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FRACTIONAL BROWNIAN MOTION AND THE MARKOV PROPERTY

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Abstract

Fractional Brownian motion belongs to a class of long memory Gaussian processes that can be represented as linear functionals of an infinite dimensional Markov process. This leads naturally to :

- *An efficient algorithm to approximate the process.*
- *An ergodic theorem which applies to functionals of the type* $\int_0^t \phi(V_h(s))\,ds$ *where* $V_h(s) = \int_0^s h(s-u)\,dB_u$ *. and* B *is a real Brownian motion.*

1 Introduction

Fractional Brownian motion of Hurst parameter $H \in (0,1)$ is a real centered Gaussian process $(W_t^H\,; t \geq 0)$ of covariance

$$\mathbb{E}\left[W_t^H W_s^H\right] = \frac{c(H)}{2}(s^{2H} + t^{2H} - |t-s|^{2H}) \qquad (s,t \geq 0)\,. \tag{1}$$

One easily sees that it starts from zero, $W_0^H = 0$ almost surely , it has stationary increments $(W_{t+s}^H - W_s^H\,; t \geq 0) \stackrel{d}{=} (W_t^H\,; t \geq 0)$ since

$$\mathbb{E}\left[(W_t^H - W_s^H)^2\right] = c(H)|t-s|^{2H} \qquad (s,t \geq 0)\,. \tag{2}$$

Fractional Brownian motion enjoys the following scaling (self-similarity) property

$$(\frac{1}{a^H} W^H_{at}\,; t \geq 0\,) \stackrel{d}{=} (W^H_t\,; t \geq 0\,) \qquad (a > 0)\,. \tag{3}$$

The increments are independent if and only if $H = \frac{1}{2}$ (Brownian motion). If $H < \frac{1}{2}$, they are negatively correlated, and if $H > \frac{1}{2}$ they are positively correlated. Furthermore it is a long memory process, since, for $H \neq \frac{1}{2}$, the covariance between far apart increments decrease to zero as a power law:

$$\mathrm{Cov}(W^H(t) - W^H(0), W^H(t+u) - W^H(u)) \sim_{u \to +\infty} constant \times u^{2H-2}\,.$$

Fractional Brownian motion has been used as a model of signal or noise in various domains: geophysical data [10, 3], communication processes [12, 11], see also the references therein. Therefore it is important to provide good numerical approximating schemes.

Problem I: approximation of fractional Brownian motion

We shall not linger on the first (naive) idea that may cross our mind. We simulate an n-dimensional Gaussian random vector with the same disribution as $(W^H_{s_1}, \ldots, W^H_{s_n})$ and we perform a linear or spline interpolation. This amounts to computing the square root of the $n \times n$ covariance matrix and is computationally inefficient.
A second solution is to use the representation introduced by Mandelbrot and Van Ness [13]: if $\alpha = H + \frac{1}{2}$,

$$W^H_t = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{t} ((t-u)^{\alpha-1} - (-u)_+^{\alpha-1})\, dB_u$$

where B is a standard Brownian motion on the real line. Without any loss in generality, we can say that the problem is to approximate the finite, but long, memory part:

$$V(t) = V_h(t) = \int_0^t h(t-s)\, dB_s$$

where $(B_t\,; t \geq 0\,)$ is a standard Brownian motion. The naive way to do this is to fix a small step $|\Delta_n| = \frac{1}{n}$ for the partitions we use

$$\Delta_n = 0 < s_1 = \frac{1}{n} < s_2 = \frac{2}{n} < \cdots\,.$$

Hence, if $t = r/n$, then $V(t)$ is approximated by

$$\begin{aligned} V(t) &\simeq \sum h(t-s_i)\,(B_{s_{i+1}} - B_{s_i}) \\ &= \frac{1}{\sqrt{n}} \sum h(t-s_i)\,\xi_i \\ &= \frac{1}{\sqrt{n}}(h(\frac{1}{n})\xi_r + h(2/n)\xi_{r-1} + \cdots) \end{aligned}$$

where $\xi_i = \sqrt{n}(B_{s_{i+1}} - B_{s_i})$ are independent identically distributed standard normal random variables.
For the next step, $t + \Delta t = t + \frac{1}{n} = (r+1)/n$, we have

