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INFINITE DIVISIBILITY OF GAUSSIAN SQUARES WITH NON–ZERO MEANS

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Abstract
Let $\eta = (\eta_1, \ldots, \eta_n)$ be an R^n valued Gaussian random variable and $c = (c_1, \ldots, c_n)$ a vector in R^n. We give necessary and sufficient conditions for $((\eta_1 + c_1\alpha)^2, \ldots, (\eta_n + c_n\alpha)^2)$ to be infinitely divisible for all $\alpha \in R^1$, and point out how this result is related to local times of Markov chains determined by the covariance matrix of η.

1 Introduction

Let $\eta = (\eta_1, \ldots, \eta_n)$ be an R^n valued Gaussian random variable. η is said to have infinitely divisible squares if $\eta^2 := (\eta_1^2, \ldots, \eta_n^2)$ is infinitely divisible, i.e. for any r we can find an R^n valued random vector Z_r such that

$$\eta^2 \stackrel{law}{=} Z_{r,1} + \cdots + Z_{r,r}, \tag{1.1}$$

where $\{Z_{r,j}\}$, $j = 1, \ldots, r$ are independent identically distributed copies of Z_r. We express this by saying that η^2 is infinitely divisible.
We are interested in characterizing Gaussian processes with infinitely divisible squares which do not have mean zero. We set $\eta_i = G_i + c_i$, $EG_i = 0$, $i = 1, , \ldots, n$. Let Γ be the covariance matrix of $(G_1, \ldots, G_n)$ and set

$$c := (c_1, \ldots, c_n). \tag{1.2}$$

Set

$$G + c := (G_1 + c_1, \ldots, G_n + c_n) \tag{1.3}$$

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and

$$(G+c)^2 := ((G_1+c_1)^2, \dots, (G_n+c_n)^2). \tag{1.4}$$

In order to continue we need to define different types of matrices. Let $A = \{a_{i,j}\}_{1\le i,j\le n}$ be an $n \times n$ matrix. We call A a positive matrix and write $A \ge 0$ if $a_{i,j} \ge 0$ for all i,j. We write $A > 0$ if $a_{i,j} > 0$ for all i,j. We say that A has positive row sums if $\sum_{j=1}^n a_{i,j} \ge 0$ for all $1 \le i \le n$.

The matrix A is said to be an M matrix if

(1) $a_{i,j} \le 0$ for all $i \ne j$.

(2) A is nonsingular and $A^{-1} \ge 0$.

This definition includes the trivial 1×1 M matrix with $a_{1,1} > 0$.

A matrix is called a signature matrix if its off diagonal entries are all zero and its diagonal entries are either one or minus one.

Our starting point is a theorem by Griffiths and Bapat [1, 7], (see also [8, Theorem 13.2.1]) that characterizes mean zero Gaussian vectors with infinitely divisible squares.

Theorem 1.1 (Griffiths, Bapat)**.** *Let $G = (G_1, \dots, G_n)$ be a mean zero Gaussian random variable with strictly positive definite covariance matrix $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$. Then G^2 is infinitely divisible if and only if there exists a signature matrix $\mathcal{N}$ such that*

$$\mathcal{N}\Gamma^{-1}\mathcal{N} \qquad \text{is an } M \text{ matrix.} \tag{1.5}$$

The role of the signature matrix is easy to understand. It simply accounts for the fact that if G has an infinitely divisible square, then so does $(\epsilon_1 G_1, \dots, \epsilon_n G_n)$ for any choice of $\epsilon_i = \pm 1$, $i = 1, \dots, n$. Therefore, if (1.5) holds for $\mathcal{N}$ with diagonal elements $n_1, \dots, n_n$

$$\left(\mathcal{N}\Gamma^{-1}\mathcal{N}\right)^{-1} = \mathcal{N}\Gamma\mathcal{N} \ge 0 \tag{1.6}$$

since the inverse of an M matrix is positive. Thus $(n_1G_1, \dots, n_nG_n)$ has a positive covariance matrix and its inverse is an M matrix.. (For this reason, in studying mean zero Gaussian vectors with infinitely divisible squares one can restrict ones attention to vectors with positive covariance.)

The question of characterizing Gaussian random variables with non-zero mean and infinitely divisible squares first came up in the work of N. Eisenbaum [2, 3] and then in joint work by Eisenbaum and H. Kaspi [4], in which they characterize Gaussian processes with a covariance that is the 0-potential density of a symmetric Markov process. This work is presented and expanded in [8, Chapter 13]. The following theorem is taken from Theorem 13.3.1 and Lemma 13.3.2 in [8].

Theorem 1.2 (Eisenbaum, Kaspi)**.** *Let $G = (G_1, \dots, G_n)$ be a mean zero Gaussian random variable with strictly positive definite covariance matrix $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$. Let $\mathbf{1} = (1, \dots, 1) \in R^n$. Then the following are equivalent:*

(1) $G + \mathbf{1}\alpha$ has infinitely divisible squares for all $\alpha \in R^1$;

(2) For $\xi = N(0, b^2)$ independent of G, $(G_1+\xi, \dots, G_n+\xi, \xi)$ has infinitely divisible squares for some $b \ne 0$. Furthermore, if this holds for some $b \ne 0$, it holds for all $b \in R^1$, with $N(0,0) = 0$.

