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# A NOTE ON A.S. FINITENESS OF PERPETUAL INTEGRAL FUNCTIONALS OF DIFFUSIONS

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*Abstract*
In this note we use the boundary classification of diffusions in order to derive a criterion for the convergence of perpetual integral functionals of transient real-valued diffusions. We present a second approach, based on Khas'minskii's lemma, which is applicable also to spectrally negative Lévy processes.
In the particular case of transient Bessel processes, our criterion agrees with the one obtained via Jeulin's convergence lemma.

## 1 Introduction

Consider a linear regular diffusion $Y$ on an open interval $I = (l\,, r)$ in the sense of Itô and McKean [11]. Let $\mathbf{P}_x$ and $\mathbf{E}_x$ denote, respectively, the probability measure and the expectation associated with $Y$ when started from $x \in I$. It is assumed that $Y$ is transient, and for all $x \in I$

$$\lim_{t\to\zeta} Y_t = r \qquad \mathbf{P}_x\text{-a.s.}, \tag{1}$$

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where $\zeta$ is the life time of $Y$. Hence, if $\zeta < \infty$ then

$$\zeta = H_r(Y) := \inf\{t : Y_t = r\}.$$

Let $f$ be a non-negative, measurable and locally bounded function on $I$, and introduce for all $t > 0$

$$A_t(f) := \int_0^t f(Y_s)\, ds. \tag{2}$$

The ultimate value of this additive functional, i.e., $A_\zeta(f)$, is often called a *perpetual integral functional.* We are interested in finding necessary and sufficient conditions for the a.s. finiteness of $A_\zeta(f)$. To derive such conditions is an important problem per se in the theory of diffusions. Functionals of type (2) appear naturally in various applications; e.g., in insurance mathematics the functional $A_\zeta(f)$ may be interpreted, in a suitable model, as the present value of a continuous stream of perpetuities, see Dufresne [7].

In case $Y$ is a Brownian motion with drift $\mu > 0$ it is known that $A_\infty(f)$ is finite a.s. if and only if $f$ is integrable at $\infty$ (see Engelbert and Senf [9] and Salminen and Yor [18]). This condition is derived in [18] via Ray–Knight theorems and the stationarity property of the local time processes (which makes Jeulin's lemma [12] applicable). Moreover, conditions for existence of moments are also given in [18]. For exponential functionals of Brownian motion, we refer to the recent surveys by Matsumoto and Yor [15] and [16].

In Theorem 2, Section 2, we present a necessary and sufficient condition for a.s. finiteness of $A_\zeta(f)$ by exploiting the fact that $A_{H_x}(f)$ for $x < r$ can, via a random time change, be seen as the first hitting time of a point for another diffusion. This approach, valid for $Y$ determined via a SDE with smooth coefficients and continuous $f$, leads naturally to an integral test for the finiteness of $A_\zeta(f)$ providing at the same time a probabilistic explanation for the test. However, having seen the simple answer to the problem, another proof essentially based on Khas'minskii's lemma was constructed proving the result for general $Y$ and $f$. This is discussed in Theorem 3.

Since the integral test for continuous $f$ can be interpreted as the condition that the right boundary point is exit for a suitably chosen diffusion we offer a short discussion on exit boundaries in Section 3. The paper is concluded with an example about Bessel processes.

## 2 The Main Results

We begin by formulating the key result connecting perpetual integral functionals to first hitting times. The result is a generalization of a result in [17, Proposition 2.1] discussed in [3, Propositions 2.1 and 2.3]. Assume that the diffusion $Y$ introduced above is in fact determined by the SDE

$$dY_t = \sigma(Y_t)\, dW_t + b(Y_t)\, dt,$$

where $W$ is a standard Wiener process defined in a complete probability space $(\Omega\,, \mathcal{F}, \{\mathcal{F}_t\}, \mathbf{P})$. It is assumed that $\sigma$ and $b$ are continuous and $\sigma(x) > 0$ for all $x \in I$. For the speed and the scale measure of $Y$ we respectively use

$$m^Y(dx) = 2\,\sigma^2(x)\mathrm{e}^{B^Y(x)}\, dx \quad \text{and} \quad S^Y(dx) = \mathrm{e}^{-B^Y(x)}\, dx, \tag{3}$$