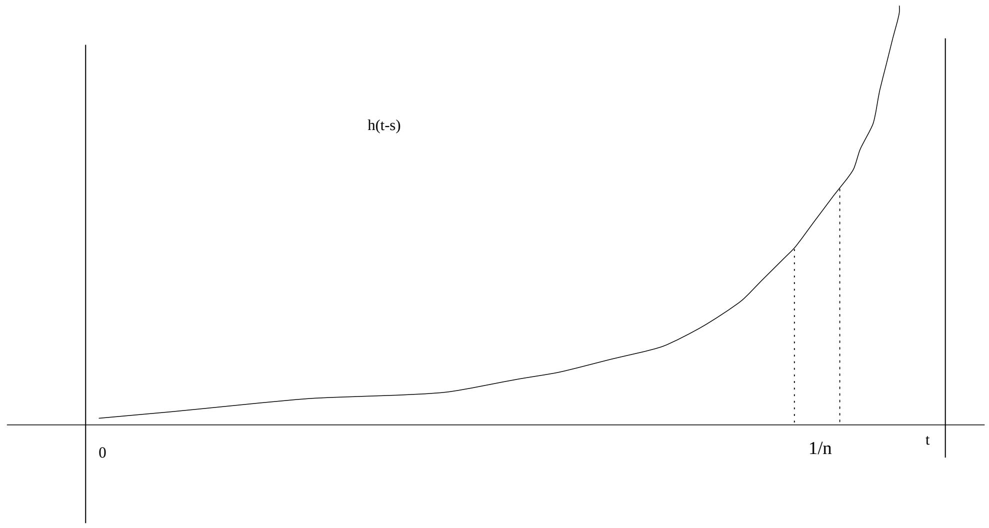

Figure 1: The function $s \to h(t-s)$ when $h(u) = \frac{1}{\Gamma(\alpha)} u^{\alpha-1} 1_{(u>0)}$

$$V(t+\Delta t) \simeq \frac{1}{\sqrt{n}}(h(1/n)\xi_{r+1} + h(2/n)\xi_r + \cdots)$$

Therefore, all we have to do is to compute the $h(i/n)$, generate the ξ_i, and keep them all in memory for the next step.
Unfortunately, if we want to keep the approximation error low (that is n big), this amounts to a lot of data to keep in memory.
The natural idea is to impose a threshold, that is to compute $h(i/n)$ and generate ξ only for $1 \le i \le M_n$. This may work well for smooth functions h, but not for

$$h(u) = \frac{1}{\Gamma(\alpha)} u^{\alpha-1} 1_{(u>0)}$$

where $\frac{1}{2} < \alpha < 1$, since h explodes near zero, and thus our partition is not dense enough near t !

Unfortunately this example is important since it is in essence fractional Brownian motion of index $H = \alpha - \frac{1}{2}$.
In [7],Carmona, Coutin and Montseny derived an approximation scheme from a representation formula inspired by the diffusive input-output model of fractional integration developed by Audounet, Montseny and Mbodje [1, 2].
More precisely Fubini's stochastic theorem implies that if h has the "spectral" representation

$$h(u) = \int e^{-ux} \mu(dx) \qquad (u > 0)$$

where μ is a positive measure on $(0,+\infty)$, finite on compact sets, and such that $\int(1 \wedge x^{-\frac{1}{2}})\,\mu(dx) < +\infty$, then

$$V_h(t) = \int_0^\infty Y_t^x \, \mu(dx)$$

where

$$Y_t^x = \int_0^t e^{-x(t-s)}\, dB_s \, .$$

Observe that for fixed $x \geq 0$, $(Y_t^x \, ; t \geq 0\,)$ is an Ornstein-Uhlenbeck process of parameter x, that is a Gaussian semi-martingale Markov process solution of the stochastic differential equation

$$dY_t^x = dB_t - x\, Y_t^x \, dt \, .$$

Therefore, $V_h = \langle \mathcal{Y} \rangle_\mu$ is a linear functional of the infinite dimensional Markov process

$$\mathcal{Y} = (\mathcal{Y}_t \, ; t \geq 0\,)\,, \qquad \mathcal{Y}_t = (Y_t^x, x \geq 0)\,.$$