(3) Γ^{-1} *is an* M *matrix with positive row sums.*

To avoid having to comment about trivial exceptions to general statements we assume that G can not be written as the direct sum of two independent Gaussian vectors G' and G''. This is equivalent to saying that the covariance matrix of G is irreducible. We have the following description of Gaussian vectors with infinitely divisible squares that doesn't require that each component of the vector has the same mean.

Theorem 1.3. *Let* $G = (G_1, \dots, G_n)$ *be a mean zero Gaussian random variable with irreducible strictly positive definite covariance matrix* $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$*. Let* $c = (c_1, \dots, c_n) \in R^n$*,* $c \neq \mathbf{0}$ *and let* C *be a diagonal matrix with* $c_i = C_{i,i}$*,* $1 \leq i \leq n$ *. Then the following are equivalent:*

(1) $G + c\alpha$ *has infinitely divisible squares for all* $\alpha \in R^1$*;*

(2) For $\xi = N(0, b^2)$ *independent of* G*,* $(G_1 + c_1\xi, \dots, G_n + c_n\xi, \xi)$ *has infinitely divisible squares for some* $b \neq 0$*. Furthermore, if this holds for some* $b \neq 0$*, it holds for all* $b \in R^1$*;*

(3) $C\,\Gamma^{-1}\,C$ *is an* M *matrix with positive row sums.*

We list several consequences of Theorem 1.3. An important step in the proof of Theorem 1.3 is to show that, under the hypotheses of this Theorem, no component of c can be equal to zero. We state this as 1. of the next Corollary, and explore its implications.

Corollary 1.1. *Let* $G = (G_1, \dots, G_n)$ *be a mean zero Gaussian random variable with irreducible strictly positive definite covariance matrix* $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$*. Let* $c = (c_1, \dots, c_n) \in R^n$*,* $c \neq \mathbf{0}$ *and let* C *be a diagonal matrix with* $c_i = C_{i,i}$*,* $1 \leq i \leq n$ *. Then*

1. *When any of the equivalent conditions (1), (2) and (3) of Theorem 1.3 hold no component of* c *can be equal to zero;*

2. *If* G^2 *is infinitely divisible none of the entries of* Γ *are equal to zero;*

3. *When any of the equivalent conditions (1), (2) and (3) of Theorem 1.3 hold and* $\Gamma \geq 0$*, (in which case, by 2.,* $\Gamma > 0$*), then* $c_ic_j > 0$*,* $1 \leq i, j \leq n$*;*

4. *When* $C\Gamma^{-1}C$ *is an* M *matrix, it follows that* $\mathcal{N}\Gamma^{-1}\mathcal{N}$ *is also an* M *matrix, where the diagonal elements of the signature matrix* $\mathcal{N}$ *are* $n_i = \mathit{sign}\ c_i$*,* $1 \leq i \leq n$*.*

To elaborate on Corollary 1.1, *4.* we know by (3) $\Rightarrow$ (1) of Theorem 1.3 and Theorem 1.1 that when $C\Gamma^{-1}C$ is an M matrix there exists a signature matrix $\mathcal{N}$ such that $\mathcal{N}\Gamma^{-1}\mathcal{N}$ is an M matrix. In Corollary 1.1, *4.* we show how C and $\mathcal{N}$ are related.

In the next corollary we use Theorem 1.3 to obtain two properties of the Gaussian vector G that hold when G^2 is infinitely divisible.

Corollary 1.2. *Let $G = (G_1, \ldots, G_n)$ be a mean zero Gaussian random variable with irreducible strictly positive definite covariance matrix $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$ and suppose that G has infinitely divisible squares. Set*

$$h_{j,n} = \frac{\Gamma_{j,n}}{\Gamma_{n,n}} \qquad 1 \le j \le n-1; \tag{1.7}$$

1. *Then*
$$(G_1 + h_{1,n}\alpha, \ldots, G_{n-1} + h_{n-1,n}\alpha, G_n + \alpha) \tag{1.8}$$
has infinitely divisible squares for all $\alpha \in R^1$.

2. *Write*
$$G = (\eta_1 + h_{1,n}G_n, \ldots, \eta_{n-1} + h_{n-1,n}G_n, G_n) \tag{1.9}$$
where
$$\eta_j = G_j - h_{j,n}G_n \qquad 1 \le j \le n-1. \tag{1.10}$$
Then
$$(\eta_1 + h_{1,n}\alpha, \ldots, \eta_{n-1} + h_{n-1,n}\alpha) \tag{1.11}$$
has infinitely divisible squares for all $\alpha \in R^1$.

3. *Let $\mathcal{E}$ denote the covariance matrix of $(\eta_1, \ldots \eta_{n-1})$. If $\Gamma \ge 0$, $\mathcal{E}^{-1}$ is an M matrix.*

Remark 1.1. In particular Corollary 1.2, 1. shows that when a Gaussian vector G in R^n has infinitely divisible squares there exists a vector $c = (c_1, \ldots, c_n)$ for which $G+c$ has infinitely divisible squares. Also Corollary 1.2, 2. shows that when a Gaussian vector G in R^n has infinitely divisible squares, then $(\eta_1, \ldots, \eta_{n-1})$, the orthogonal complement of the projection of G onto G_n, has infinitely divisible squares.

We next list several properties of the elements of the covariance matrix $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$ that hold when G^2 is infinitely divisible, or when any of the equivalent conditions (1), (2) and (3) of Theorem 1.2 hold. The first two are known, references for them are given in Remark 2.1.