where

$$B^Y(x) = 2\int^x \frac{b(z)}{\sigma^2(z)}\, dz. \tag{4}$$

**Proposition 1.** *Let $Y$ and $f$ be as above, and assume that there exists a two times continuously differentiable function $g$ such that*

$$f(x) = \big(g'(x)\sigma(x)\big)^2, \quad x \in I. \tag{5}$$

*Let $\{a_t \,:\, 0 \le t < A_\zeta\}$ denote the inverse of $A$, that is,*

$$a_t := \min\big\{s : A_s > t\big\}, \qquad t \in [0\,, A_\zeta).$$

*Define $Z_t := g(Y_{a_t})$ for all $t \in [0\,, A_\zeta)$. Then, $Z$ is a diffusion that satisfies the SDE*

$$dZ_t = d\widetilde{W}_t + G(g^{-1}(Z_t))\,dt, \qquad t \in [0\,, A_\zeta).$$

*Here, $\widetilde{W}_t$ is a Brownian motion and*

$$G(x) = \frac{1}{f(x)}\left(\frac{1}{2}\,\sigma(x)^2\, g''(x) + b(x)\, g'(x)\right).$$

*Moreover, for $l < x < y < r$*

$$A_{H_y(Y)} = \inf\{t : \, Z_t = g(y)\} =: H_{g(y)}(Z) \quad a.s. \tag{6}$$

*with $Y_0 = x$ and $Z_0 = g(x)$.*

In order to fix ideas, let us assume that the function $g$ of Proposition 1 is increasing. We define $g(r) := \lim_{x\to r} g(x)$, and use the same convention for any increasing function defined on $(l\,, r)$. The state space of the diffusion $Z$ is the interval $(g(l), g(r))$ and $\lim_{t\to\zeta(Z)} Z_t = g(r)$ a.s.. We can let $y \to r$ in (6) to find that

$$A_{H_r(Y)} = \inf\{t : \, Z_t = g(r)\} \quad \text{a.s.}, \tag{7}$$

where both sides in (7) are either finite or infinite. Now we have

**Theorem 2.** *For $Y$, $A$, $f$ and $g$ as above it holds that $A_\zeta$ is finite $\mathbf{P}_x$-a.s. for all $x \in (l, r)$ if and only if for the diffusion $Z$ the boundary point $g(r)$ is an exit boundary, i.e.,*

$$\int^{g(r)} S^Z(d\alpha) \int^{\alpha} m^Z(d\beta) < \infty, \tag{8}$$

*where the scale $S^Z$ and the speed $m^Z$ of the diffusion $Z$ are given by*

$$S^Z(d\alpha) = \mathrm{e}^{-B^Z(\alpha)}\, d\alpha \quad \text{and} \quad m^Z(d\beta) = 2\,\mathrm{e}^{B^Z(\beta)}\, d\beta,$$

*with*

$$B^Z(\beta) = 2\int^{\beta} G \circ g^{-1}(z)\, dz.$$

*Condition (8) is equivalent with the condition*

$$\int_x^r \big(S^Y(r) - S^Y(v)\big)\, f(v)\, m^Y(dv) < \infty \qquad \text{for all } x \in (l\,, r). \tag{9}$$

*Proof.* As is well known from the standard diffusion theory, a diffusion hits its exit boundary with positive probability and an exit boundary cannot be unattainable (see [11] or [2]). This combined with (7) and the characterization of an exit boundary (see [2, No. II.6, p. 14]) proves the first claim. It remains to show that (8) and (9) are equivalent. We have

$$\begin{aligned} B^Z(\alpha) &= 2\int^{g^{-1}(\alpha)} G(u)\, g'(u)\, du \\ &= 2\int^{g^{-1}(\alpha)} \left(\frac{1}{2}\frac{g''(u)}{g'(u)} + \frac{b(u)}{\sigma^2(u)}\right) du \\ &= \log(g'(g^{-1}(\alpha))) + B^Y(g^{-1}(\alpha)). \end{aligned}$$

Consequently,

$$S^Z(d\alpha) = \mathrm{e}^{-B^Z(\alpha)}\, d\alpha = \frac{1}{g'(g^{-1}(\alpha))}\, \exp\left(-B^Y(g^{-1}(\alpha))\right)\, d\alpha,$$

and

$$m^Z(d\alpha) = 2\,\mathrm{e}^{B^Z(\alpha)}\, d\alpha = 2\, g'(g^{-1}(\alpha))\, \exp\left(B^Y(g^{-1}(\alpha))\right)\, d\alpha.$$