The idea behind the approximation scheme is to first do a spatial approximation

$$V_h(t) \simeq V_h^\pi(t) = \sum_\pi c_i \, Y_t^{\eta_i} \,,$$

where $\pi = \{\eta_i\}_i$ is a chosen finite partition converging to identity, and c_i are coefficients depending only on h. This approximation is uniformly good with respect to time, that is $\sup_{t \leq T} |V_h(t) - V_h^\pi(t)|$ goes to zero when π grows.
The second step is the time discretization, which can be performed in the following way :

$$Y^{\eta_i}(t + \Delta t) = Y^{\eta_i}(t) + \Delta B_t - \eta_i Y^{\eta_i}(t)\,. \tag{4}$$

This second step is a classical one : see e.g. the book of Bouleau and Lepingle [5], chapter V, section B, pages 271–278, where it is stated that to get a precision of ϵ, one has to consider at most a constant C times $N = \log(1/\epsilon)$ time points. Nevertheless, when one needs to do the whole family of approximations (4), one needs to control precisely the dependence of the constant $C = C(\eta_i, T)$ with respect to the space and time variables. The classical method, combined with Gronwall's lemma, gives an exponential upper bound for the L^p-norm of the approximation error $C(\eta, T) \leq c_1(T,p)\eta \exp(c_2(T,p)\eta^p)$. Therefore, we have to use another time discretization scheme for which we have polynomial constants (see e.g. [6] or [8]).
The first step amounts to approximate a linear functional of an infinite dimensional Markov process by a linear functional of a finite dimensional Markov process. The secret of the success of this method is that for a large class of functions h, this dimension can be kept low : for a precision of ϵ, the cardinal of the partition $\pi(\epsilon)$ is roughly

$$\mathcal{N}(\epsilon) = |\pi(\epsilon)| \sim c\sqrt{\frac{1}{\epsilon}}\, \log(\frac{1}{\epsilon})\,.$$

We do not say that this approximation method is the best, but we want to stress the fact that the Markov property of the underlying process $\mathcal{Y}$ allows an algorithm that is very economic in memory allocation. You only have to store $\mathcal{N}(\epsilon)$ values at each time step, then simulate one Gaussian random variable, and then update these values for the next step (according to rule (4)).

The example of fractional Brownian motion

We can take advantage not only of the Gaussian nature of the processes Y^x, but also of their semi-martingale nature (via Itô's formula). We have been able to give exact L^2 and almost sure convergence rates for fractional Brownian motion of index $H = \alpha - \frac{1}{2} \in (0, \frac{1}{2})$.
Indeed, to obtain the precision

$$\sup_{t \leq T} \|V_h(t) - V_h^{\pi}(t)\|_{L^2} \leq \epsilon ,$$

we may choose the compact

$$K(\epsilon) \simeq \left[\epsilon^{\frac{1}{1-\alpha}}, (\frac{1}{\epsilon})^{\frac{2}{2\alpha+3}} \right]$$

and a geometric partition

$$\pi(\epsilon) = \left\{ r^{-m}, r^{-m+1}, \ldots, r^{n} \right\}$$

of ratio $r = r(\epsilon) = 1 + c(\alpha)\sqrt{\epsilon}$.
Furthermore, if ϵ_n decreases fast enough to 0,

$$\epsilon_n = o\Big((\log n)^{-\frac{2\alpha+3}{8(2-\alpha)}} \Big) \tag{5}$$

then, we have the uniform almost sure convergence

$$\sup_{t \leq T} \left| V_h(t) - V_h^{\pi(\epsilon_n)}(t) \right| \underset{n \to \infty}{\longmapsto} 0 \qquad \text{almost surely} .$$

Theoretical results (see [7] or [8]) show then that to obtain a precision of $\epsilon = 10^{-3}$, we need to consider a few hundred points. Practical results show that forty points are enough. Furthermore, we have been able to check that (almost surely) the numerical approximations stick exponentially fast to the actual random process (and this happens uniformly on compact sets), when ϵ_n goes faster to 0 than indicated in (5).