Corollary 1.3. *Let $G = (G_1, \ldots, G_n)$ be a mean zero Gaussian random variable with irreducible strictly positive definite covariance matrix $\Gamma = \{\Gamma_{i,j}\} = \{E(G_iG_j)\}$. Let $c = (c_1, \ldots, c_n) \in R^n$, $c \ne \mathbf{0}$. Then*

1. *When G^2 is infinitely divisible, $\Gamma \ge 0$ and $n \ge 3$*
$$\Gamma_{j,l}\Gamma_{k,l} \le \Gamma_{j,k}\Gamma_{l,l} \qquad \forall\, 1 \le j,k,l \le n. \tag{1.12}$$

2. *When any of the equivalent conditions (1), (2) and (3) of Theorem 1.2 hold,*
$$0 < \Gamma_{i,j} \le \Gamma_{i,i} \wedge \Gamma_{j,j}. \tag{1.13}$$

3. *When any of the equivalent conditions (1), (2) and (3) of Theorem 1.3 hold*
$$\Gamma_{i,i} \ge \frac{c_i}{c_j}\Gamma_{i,j} \qquad \forall\, 1 \le i,j \le n; \tag{1.14}$$

4. *When $n = 2$ and the covariance matrix of G is invertible, $(G+c\alpha)^2$ is infinitely divisible for all $\alpha \in R^1$ if and only if*

$$\Gamma_{i,i} \geq \frac{c_i}{c_j}\Gamma_{i,j} > 0 \qquad \forall\, 1 \leq i,j \leq 2. \tag{1.15}$$

5. *When $n \geq 2$ there is no Gaussian vector G for which $(G+c\alpha)^2$ is infinitely divisible for all $\alpha \in R^1$ and all $c \in R^n$. (Recall that we are assuming that the covariance matrix of G is irreducible. This rules out the possibility that all the components of G are independent.)*

By definition, when $(G+c)^2$ is infinitely divisible, it can be written as in (1.1) as a sum of r independent identically distributed random variables, for all $r \geq 1$. Based on the work of Eisenbaum and Kaspi mentioned above and the joint paper [5] we can actually describe the decomposition. (In fact this decomposition plays a fundamental role in the proofs of Lemma 4.1 and Theorem 4.2, [2] and Theorem 1.1, [3].) We give a rough description here. For details see [2, 3, 4] and [8, Chapter 13].
Assume that (1), (2) and (3) of Theorem 1.3 hold. Consider G/c, (see (2.1)). Let Γ_c denote the covariance matrix of G/c. Theorem 1.2 holds for G/c and Γ_c, so Γ_c^{-1} is an M matrix with positive row sums. To be specific let $G/c \in R^n$. Set $S = \{1,\ldots,n\}$. By [8, Theorem 13.1.2] Γ_c is the 0-potential density of a strongly symmetric transient Borel right process, say X, on S. We show in the proof of [8, Theorem 13.3.1] that we can find a strongly symmetric recurrent Borel right process Y on $S \cup \{0\}$ with $P^x(T_0 < \infty) > 0$ for all $x \in S$ such that X is the process obtained by killing Y the first time it hits 0. Let $L_t^x = \{L_t^x; t \in R_+, x \in S \cup \{0\}\}$ denote the local time of Y. It follows from the generalized second Ray-Knight Theorem in [5], see also [8, Theorem 8.2.2] that under $P^0 \times P_G$,

$$\left\{L^x_{\tau(t)} + \frac{1}{2}\left(\frac{G_x}{c_x}\right)^2; x \in S\right\} \stackrel{law}{=} \left\{\frac{1}{2}\left(\frac{G_x}{c_x} + \sqrt{2t}\right)^2; x \in S\right\} \tag{1.16}$$

for all $t \in R_+$, where $\tau(t) = \inf\{s > 0 | L_s^0 > t\}$, the inverse local time at zero, and Y and G are independent. Consequently

$$\left\{c_x^2 L^x_{\tau(\alpha^2/2)} + \frac{1}{2}G_x^2; x \in S\right\} \stackrel{law}{=} \left\{\frac{1}{2}\left(G_x + c_x\alpha\right)^2; x \in S\right\} \tag{1.17}$$

for all $\alpha \in R^1$. (We can extend α from R_+ to R^1 because G is symmetric.) $\{c_x^2 L^x_{\tau(\alpha^2/2)}; x \in S\}$ and $\{\frac{1}{2}G_x^2; x \in S\}$ are independent. G^2 is infinitely divisible and for all integers $r \geq 1$

$$c_\cdot^2 L^\cdot_{\tau(\alpha^2/2)} \stackrel{law}{=} c_\cdot^2 L^\cdot_{\tau(\alpha^2/(2r)),1} + \cdots + c_\cdot^2 L^\cdot_{\tau(\alpha^2/(2r)),r} \tag{1.18}$$

where $\{L^\cdot_{\tau(\alpha^2/(2r)),j}\}$, $j = 1,\ldots,r$ are independent.
Note that in (1.17) we identify the components of the decomposition of $\{(G_x + c_x\alpha)^2; x \in S\}$ that mark it as infinitely divisible. The profound connection between Gaussian processes with infinitely divisible squares and local times of strongly symmetric Markov processes shows that the question of Gaussian processes with infinitely divisible squares is more than a mere curiosity.