Substituting first $\alpha = g(u)$ in the outer integral in (8) and after this $\beta = g(v)$ in the inner integral yields

$$\begin{aligned} \int^{g(r)} S^Z(d\alpha)\int^{\alpha} m^Z(d\beta) &= 2\int^{r} du\, \mathrm{e}^{-B^Y(u)}\int^{u} dv\, (g'(v))^2\, \mathrm{e}^{B^Y(u)} \\ &= 2\int^{r} dv\, (g'(v))^2\, \mathrm{e}^{B^Y(v)}\int_v^{r} du\, \mathrm{e}^{-B^Y(u)} \end{aligned}$$

by Fubini's theorem. Using the expressions given in (3) for the speed and the scale of $Y$ and the relation (5) between $f$ and $g$ completes the proof. □

It is easy to derive a condition that the mean of $A_\zeta(f)$ is finite:

$$\begin{aligned} \mathbf{E}_x\left(A_\zeta(f)\right) &= \int_0^\infty \mathbf{E}_x\left(f(Y_s)\right)\, ds \\ &= \int_l^r G_0^Y(x,y)\, f(y)\, m^Y(dy) < \infty, \end{aligned} \tag{10}$$

where $G_0^Y$ is the Green kernel of $Y$ with respect to $m^Y$. Under the assumption (1) we may take for $x \le y$

$$G_0^Y(x,y) = S^Y(r) - S^Y(y).$$

Consequently, condition (9) may be viewed as a part of condition (10). This point of view can be elaborated further and, indeed, we have the following Theorem 3 extending criterion (9) in Theorem 2 for non-negative, measurable and locally bounded functions which are not neccessarily continuous. An essential tool hereby is Khas'minskii's lemma (see Khas'minskii [14], Simon [19], Durrett [8], Chung and Zhao [5], Stummer and Sturm [20] and Salminen and Yor [18] with a reference to Dellacherie and Meyer [6]) from which we may deduce that (iv) and (v) below are equivalent. Also Lemma 4 is closely related to Khas'minskii's result.

**Theorem 3.** *Let $Y$, $f$, and $A_\zeta(f)$ be as in the introduction. For all $x \in (l\,, r)$ define $f_x(y) := f(y)\mathbf{1}_{[x,r)}(y)$. Then, the following are equivalent:*

(i) $\mathbf{P}_x\{A_\zeta(f) < \infty\} = 1$ *for all* $x \in (l\,, r)$*;*

(ii) $\mathbf{P}_x\{A_\zeta(f_x) < \infty\} = 1$ *for all* $x \in (l\,, r)$*;*

(iii) $(G_0^Y f_x)(x) := \int_x^r G_0^Y(x,v)\, f(v)\, m^Y(dv) < \infty$ *for all* $x \in (l\,, r)$*;*

(iv) $\sup_{l<z<r}(G_0^Y f_x)(z) < \infty$ *for all* $x \in (l\,, r)$*;*

(v) $\sup_{l<z<r} \mathbf{E}_z[\exp(cA_\zeta(f_x))] < \infty$ *for some* $c \in (0\,,\infty)$ *and for all* $x \in (l\,, r)$*.*

In our proof of Theorem 3 we make use of an observation about Hunt processes presented below as Lemma 4. For this, let $\{A_t\}_{t\geq 0}$ be a continuous additive functional of a Hunt process with shifts $\{\theta_t\}_{t\geq 0}$. Define $\{\tau(\lambda)\}_{\lambda\geq 0}$ to be the right-continuous inverse to $\{A_t\}_{t\geq 0}$. Then, for all integers $n \geq 0$, and all reals $\lambda > 0$,

$$\begin{aligned} \mathbf{P}_x\left\{A_\zeta \geq (n+1)\lambda\right\} &= \mathbf{P}_x\{\tau(n\lambda) < \zeta\ ,\ A_\zeta \circ \theta_{\tau(n\lambda)} \geq \lambda\} \\ &\leq \mathbf{P}_x\{A_\zeta \geq n\lambda\} \cdot \sup_{l<y<r} \mathbf{P}_y\{A_\zeta \geq \lambda\}. \end{aligned}$$

Consequently,

$$\sup_{l<y<r} \mathbf{P}_y\left\{A_\zeta \geq n\lambda\right\} \leq \left(\sup_{l<y<r} \mathbf{P}_y\left\{A_\zeta \geq \lambda\right\}\right)^n .$$