Problem II : an ergodic theorem

It is well known (Birkhoff's ergodic theorem) that for a stationary process U, we have the convergence, almost sure and in L^1,

$$\frac{1}{t} \int_0^t \phi(U(s))\, ds \underset{t \to \infty}{\longmapsto} \mathbb{E}[\phi(U(0))] , \tag{6}$$

as soon as ϕ is an integrable Borel function: $\mathbb{E}[|\phi(U(0))|] < +\infty$.
Of course, we first have to check the ergodicity, that is that the invariant σ-field is almost surely trivial, which is not so easy to do in general (see e.g. Rozanov [14], page 25, Theorem 6.1).
But what we actually want to do is to answer the following question. Given $(V(t)\,; t \geq 0)$ a real Gaussian process, a priori non stationary, such that $V^r = (V(t+r)\,; t \geq 0)$ converges in distribution to the stationary process U. We want to show that, as soon as ϕ is an integrable Borel function: $\mathbb{E}[|\phi(U(0))|] < +\infty$, we have the convergence, almost sure and in L^1,

$$\frac{1}{t} \int_0^t \phi(V(s))\, ds \underset{t \to \infty}{\longmapsto} \mathbb{E}[\phi(U(0))] . \tag{7}$$

To be more precise, given $h \in L^2(\mathbb{R}_+)$ and $(B_t, t \in \mathbb{R})$ a standard Brownian motion, we let

$$V(t) = \int_0^t h(t-s)\, dB_s\,, \quad \text{and} \quad U(t) = \int_{-\infty}^t h(t-s)\, dB_s\,. \tag{8}$$

We shall show in Proposition 5 that (7) holds if ϕ is Lipschitz and h is the Laplace transform

$$h(u) = \int e^{-ux}\, \mu(dx)\,, \tag{9}$$

of a positive measure μ, finite on compact sets and such that

$$\int \frac{1}{\sqrt{x}}\, \mu(dx) < +\infty\,. \tag{10}$$

2 The Markov and ergodic properties of the underlying infinite dimensional process

Let $B = (B_t\,; t \geq 0\,)$ be a standard Brownian motion. Given $x \geq 0$, $Y^x = (Y_t^x\,; t \geq 0\,)$ is the Ornstein-Uhlenbeck process driven by B of parameter x (starting from 0):

$$Y_t^x = \int_0^t e^{-x(t-s)}\, dB_s\,.$$

It is also the solution of the stochastic differential equation

$$dY_t^x = dB_t - xY_t^x\, dt\,.$$

For every $t \geq 0$, we consider $\mathcal{Y}_t = (Y_t^x, x > 0)$; it is a centered Gaussian process of covariance

$$\Gamma_{\mathcal{Y}_t}(x,y) = \frac{1 - e^{-t(x+y)}}{x+y} \qquad (x, y > 0)\,.$$

The natural candidate for a limiting process is $\mathcal{Y}_\infty$ the centered Gaussian process of covariance

$$\Gamma_{\mathcal{Y}_\infty}(x,y) = \frac{1}{x+y} \qquad (x, y > 0)\,.$$

Proposition 1. *Suppose that $\int x^{-\frac{1}{2}} d\mu(x) < +\infty$. Then, the process $\mathcal{Y} = (\mathcal{Y}_t\,; t \geq 0\,)$ is an infinite dimensional Markov process with values in $L^1(\mu)$; $\mathcal{Y}_\infty$ is a random variable with values in $L^1(\mu)$.*