In this same vein we can describe a decomposition of mean zero Gaussian vectors with infinitely divisible squares that is analogous to (1.17). Suppose G^2 is infinitely divisible. Write G and $\eta_1,\dots,\eta_{n-1}$ as in (1.9) and (1.10) and set

$$c_i = \frac{\Gamma_{i,n}}{\Gamma_{n,n}}, \qquad i = 1,\dots,n. \tag{1.19}$$

Let $\mathcal{P} := (\eta_1,\dots \eta_{n-1})$ and $c := c_1,\dots,c_{n-1}$.
It follows from Corollary 1.2, 2, that $\eta + c\alpha$ has infinitely divisible squares for all $\alpha \in R^1$. Therefore, as in (1.17),

$$\left\{c_x^2 L^x_{\tau(\alpha^2/2)} + \frac{1}{2}\eta_x^2;\, x \in S\right\} \stackrel{law}{=} \left\{\frac{1}{2}\left(\eta_x + c_x\alpha\right)^2;\, x \in S\right\} \tag{1.20}$$

for all $\alpha \in R^1$. Here L^x_t is the local time determined by the process $\mathcal{P}/c$, in the same way as the local time is determined by G/c in the paragraph containing (1.16). Let ξ be a normal random variable with mean zero and variance $EG_n^2/2$ that is independent of everything else in (1.20). It follows from (1.20) and (1.9) that

$$\left\{c_x^2 L^x_{\tau(\xi^2)} + \frac{1}{2}\eta_x^2;\, x \in S\right\} \stackrel{law}{=} \left\{\frac{1}{2}G_x^2;\, x \in S\right\}. \tag{1.21}$$

This isomorphism identifies the components of the decomposition of $\{G_x^2;\, x \in S\}$ that mark it as infinitely divisible.

There remains an interesting question. Assume that G^2 has infinitely divisible squares. Is it possible for $(G_1+\alpha,\dots,G_n+\alpha)$ to have infinitely divisible squares for some $\alpha \neq 0$ but not for all $\alpha \in R^1$? We do know that if this is the case there would exist an $0 < \alpha_0 < \infty$ such that $(G_1+\alpha,\dots,G_n+\alpha)$ would have infinitely divisible squares for all $0 \le |\alpha| \le \alpha_0$ but not for $|\alpha| > \alpha_0$. We can show that such an α_0 always exists for $(G_1+\alpha, G_2+\alpha)$, as long as $EG_1G_2 \neq 0$, [9].

2 Proof of Theorem 1.3

Proof of Theorem 1.3 If none of the $c_i, i = 1,\dots,n$ are equal to 0, the theorem follows directly from Theorem 1.2 applied to the Gaussian process

$$\frac{G}{c} = \left(\frac{G_1}{c_1},\dots,\frac{G_n}{c_n}\right) \tag{2.1}$$

which is a mean zero Gaussian random variable with strictly positive definite covariance matrix

$$\Gamma_c = \left\{\frac{\Gamma_{i,j}}{c_i c_j}\right\} = \left\{E\left(\frac{G_i}{c_i}\frac{G_j}{c_j}\right)\right\} = C^{-1}\Gamma C^{-1}, \tag{2.2}$$

so that $\Gamma_c^{-1} = C\Gamma^{-1}C$. Applying Theorem 1.2 gives Theorem 1.3 except that (1) and (2) are replaced by

(1′)

$$\left(\frac{G_1}{c_1}+\alpha,\dots,\frac{G_n}{c_n}+\alpha\right) \qquad \text{has infinitely divisible squares.} \tag{2.3}$$

(2′)

$$\left(\frac{G_1}{c_1}+\xi,\dots,\frac{G_n}{c_n}+\xi,\xi\right)\qquad \text{has infinitely divisible squares.} \tag{2.4}$$

It is easy to see that these hold if and only if (1) and (2) hold.
Therefore, to complete the proof of Theorem 1.3 we need only show that each of the conditions (1), (2), and (3) imply that none of the $c_i, i=1,\dots,n$ are equal to 0. This is obvious for (3) since an M matrix is invertible.

To proceed we give a modification of [8, Lemma 13.3.2].

Lemma 2.1. *Let $G=(G_1,\dots,G_n)$ be a mean zero Gaussian process with covariance matrix Γ which is invertible. Consider $\overline{G}=(G_1+c_1\xi,\dots,G_n+c_n\xi,\xi)$ where $\xi=N(0,b^2)$, $b\neq 0$, is independent of G. Denote the covariance matrix of $\overline{G}$ by $\overline{\Gamma}$. Then*

$$\begin{aligned}
\overline{\Gamma}^{j,k} &= \Gamma^{j,k} && j,k=1,\dots,n\\
\overline{\Gamma}^{n+1,k} &= -\textstyle\sum_{j=1}^n c_j\Gamma^{j,k} && k=1,\dots,n\\
\overline{\Gamma}^{n+1,n+1} &= \frac{1}{b^2}+\sum_{j,k=1}^n c_jc_k\Gamma^{j,k}
\end{aligned} \tag{2.5}$$

where, for an invertible matrix A we use $A^{i,j}$ to denote $\{A^{-1}\}_{i,j}$.

Note that it is possible for some or all of the components of c to be equal to zero.