Hence, we have proved

**Lemma 4.** *The following are equivalent:*

(a) *There exists* $\lambda > 0$ *such that* $\sup_{l<y<r} \mathbf{P}_y\{A_\zeta > \lambda\} < 1$*;*

(b) *There exists* $c > 0$ *such that* $\sup_{l<y<r} \mathbf{E}_y[\exp(cA_\zeta)] < \infty$*.*

*Proof of Theorem 3.*
(i)$\Leftrightarrow$(ii): Choose and fix $x \in (l\,, r)$. Let $L_x$ denote the last exit time from $x$. Then, by transience, $\mathbf{P}_x\{0 < L_x < \infty\} = 1$, and, under $\mathbf{P}_x$,

$$\begin{aligned} A_\zeta(f) := \int_0^\zeta f(Y_s)\, ds &= \int_0^{L_x} f(Y_s)\, ds + \int_{L_x}^\zeta f(Y_s)\, ds \\ &= \int_0^{L_x} f(Y_s)\, ds + \int_{L_x}^\zeta f_x(Y_s)\, ds, \end{aligned}$$

from which the claim follows.
(ii)$\Rightarrow$(v): Let $x \in (l\,, r)$ be fixed. Applying the strong Markov property, the continuity of $t \mapsto Y_t$ and assumption (1) we obtain for $y \leq x$

$$\mathbf{P}_y\{A_\zeta(f_x) \geq \lambda\} = \mathbf{P}_x\{A_\zeta(f_x) \geq \lambda\}, \tag{11}$$

and for $y \geq x$

$$\mathbf{P}_y\{A_\zeta(f_x) \geq \lambda\} \leq \mathbf{P}_x\{A_\zeta(f_x) \geq \lambda\}. \tag{12}$$

Consequently,

$$\sup_{l<y<r} \mathbf{P}_y\{A_\zeta(f_x) \geq \lambda\} = \mathbf{P}_x\{A_\zeta(f_x) \geq \lambda\}.$$

From (ii) it follows that there exists $\lambda'$ such that

$$\mathbf{P}_x\{A_\zeta(f_x) \geq \lambda'\} < 1,$$

and, hence, validity of (v) is proved by using Lemma 4.
(v)⇒(iv): Since (cf. (10))

$$\mathbf{E}_y[A_\zeta(f_x)] = (G_0^Y f_x)(y)$$

statement (iv) is obtained from (v) by series expansion of the exponential function.
(iv)⇒(iii): By reasoning as we did in the proof of (ii)⇒(v), when deriving (11) and (12), we deduce

$$\sup_{l<y<r} \mathbf{E}_y[A_\zeta(f_x)] = \mathbf{E}_x[A_\zeta(f_x)] = (G_0^Y f_x)(x). \tag{13}$$

(iii)⇒(ii): This means that $\mathbf{E}_x[A_\zeta(f_x)] < \infty$ implies $\mathbf{P}_x\{A_\zeta(f_x) < \infty\} = 1$, which is obvious.
□

**Remark 5.** *In accord with (11) and (12) we are using only the fact that the process is continuous when it hits points from below. Therefore, Theorem 3 is valid also when $Y$ is a spectrally negative Lévy process. In this case, we recall the result in Bertoin [1, p. 212] which states that the Green kernel associated with $Y$ when killed at rate $q > 0$ exists and has the form*

$$G_q^Y(x\,,y) = \Phi'(q)e^{-\Phi(q)(y-x)}, \qquad y > x, \tag{14}$$

*where $\Phi$ is an appropriately-defined inverse to the Laplace exponent of $Y$. If, in addition, $\lim_{t\to\infty} Y_t = \infty$ then $\Phi(0) = 0$ and $\Phi'(0) > 0$. In this case we can choose $q \equiv 0$ and find that $G_0^Y(x\,,y)$ is a constant for all $y > x$. Consequently, Theorem 3(iii) implies that*

$$\int_0^\infty f(Y_s)\,ds < \infty \ \textit{a.s.} \quad \Leftrightarrow \quad \int^\infty f(x)\,dx < \infty.$$

*This condition is derived in Erickson and Maller [10] under a more restrictive condition on $f$. However, Erickson and Maller treat more general Lévy processes than those considered here.*