Proof. Since $E = L^1(\mu)$ is a separable Banach space, the Borel sigma-fields of weak and strong topologies on E coincide with

$$\mathcal{E} = \sigma(\chi : \chi \in L^1(\mu)^*) = \sigma(\chi : \chi \in L^\infty(\mu))\,,$$

(see e.g. Borkar [4], Lemma 2.5.2).
Therefore we first have to check that $\mathcal{Y}_t$ takes its values in $L^1(\mu)$. This is the case since

$$\mathbb{E}\left[\int |Y_t^x|\, d\mu(x)\right] = c \int d\mu(x) \left(\frac{1 - e^{-2xt}}{2x}\right)^{\frac{1}{2}} < +\infty\,.$$

Similarly, $\mathcal{Y}_\infty$ takes its values in $L^1(\mu)$ since

$$\mathbb{E}\left[\int |Y^x_\infty|\, d\mu(x)\right] = c\int d\mu(x)\left(\frac{1}{2x}\right)^{\frac{1}{2}} < +\infty\,.$$

Now we have to check that for every $\chi \in L^\infty(\mu)$, $\langle \mathcal{Y}_t, \chi\rangle$ is a random variable. This is indeed the case since it is a centered Gaussian random variable living in the Gaussian space generated by the Brownian motion B, with covariance, given by an application of Fubini's Theorem,

$$\begin{aligned}
\mathbb{E}\left[\left(\int \chi(x) Y^x_t\, \mu(dx)\right)^2\right] &= \int\int \chi(x)\chi(y)\mu(dx)\mu(dy)\mathbb{E}\left[Y^x_t Y^y_t\right] \\
&= \int\int \chi(x)\chi(y)\mu(dx)\mu(dy)\frac{1-e^{-t(x+y)}}{x+y} \\
&\underset{t\to\infty}{\longmapsto} \int\int \chi(x)\chi(y)\frac{\mu(dx)\mu(dy)}{x+y}
\end{aligned}$$

This last integral is finite, Indeed, since $ab \le \frac{1}{2}(a^2+b^2)$,we can majorize it by

$$\cdots \le \|\chi\|^2_{L^\infty}\int\int \frac{1}{2}\frac{\mu(dx)\mu(dy)}{\sqrt{x}\sqrt{y}} = \frac{1}{2}\|\chi\|^2_{L^\infty}\left(\int \frac{\mu(dx)}{\sqrt{x}}\right)^2 < +\infty\,.$$

Therefore, $\langle \chi, \mathcal{Y}_t\rangle$ converges in distribution to a centered Gaussian variable with variance

$$\int\int \chi(x)\chi(y)\frac{\mu(dx)\mu(dy)}{x+y}\,,$$

and this is easily seen to be the variance of $\langle \chi, \mathcal{Y}_\infty\rangle$.
The strong Markov property is quite easy to prove. Indeed, let T be an almost surely finite stopping time (with respect to $(\mathcal{F}_t\,; t\ge 0)$ the completed, hence right-continuous, filtration generated by the Brownian motion B). Then, $\bar{B} = (\bar{B}_t = B_{t+T} - B_T\,; t \ge 0)$ is a standard Brownian motion independent of $(B_u, u \le T)$, and $(Y^x_{t+T},\,; t\ge 0)$ is the solution of the stochastic differential equation

$$dY^x_{t+T} = d\bar{B}_t - x\,Y^x_{t+T}\,dt\,.$$

Therefore,

$$Y^x_{t+T} = e^{-xt}Y^x_T + \int_0^t e^{-x(t-s)}\,d\bar{B}_s\,.$$

Hence, for any bounded measurable function ψ,

$$\mathbb{E}\left[\psi(Y_{t+T}\,; t\ge 0) \mid \mathcal{F}_T\right] = \mathbb{E}_{\mathcal{Y}_T}\left[\psi(Y_t\,; t \ge 0)\right],$$

with the family $(\mathbb{P}_y, y \in L^1(\mu))$ of probabilities defined by

$$\mathbb{E}_y\left[\psi(\mathcal{Y}_t\,; t\ge 0)\right] = \mathbb{E}\left[\psi((Y^{x,y(x)}, x \ge 0)\,; t \ge 0)\right],$$