Proof To prove (2.5) we simply go through the elementary steps of taking the inverse of $\overline{\Gamma}$. We begin with the array

$$\begin{array}{cccc|cccc}
\Gamma_{1,1}+c_1^2b^2 & \dots\dots & \Gamma_{1,n}+c_1c_nb^2 & c_1b^2 & 1 & \dots\dots & 0 & 0\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\Gamma_{n,1}+c_nc_1b^2 & \dots\dots & \Gamma_{n,n}+b^2 & c_nb^2 & 0 & \dots\dots & 1 & 0\\
c_1b^2 & \dots\dots & c_nb^2 & b^2 & 0 & \dots\dots & 0 & 1
\end{array} \tag{2.6}$$

Next, for each $j=1,\dots,n$ subtract c_j times the last row from the j–th row and then divide the last row by b^2 to get

$$\begin{array}{cccc|cccc}
\Gamma_{1,1} & \dots\dots & \Gamma_{1,n} & 0 & 1 & \dots\dots & 0 & -c_1\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
\Gamma_{n,1} & \dots\dots & \Gamma_{n,n} & 0 & 0 & \dots\dots & 1 & -c_n\\
c_1 & \dots\dots & c_n & 1 & 0 & \dots\dots & 0 & 1/b^2
\end{array} \tag{2.7}$$

This shows that $\det(\overline{\Gamma})=b^2\det(\Gamma)$ and consequently Γ is invertible if and only if $\overline{\Gamma}$ is invertible.
We now work with the first n rows to get the inverse of Γ so that the array looks like

$$\begin{array}{cccc|cccc}
1 & \dots\dots & 0 & 0 & \Gamma^{1,1} & \dots\dots & \Gamma^{1,n} & a_1\\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
0 & \dots\dots & 1 & 0 & \Gamma^{n,1} & \dots\dots & \Gamma^{n,n} & a_n\\
c_1 & \dots\dots & c_n & 1 & 0 & \dots\dots & 0 & 1/b^2
\end{array} \tag{2.8}$$

At this stage we don't know what are the a_j, $j = 1, \ldots, n$.
Finally, for each $j = 1, \ldots, n$ we subtract c_j times the j–th row from the last row to obtain

$$\left.\begin{array}{cccc} 1 & \ldots\ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \ldots\ldots & 1 & 0 \\ 0 & \ldots\ldots & 0 & 1 \end{array}\right| \begin{array}{cccc} \Gamma^{1,1} & \ldots\ldots & \Gamma^{1,n} & a_1 \\ \vdots & \ddots & \vdots & \vdots \\ \Gamma^{n,1} & \ldots\ldots & \Gamma^{n,n} & a_n \\ -\sum_{j=1}^n c_j \Gamma^{j,1} & \ldots\ldots & -\sum_{j=1}^n c_j \Gamma^{j,n} & a_{n+1} \end{array} \tag{2.9}$$

where

$$a_{n+1} = (1/b^2 - \sum_{k=1}^n c_k a_k). \tag{2.10}$$

Since the inverse matrix is symmetric we see that $a_k = -\sum_{j=1}^n c_j \Gamma^{j,k}$, $k = 1, \ldots, n$. This verifies (2.5). □

We now show that (2) implies that no component of c can be zero. Recall that $c \neq \mathbf{0}$.ÊÊ Suppose that some of the components of c are equal to zero. We can separate the components of $G_1, \ldots, G_n$ into two sets. Those that have the associated $c_i \neq 0$, $1 \le i \le n$ and those that have the associated $c_j = 0$, $1 \le j \le n$. There must be at least one member of the first set that is not independent of some member of the second set. Otherwise Γ is not irreducible. (Recall that this is an hypothesis, stated just before Theorem 1.3.) We take these two members and relabel them G_1 and G_2 . If (2) Êholds then $\widetilde{G} = (G_1 + c_1\xi, G_2, \xi)$ has an infinitely divisible square. LetÊ $\widetilde{\Gamma}$ denote the covariance matrix of $\widetilde{G}$. Then by Theorem 1.1 there exists a signature matrix $\mathcal{N}$ such that that $\mathcal{N}\widetilde{\Gamma}^{-1}\mathcal{N}$ is an M matrix.Ê It follows from (2.9) that

$$\mathcal{N}\widetilde{\Gamma}^{-1}\mathcal{N} = \begin{pmatrix} \Gamma_{(2)}^{1,1} & \Gamma_{(2)}^{1,2} n_1 n_2 & -c_1 \Gamma_{(2)}^{1,1} n_1 n_3 \\ \Gamma_{(2)}^{2,1} n_1 n_2 & \Gamma_{(2)}^{2,2} & -c_1 \Gamma_{(2)}^{1,2} n_2 n_3 \\ -c_1 \Gamma_{(2)}^{1,1} n_1 n_3 & -c_1 \Gamma_{(2)}^{1,2} n_2 n_3 & 1/b^2 + c_1^2 \Gamma_{(2)}^{1,1} \end{pmatrix}. \tag{2.11}$$

where $\Gamma_{(2)}$ is the covariance matrix ofÊ (G_1, G_2) and $\Gamma_{(2)}^{i,j} = (\Gamma_{(2)}^{-1})_{i,j}$. Since $\mathcal{N}\widetilde{\Gamma}^{-1}\mathcal{N}$ is an M matrix, it must have negative off diagonal elements. In fact they are strictly negative because G_1 and G_2 are not independent. Therefore