# 3 Reminder on exit boundaries

Since the exit condition (8) plays a crucial rôle in our approach we discuss here shortly two proofs of this condition, thus making the paper as self-contained as possible.
Let $Y$ be an arbitrary regular diffusion living on the interval $I$ with the end points $l$ and $r$. The scale function of $Y$ is denoted by $S$ and the speed measure by $m$. It is also assumed that the killing measure of $Y$ is identically zero. Recall the definition due to Feller

$$r \text{ is exit} \quad \Leftrightarrow \quad \int^r S(d\alpha)\int^\alpha m(d\beta) < \infty. \tag{14}$$

Note that by Fubini's theorem

$$\int^r S(d\alpha)\int^\alpha m(d\beta) = \int^r m(d\beta)(S(r) - S(\beta)),$$

and, hence, $S(r) < \infty$ if $r$ is exit. Moreover, if $r$ is exit then $H_r < \infty$ with positive probability.

**3.1.** We give now some details of the proof of (14) following closely Kallenberg [13] (see also Breiman [4]). For $l < a < b < r$ let $H_{ab} := \inf\{t : Y_t = a \text{ or } b\}$. Then for $a < x < b$

$$\mathbf{E}_x\,(H_{ab}) = \int_a^b \widehat{G}_0^Y(x,z)\,m(dz), \tag{15}$$

where $\widehat{G}_0^Y$ is the (symmetric) Green kernel of $Y$ killed when it exits $(a,b)$, i.e.,

$$\widehat{G}_0^Y(x,z) = \frac{(S(b)-S(x))(S(y)-S(a))}{S(b)-S(a)} \qquad x \geq y.$$

If $r$ is exit there exists $h > 0$ such that $\mathbf{P}_x(H_r < h) > 0$ for any fixed $x \in (a,r)$. Using this property it can be deduced (see [13, p. 377]) that for any $a \in (l\,,r)$

$$\mathbf{E}_x\,(H_{ar}) < \infty,$$

which, from (15), is seen to be equivalent with (14).

**3.2.** Another proof of (14) can be found in Itô and McKean [11, p. 130]). To also present this briefly recall first the formula

$$\mathbf{E}_x(\exp(-\lambda\,H_b)) = \frac{\psi_\lambda(x)}{\psi_\lambda(b)}, \tag{16}$$

where $\lambda > 0$ and $\psi_\lambda$ is an increasing solution of the generalized differential equation

$$\frac{d}{dm}\frac{d}{dS}u = \lambda u. \tag{17}$$

Letting $b \to r$ in (16) it is seen that

$$r \text{ is exit} \quad \Leftrightarrow \quad \lim_{b\to r} \psi_\lambda(b) < \infty.$$

Let $\psi_\lambda^+$ denote the (right) derivative of $\psi_\lambda$ with respect to $S$. Since $\psi_\lambda$ is increasing it holds that $\psi_\lambda^+ > 0$. The fact that $\psi_\lambda$ solves (17) yields for $z < r$

$$\psi_\lambda^+(r) - \psi_\lambda^+(z) = \lambda \int_z^r \psi_\lambda(a)\,m(da).$$

In particular, $\psi_\lambda^+$ is increasing and $\psi_\lambda^+(r) > 0$. Hence, assuming now that $\psi_\lambda(r) < \infty$ we obtain $S(r) < \infty$, and, further,

$$\begin{aligned}\lambda\,\psi_\lambda(z) \int_z^r S(d\alpha) \int_z^\alpha m(d\beta) &\leq \lambda \int_z^r S(d\alpha) \int_z^\alpha \psi_\lambda(\beta) m(d\beta) \\ &= \int_z^r S(d\alpha)\left(\psi_\lambda^+(\alpha) - \psi_\lambda^+(z)\right) \\ &= \psi_\lambda(r) - \psi_\lambda(z) - \psi_\lambda^+(z)\,(S(r) - S(z)) < \infty,\end{aligned}$$

which yields the condition on the right hand side of (14). Assume next that the condition on the right hand side of (14) holds, and consider for $z < \beta$

$$0 \leq (\psi_\lambda(\beta))^{-1}\left(\psi_\lambda^+(\beta) - \psi_\lambda^+(z)\right) = (\psi_\lambda(\beta))^{-1} \int_z^\beta \psi_\lambda(\alpha) m(d\alpha).$$

Integrating over $\beta$ gives

$$\begin{aligned}\log(\psi_\lambda(r)) - \log(\psi_\lambda(z)) - \psi_\lambda^+(z)\int_z^r (\psi_\lambda(\beta))^{-1}\, S(d\beta)\\ &= \int_z^r S(d\beta)\,(\psi_\lambda(\beta))^{-1}\int_z^\beta \psi_\lambda(\alpha) m(d\alpha)\\ &\le \int_z^r S(d\beta)\int_z^\beta m(d\alpha) < \infty,\end{aligned}$$

which implies that $\psi_\lambda(r) < \infty$, thus completing the proof.