where $(Y^{x,y}_t\,; t\ge 0)$ denotes the Ornstein-Uhlenbeck process driven by B, of parameter x, starting from y:

$$Y^{x,y}_t = e^{-xt}y + \int_0^t e^{-x(t-u)}\,dB_u\,.$$

□

Proposition 2. *Suppose that μ is a sigma-finite measure on $(0,+\infty)$ such that,*

$$\int \mu(dx)\, x^{-\frac{1}{2}} < +\infty\,.$$

Then, for every $y \in L^1(\mu)$, under $\mathbb{P}_y$, we have the convergence in distribution

$$\mathcal{Y}_t \underset{t\to\infty}{\overset{d}{\longmapsto}} \mathcal{Y}^\infty\,.$$

Proof. We shall first prove the convergence of characteristic functions, since this will characterize the distribution of a possible limit (see [4]), and then prove the tightness of the family of distributions of $(\mathcal{Y}_t\,; t\geq 0\,)$.
The characteristic function is defined by

$$\phi_{\mathcal{Y}_t}(\chi) = \mathbb{E}_y\left[e^{i\langle\chi,\mathcal{Y}_t\rangle}\right] \qquad (\chi \in L^\infty)\,.$$

Observe that $\langle\chi,\mathcal{Y}_t\rangle$ is, under $\mathbb{P}_y$, a Gaussian random variable of mean

$$\int \chi(x)y(x)e^{-xt}\,\mu(dx) \underset{t\to\infty}{\longmapsto} 0 \quad \text{by dominated convergence}$$

and of variance

$$\begin{aligned}\mathbb{E}\left[\left(\int \chi(x)Y_t^x\,\mu(dx)\right)^2\right] &= \int\int \chi(x)\chi(y)\mu(dx)\mu(dy)\mathbb{E}\left[Y_t^xY_t^y\right]\\ &= \int\int \chi(x)\chi(y)\mu(dx)\mu(dy)\frac{1-e^{-t(x+y)}}{x+y}\\ &\underset{t\to\infty}{\longmapsto} \int\int \chi(x)\chi(y)\frac{\mu(dx)\mu(dy)}{x+y}\end{aligned}$$

Therefore, $\langle\chi,\mathcal{Y}_t\rangle$ converges in distribution to a centered Gaussian variable with variance

$$\int\int \chi(x)\chi(y)\frac{\mu(dx)\mu(dy)}{x+y}\,,$$

and this is easily seen to be the variance of $\langle\chi,\mathcal{Y}_\infty\rangle$.
We shall now prove the tightness of the family of laws of the processes $(\mathcal{Y}_t, t\geq 0)$. We need to introduce a little topology: we consider that for every $t>0$, $\mathcal{Y}_t$ is a random variable taking its values in $L^1(\mu)$ endowed with the weak topology $\sigma(L^1,L^\infty)$; this is the coarsest topology for which the functions

$$y \to \langle\chi, y\rangle \quad (\chi \in L^\infty(\mu))$$

are continuous.
Let $\phi(x) = \frac{1}{\sqrt{x}}$. Given $M>0$, Lemma 4 implies that, given $p>1$, if $1 = 1/p + 1/q$, then

$$K = \left\{ y \in L^1 : \int \frac{|y|^p}{\phi^{p/q}}\,d\mu \leq M \right\}$$

is a relatively compact set.

Let us notice that, by construction, under $\mathbb{P}_y$ the law of $\mathcal{Y}_t$ is the law , under $\mathbb{P}_0$ of $\mathcal{Y}_t + (x \to e^{-xt}y(x))$. Since this last function converges weakly to the null function, we can suppose $y = 0$, without any loss in generality. We have the sequence of inequalities,

$$\begin{aligned}
\mathbb{P}_0(\mathcal{Y}_t \notin K) &= \mathbb{P}_0\Big(\int |Y_t^x|^p \frac{1}{\phi(x)^{p/q}}\,\mu(dx) > M\Big)\\
&\le \frac{1}{M}\mathbb{E}_0\Big[\int |Y_t^x|^p \frac{1}{\phi(x)^{p/q}}\,\mu(dx)\Big]\\
&\le \frac{c}{M}\int \Big(\frac{1-e^{-2xt}}{2x}\Big)^{p/2}\frac{1}{\phi(x)^{p/q}}\,\mu(dx)\\
&\le \frac{c}{M}\int \Big(\frac{1}{2x}\Big)^{p/2}\frac{1}{\phi(x)^{p/q}}\,\mu(dx)\,.
\end{aligned}$$