$$0 < (\mathcal{N}\widetilde{\Gamma}^{-1}\mathcal{N})_{1,3}(\mathcal{N}\widetilde{\Gamma}^{-1}\mathcal{N})_{2,3} = c_1^2 \Gamma_2^{1,1} \Gamma_2^{1,2} n_1 n_2$$

which implies that $(\mathcal{N}\widetilde{\Gamma}^{-1}\mathcal{N})_{1,2} = \Gamma_2^{1,2} n_1 n_2 > 0$.Ê This contradiction shows that when (2) holds no component of c can be zero.
We complete the proof of the Theorem by showing that (1) $\Rightarrow$ (2) , which, in particular, implies that when (1) holds no component of c can be zero.
It follows from (1) that $(\sqrt{m}\alpha, G_1 + \sqrt{m}\alpha c_1, \ldots, G_n + \sqrt{m}\alpha c_n)$ has infinitely divisible squares for all integers $m \ge 0$ and $\alpha \in R^1$. Let $G_0 := 0$, $c_0 := 1$ and let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be an n-dimensional vector and Λ an $n \times n$ diagonal matrix with λ_j as

its j-th diagonal entry. By [8, Lemma 5.2.1]

$$E\exp\left(-\sum_{i=0}^{n}\lambda_i(G_i+\sqrt{m}\alpha c_i)^2/2\right) \tag{2.12}$$
$$=\frac{1}{(\det(I+\Gamma\,\Lambda))^{1/2}}\exp\left(m\,\alpha^2\left(-\frac{\lambda_0}{2}-\frac{c\Lambda c^t}{2}+\frac{(c\Lambda\widetilde{\Gamma}\,\Lambda c^t)}{2}\right)\right),$$

where

$$\widetilde{\Gamma}:=(\Gamma^{-1}-\Lambda)^{-1}=(I-\Gamma\,\Lambda)^{-1}\Gamma. \tag{2.13}$$

By the same lemma

$$E\exp\left(-\sum_{i=0}^{n}\lambda_i(G_i+c_i\xi)^2/2\right) \tag{2.14}$$
$$=\frac{1}{(\det(I+\Gamma\,\Lambda))^{1/2}}\exp\left(\xi^2\left(-\frac{\lambda_0}{2}-\frac{c\Lambda c^t}{2}+\frac{(c\Lambda\widetilde{\Gamma}\,\Lambda c^t)}{2}\right)\right).$$

Compare (2.12) and (2.14) with (13.82) and (13.83) in the proof of (2) $\Rightarrow$ (3) of [8, Theorem 13.3.1]. Following that proof we see that (1) $\Rightarrow$ (2). $\square$

Proof of Corollary 1.1

1. As observed in the proof of Theorem 1.3, (3), $C\,\Gamma^{-1}\,C$ is invertible and thus $\det C\neq 0$.

2. Ê Pick any component of G; for convenience we take G_n. LetÊ

$$\eta_j=G_j-\frac{\Gamma_{j,n}}{\Gamma_{n,n}}G_n \qquad 1\le j\le n-1. \tag{2.15}$$

We write

$$G=\left(\eta_1+\frac{\Gamma_{1,n}}{\Gamma_{n,n}}G_n,\ldots,\eta_{n-1}+\frac{\Gamma_{n-1,n}}{\Gamma_{n,n}}G_n,G_n\right) \tag{2.16}$$

and note that this has the form of

$$G=(\eta_1+c_1\xi,\ldots,\eta_{n-1}+c_{n-1}\xi,\xi) \tag{2.17}$$

where $\xi=N(0,EG_n^2)$ is independent of $\eta=(\eta_1,\ldots,\eta_{n-1})$.

If the covariance matrix of $(\eta_1,\ldots,\eta_{n-1})$ is irreducible then the fact that G^2 is infinitely divisible and 1. of this corollary imply that $c_j\neq 0$, $1\le j\le n-1$. Thus none of $\Gamma_{j,n}$, $1\le j\le n-1$, are equal to zero. Since the initial choice of G_n is arbitraryÊ we get 2.

Now suppose that the covariance matrix of $(\eta_1,\ldots,\eta_{n-1})$ is not irreducible. Assume that $\Gamma_{1,n}=0$. If η_1 is independent of $(\eta_2,\ldots,\eta_{n-1})$, then since $\eta_1=G_1$, and it is independent of G_n, we get a contradiction of the hypothesis that the covariance matrix of G is irreducible. Therefore let $(\eta_1,\ldots,\eta_l)$, $2\le l\le n-2$, be the smallest set of components in $(\eta_1,\ldots,\eta_{n-1})$ that both contains η_1 and has an irreducible covariance matrix. Consider the corresponding Gaussian vector $(G_1,\dots,G_l)$, Note that at least one of the components in this vector is not independent of G_n, since if this is not the case $(G_1,\dots,G_l)$ would be independent of $(G_{l+1},\dots,G_n)$ which contradicts the hypothesis that the covariance matrix of G is irreducible.

Suppose G_2 depends on G_n, and consider the vector (G_1,G_2,G_n). This vector has infinitely divisible squares because G has infinitely divisible squares. Let $G_n := \xi$. Then we can write (G_1,G_2,G_n) as $(\eta_1,\eta_2+\alpha\xi,\xi)$, where $\alpha \neq 0$. However, by Theorem 1.3, or 1. of this corollary, this vector does not have infinitely divisible squares. This contradiction shows that none of $\Gamma_{j,n}$, $1 \le j \le n-1$, are equal to zero, and as above this establishes 2. even when the covariance matrix of $(\eta_1,\dots,\eta_{n-1})$ is not irreducible.