# 4 An example

As an application of Theorem 2, we consider a Bessel process with dimension parameter $\delta > 2$. Let $R$ denote this process. It is well known that $\lim_{t\to\infty} R_t = \infty$ and that $R$ solves the SDE

$$dR_t = dW_t + \frac{\delta - 1}{2R_t}\, dt,$$

where $W$ is a standard Brownian motion. Here the function $B^R$ (cf. (4)) takes the form

$$B^R(v) = (\delta - 1)\log v,$$

and, consequently,

$$\begin{aligned}\int^\infty dv\, f(v)\, \mathrm{e}^{B^R(v)} \int_v^\infty du\, \mathrm{e}^{-B^R(u)}\\ &= \int^\infty dv\, f(v)\, v^{\delta-1}\int_v^\infty du\, u^{-\delta+1}\\ &= \int^\infty dv\, f(v)\, v^{\delta-1}\,\frac{1}{\delta-2}\, v^{-\delta+2}\end{aligned}$$

leading to

$$\int_0^\infty f(R_t)\, dt < \infty \quad \text{a.s.} \quad \Leftrightarrow \quad \int^\infty u\, f(u)\, du < \infty.$$

Another way to derive this condition is via local times and Jeulin's lemma [12]. Indeed, by the occupation time formula and Ray–Knight theorem for the total local times of $R$ (see, e.g. [21, Theorem 4.1 p. 52]) we have

$$\begin{aligned}\int_0^\infty f(R_s)\, ds &\overset{\text{(d)}}{=} \int_0^\infty f(a)\,\frac{\rho_{a^\gamma}}{\gamma\, a^{\gamma-1}}\, da\\ &= \frac{1}{\gamma}\int_0^\infty a\, f(a)\,\frac{\rho_{a^\gamma}}{a^\gamma}\, da\end{aligned}$$

where $\delta = 2 + \gamma$ and $\rho$ is a squared 2-dimensional Bessel process. Using the scaling property, it is seen that the distribution of the random variable $\rho_{a^\gamma}/a^\gamma$ does not depend on $a$. Hence, we obtain by Jeulin's lemma that if the function $a \mapsto a\, f(a)$, $a > 0$, is locally integrable on $[0\,,\infty)$ then

$$\int_0^\infty f(R_s)\, ds < \infty \quad \Leftrightarrow \quad \int^\infty a\, f(a)\, da < \infty. \tag{18}$$

The same argument allows us to recover the result in [18], that is,

$$\int_0^\infty g(W_s^{(\mu)})\,ds < \infty \quad \Leftrightarrow \quad \int^\infty g(x)\,dx < \infty. \tag{19}$$

where $g$ is any non-negative locally integrable function and $W^{(\mu)}$ denotes a Brownian motion with drift $\mu > 0$. To see this, write $g(x) = f(\mathrm{e}^x)$ and use Lamperti's representation

$$\exp(W_s^{(\mu)}) = R^{(\mu)}_{A_s^{(\mu)}}, \quad s \geq 0,$$

where

$$A_s^{(\mu)} = \int_0^s du\ \exp(2W_u^{(\mu)}),$$

and $R^{(\mu)}$ is a Bessel process with dimension $d = 2(1+\mu)$ starting from 1, we obtain (cf. [17, Remark 3.3.(3)])

$$\int_0^\infty f(\exp(W_s^{(\mu)}))\,ds = \int_0^\infty \left(R_u^{(\mu)}\right)^{-2} f(R_u^{(\mu)})\,du \quad \text{a.s.},$$

and, in order to get (19) it now only remains to use the equivalence (18).
We wish to underline the fact that in Theorem 3 it is assumed that the function $f$ is locally bounded whereas the approach via Jeulin's lemma, which we developed above, demands only local integrability.

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