{(1)}(\lambda_0, y) = 0$ $\mu - a.s$ and $g^{(2)}(\lambda_0, y) \geq 0$ $\mu - a.s.$ By induction we have that $g^{(j)}(\lambda_0, y) = 0$ $\mu - a.s$ and $\mu\{y, g(\lambda, y) \in \mathcal{A}\} = 0$. We have proved that $f \notin \mathcal{M}(\mathcal{A})$. □

The class of analytic functions only rarely allows for disaggregation. A slight modification of the proof presented above shows that spectral densities which are polynomials (of given degree) by pieces cannot be in $\mathcal{M}(\mathcal{A})$ except if they are themselves elements of $\mathcal{A}$, that is, if they are polynomials.

5 Case of Non Independent Elementary Processes

In this section we consider the disaggregation procedure for the case of non-independent elementary processes. We suppose that

$$Z_t^i(Y^i) = \int_\Lambda h(\lambda, y)e^{it\lambda}d\zeta(\lambda)$$

where $\{\zeta_\lambda\}$ is a second order stationary process with orthogonal increment, $h \in L^2(d\lambda)$ is a square root of the spectral density of $Z^i(Y^i)$ and $Y = \{Y^i\}$ is an ergodic process with distribution ν. We prove in [2] that the sequence of partial aggregation

$$X_t^N(Y) = \frac{1}{N}\sum_{i=1}^{N} Z_t^i(Y^i),$$

converges $\nu - a.s.$ in distribution to centered stationary gaussian process X with spectral density $F(\lambda) = |H(\lambda)|^2$, where

$$H(\lambda) = \int h(\lambda, y)d\mu(y). \tag{4}$$

Now let $F = |H|^2$ and $\mathcal{G}$ be given class, then we want to have (4) with $|h(\lambda, y)|^2 \in \mathcal{G}$ $\mu - a.s.$ Singularities of F give singularities on H and it is not very difficult to prove that the main theorems remain valid taking dependent elementary processes instead of independent processes. We can also deduce the results for interactive elementary processes from the two previous cases, see [2].

Remark 3 *We can define mixtures of Wold regular densities, in the Wold Theorem sense, [1], which verify the following condition*

$$\int_\Lambda \log g(\lambda, y)d\lambda > -\infty, \quad \mu - a.s. \tag{5}$$

In this case, we say that the processes with spectral densities $g(\lambda, y)$ are regular. If $f(\lambda)$ is the mixture of the densities $g(\lambda, y)$, condition (5) does not imply that $\int_\Lambda \log f(\lambda)d\lambda > -\infty$. But if f is regular then $g(\lambda, y)$ is regular $\mu - a.s.$ by Jensen's inequality. The main point of this topic is that we can choose the killer kernel $[K; \phi]$ such that if f is Wold regular then all the elementary processes used in the aggregation are also Wold regular.

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References

[1] Azencott R. and Dacunha-Castelle D. *Series of irregular observations.* Springer Verlag, New York. (1986).

[2] Dacunha-Castelle D. and Fermín L. *Aggregations of Doubly Stochastic Interactive Gaussian Processes and Toeplitz forms of U-Statistics.* Dependence in Probability and Statistics, Series: Lecture Notes in Statistics, Vol. 187 . (2006).

[3] Dacunha-Castelle D. and Fermín L. *Aggregations of Ornstein-Uhlenbeck processes or AR processes and long memory.* Preprint Univ. Paris-Sud. (2005).

[4] Dym H. and McKean H.P. *Fourier Series and Integrals* Academic Press New York and London. (1972).

[5] Gonçalvez E. and Gourieroux C. *Agrégations de processus autorégressives d'ordre 1.* Annales d'Economie et de Statistique **12** (1988), 127-149.

[6] Granger C.W.J *Long Memory relationships and the aggregation of dinamic models.* Journal of Econometrics**14** (1980), 227-238.

[7] Kahane J.P. *Emsembles parfaits et séries trigonométriques*. Herman. (1994).

[8] Linden M. *Time series properties of aggregated AR(1) processes with uniformly distributed coefficients.* Economics Letters **64** (1999), 31-36.

[9] Lippi M. and Zaffaroni P. *Aggregation of simple linear dynamics: exact asymptotic results.* Econometrics Discussion Paper 350, STICERD-LSE. (1998).

[10] Oppenheim G. and Viano M.-C. *Aggregation of ramdom parameters Ornstein-Uhlenbeck or AR processes: some convergence results.* Journal of Time Series Analysis **25 (3)** (2004), 335-350.

[11] Terence, T.C and Kwan-to W. *Time series properties of aggregated AR(2) processes.* Economics Letters **73** (2001), 325-332.