3. By Theorem 1.3, (3)
$$\left(C\,\Gamma^{-1}\,C\right)^{-1}_{j,k} = \frac{\Gamma_{j,k}}{c_jc_k} \ge 0. \tag{2.18}$$
and by hypothesis and 2., $\Gamma_{j,k}>0$. (Also by 1. neither c_j nor c_k are equal to zero.) Therefore under the hypotheses of 3. we actually have
$$\left(C\,\Gamma^{-1}\,C\right)^{-1}_{j,k} = \frac{\Gamma_{j,k}}{c_jc_k} > 0, \tag{2.19}$$
so, since $\Gamma_{j,k}>0$, $c_jc_k>0$.

4. Write $C = C\mathcal{N}\mathcal{N}$ for $\mathcal{N}$ as given. Then consider
$$C\mathcal{N}\mathcal{N}\Gamma^{-1}\mathcal{N}C\mathcal{N} = C\Gamma^{-1}C. \tag{2.20}$$
Since $C\mathcal{N}>0$ we see that $\mathcal{N}\Gamma^{-1}\mathcal{N}$ is an M matrix.

Proof of Corollary 1.2

1. To begin suppose that $\Gamma \ge 0$. In this case we see by Corollary 1.1, 2. that $h_{j,n}>0$, $1\le j\le n-1$. Since G has infinitely divisible squares so does
$$\frac{G}{h} = \left(\frac{G_1}{h_{1,n}},\dots,\frac{G_{n-1}}{h_{n-1,n}},G_n\right). \tag{2.21}$$
Write G as in (2.16) so that
$$\frac{G}{h} = \left(\frac{\eta_1}{h_{1,n}}+G_n,\dots,\frac{\eta_{n-1}}{h_{n-1,n}}+G_n,G_n\right). \tag{2.22}$$
Let Θ be the covariance matrix of G/h. Since $\Gamma\ge 0$ and $h_{j,n}>0$, $1\le j\le n-1$, we have $\Theta\ge 0$. Therefore by Theorem 1.1, Θ^{-1} is an M-matrix. It is also a matrix of the type $\overline{\Gamma}^{-1}$ considered in [8, Lemma 13.4.1]. It follows from (4) $\implies$ (3) of this Lemma that Θ^{-1} has positive row sums. Therefore by Theorem 1.2
$$\left(\frac{\eta_1}{h_{1,n}}+G_n+\alpha,\dots,\frac{\eta_{n-1}}{h_{n-1,n}}+G_n+\alpha,G_n+\alpha\right) \tag{2.23}$$

has infinitely divisible squares for all $\alpha \in R^1$. This is equivalent to (1.8).

In general let $\mathcal{N}$ be the signature matrix such that $\mathcal{N}\Gamma\mathcal{N} \geq 0$. Without loss of generality we can take $n_n = 1$. Applying the result in the preceding paragraph to $(n_1G_1, \ldots, n_{n-1}$
$G_{n-1}, G_n)$ we see that

$$(n_1G_1 + n_1h_{1,n}\alpha, \ldots, n_{n-1}G_{n-1} + n_{n-1}h_{n-1,n}\alpha, G_n + \alpha) \tag{2.24}$$

has infinitely divisible squares for all $\alpha \in R^1$. This implies that

$$(G_1 + h_{1,n}\alpha, \ldots, G_{n-1} + h_{n-1,n}\alpha, G_n + \alpha) \tag{2.25}$$

has infinitely divisible squares, which gives (1.8).

2. ÊWrite G as in (1.9). If the covariance matrix of $\eta = (\eta_1, \ldots, \eta_{n-1})$ is irreducible then it follows from Theorem 1.3 and the hypothesis that G^2 is infinitely divisible, that $(\eta + h\alpha)^2$ is infinitely divisible for all α. However, it is easy to see that the covariance matrix of $(\eta_1, \ldots, \eta_{n-1})$ need not be irreducible. (Consider $G = (\eta_1 + \xi, \eta_2 + \xi, \xi)$ where η_1, η_2, ξ are i.i.d. $N(0, 1)$. G^2 is infinitely divisible but, obviously, the covariance matrix of (η_1, η_2) is not irreducible.)

   If the covariance matrix of $(\eta_1, \ldots, \eta_{n-1})$ is not irreducible, consider a subset of the components say $(\eta_1, \ldots, \eta_l)$ such that its covariance matrix is irreducible. Since G^2 is infinitely divisible

$$G = (\eta_1 + h_{1,n}G_n, \ldots, \eta_l + h_{l,n}G_n, G_n) \tag{2.26}$$

   is infinitely divisible. It follows from the argument in the beginning of the preceding paragraph that $(\eta_1 + h_1\alpha, \ldots, \eta_l + h_l\alpha,)$ has infinitely divisible squares for all α. This holds for all the blocks of irreducible components of η. Therefore $(\eta + h\alpha)^2$ is infinitely divisible for all α.

3. As we point out in 2., immediately above, the covariance matrix of $(\eta_1, \ldots, \eta_{n-1})$ is not necessarily irreducible. If it is not, as above, consider a subset of the components, say $(\eta_1, \ldots, \eta_l)$, such that its covariance matrix is irreducible. Since G^2 is infinitely divisible

$$G = (\eta_1 + h_{1,n}G_n, \ldots, \eta_l + h_{l,n}G_n, G_n) \tag{2.27}$$

   is infinitely divisible.Ê Let $\mathcal{E}'$ be the covariance matrix of $(\eta_1, \ldots, \eta_l)$. It follows from Theorem 1.3, that $C'\mathcal{E}'^{-1}C'$ is an M matrix, where C' is a diagonal matrix with entries $c_i = h_{i,n}$, $1 \leq i \leq l$. Since $\Gamma \geq 0$, the diagonal elements of C' are strictly positive. Thus $\mathcal{E}'^{-1}$ is an M matrix. It is easy to see that if the covariance matrix of each irreducible block of $(\eta_1, \ldots, \eta_{n-1})$ is an M matrix, then $\mathcal{E}$ is an M matrix.

□

Proof of Corollary 1.3

1. Let $\mathcal{E}$ be as in Corollary 1.2, 3. Since $\mathcal{E} \geq 0$ we see that

$$\mathcal{E}_{j,k} = E\eta_j\eta_k = \Gamma_{j,k} - \frac{\Gamma_{j,n}\Gamma_{k,n}}{\Gamma_{n,n}} \geq 0. \tag{2.28}$$

This gives (1.12) when $l = n$. Since the choice of G_n in 8. is arbitrary we get (1.12) as stated.

3. By (2.19)
$$\frac{\Gamma_{j,k}}{c_j c_k} > 0 \qquad 1 \le j,k \le n. \tag{2.29}$$
Consider $(G_i + c_i, G_j + c_j)$, $i \neq j$. Let $\Gamma_{(2)}$ be the covariance matrix of (G_i, G_j) and Γ_c be the covariance matrix of $(G_i/c_i, G_j/c_j)$ so that $\Gamma_c = \widetilde{C}^{-1}\Gamma_{(2)}\widetilde{C}^{-1}$ where $\widetilde{C}$ is a diagonal matrix with entries c_i, c_j. Hence $\Gamma_c^{-1} = \widetilde{C}\Gamma_{(2)}^{-1}\widetilde{C}$. Considering the nature of the inverse of the 2×2 matrix $\Gamma_{(2)}$ we have that
$$\Gamma_c^{i,i} = c_i^2 \Gamma^{i,i} = \frac{c_i^2 \Gamma_{j,j}}{\det \Gamma} > 0 \qquad i \neq j \tag{2.30}$$
and
$$\Gamma_c^{i,j} = c_i c_j \Gamma^{i,j} = -\frac{c_i c_j \Gamma_{i,j}}{\det \Gamma}. \tag{2.31}$$
Since $(G_i + c_i, G_j + c_j)$ has infinitely divisible squares $C\Gamma^{-1}C$ has positive row sums. Therefore
$$c_i^2 \Gamma_{j,j} \ge c_i c_j \Gamma_{i,j} \tag{2.32}$$
which gives (1.14).

4. If (1.15) holds no component of c can be equal to zero and $\Gamma_{i,j} \neq 0$. The proof of 3. shows that when $n = 2$ the first inequality in (1.15) is necessary and sufficient for $C\Gamma^{-1}C$ to have positive row sums. The second inequality in (1.15) is necessary and sufficient for $C\Gamma^{-1}C$ to be an M-matrix.

2. This follows from 3.

5. This follows from 3. and Corollary 1.1, 1. □

Remark 2.1. Item 1. in Corollary 1.3 is given in [8, (13.39)]. An interesting consequence of this property is given in [8, (13.42)]. Item 1. in Corollary 1.2 follows from [8, Theorem 13.3.3 and Theorem 13.3.1]. Item 2. in Corollary 1.3 follows from [8, Theorem 13.3.1 and (13.2)]. These are all are consequences of the relationship between M matrices with positive row sums and the 0-potential density of Borel right process. Item 5. in Corollary 1.3 is also a consequence of [3], Remark 2.3.

Example 2.1. Let $G = (G_1, G_2)$ have mean zero and have covariance matrix
$$\Gamma = \begin{pmatrix} 1 & \frac{4}{3} \\ \frac{4}{3} & 2 \end{pmatrix}, \qquad \Gamma^{-1} = \begin{pmatrix} 9 & -6 \\ -6 & 9/2 \end{pmatrix}. \tag{2.33}$$
By Corollary 1.3, 4. $(G_1 + \alpha, G_2 + \alpha)$ does not have infinitely divisible squares for all $\alpha \in R^1$, and obviously, Γ^{-1} does not have positive row sums. However, by Corollary 1.3, 4. again or Corollary 1.2, 1., $(G_1 + \alpha, G_2 + (4/3)\alpha)$ does have infinitely divisible squares for all $\alpha \in R^1$. (Let C be the diagonal matrix with $C_{1,1} = 1$ and $C_{2,2} = 4/3$. Then
$$C\Gamma^{-1}C = \begin{pmatrix} 9 & -8 \\ -8 & 8 \end{pmatrix} \tag{2.34}$$
is an M matrix with positive row sums.)
Moreover by Corollary 1.3, 4. $(G_1 + \alpha, G_2 + c_2\alpha)$ has infinitely divisible squares for all $\alpha \in R^1$, for all $1 \le c_2 \le 2$.

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