This last quantity goes uniformly to zero as $M \to \infty$, since the integral is finite.

□

Proposition 3. *Assume that μ is a sigma finite measure on $(0, +\infty)$ such that $\int x^{-\frac{1}{2}}\,d\mu(x) < +\infty$, and let ν_∞ denote the law of $\mathcal{Y}_\infty$. Let f be a function depending only on a finite number of coordinates. Then*

1. *If $t > 0$, then $y \to P_t f(y)$ is continuous for the weak topology.*
2. *for every $y \in L^1(\mu)$, $\lim_{t\to\infty} P_t f(y) = \nu_\infty(f)$.*

Consequently, ν_∞ is an invariant probability measure for the semi group $(P_t\,; t \ge 0)$ of $(\mathcal{Y}_t\,; t \ge 0)$; that is, for every bounded (or positive) measurable function g, $\nu_\infty(P_t g) = \nu_\infty(g)$.

Proof. Indeed, we have, for a bounded measurable function $g : \mathbb{R}^n \to \mathbb{R}$, and $\{\chi_i, 1 \le i \le n\} \subset L^\infty(\mu)$

$$f(y) = g(\langle \chi_i, y\rangle, 1 \le i \le n)\,.$$

Hence,

$$\begin{aligned}
P_t f(y) &= \mathbb{E}_y\left[g(\langle \chi_i, \mathcal{Y}_t\rangle, 1 \le i \le n)\right]\\
&= \mathbb{E}_0\left[g(\left\langle \chi_i, y(.)e^{-t.}\right\rangle + \langle \chi_i, \mathcal{Y}_t\rangle, 1 \le i \le n)\right].
\end{aligned}$$

But we know that if $X = (X_1, \dots, X_n)$ is a non degenerate centered Gaussian vector, then we have the absolute continuity relationship

$$\mathbb{E}[g(z + X)] = \mathbb{E}[\Lambda(z)g(X)] \qquad (z \in \mathbb{R}^n)\,,$$

with Λ a continuous bounded function. Applying this to $(\langle \chi_i, \mathcal{Y}_t\rangle, 1 \le i \le n)$, for $t > 0$, yields

$$P_t f(y) = \mathbb{E}\left[\Lambda(\left\langle \chi_i, y(.)e^{-t.}\right\rangle, 1 \le i \le n)g(\langle \chi_i, \mathcal{Y}_t\rangle, 1 \le i \le n)\right].$$

Now if then net $y_p \to y$ weakly in $L^1(\mu)$, then, for each i,

$$\left\langle \chi_i, y_p(.)e^{-t.}\right\rangle \to \left\langle \chi_i, y(.)e^{-t.}\right\rangle$$

and $P_t f(y_p) \to P_t f(y)$. Therefore, if $t > 0$, then $P_t f$ is a continuous bounded function. Since, under $\mathbb{P}_y$, $\mathcal{Y}_t \overset{d}{\underset{t\to\infty}{\longmapsto}} \mathcal{Y}_\infty$, (see Proposition 2), we have

$$\nu_\infty(P_t f) = \lim_{s\to\infty} P_s(P_t f)(y) = \lim_{u\to\infty} P_u f(y) = \nu_\infty(f)\,.$$

We let $\mathcal{H}$ be the space of bounded measurable functions $g : L^1(\mu) \to \mathbb{R}$ such that $\nu_\infty(P_t g) = \nu_\infty(g)$. $\mathcal{H}$ is obviously a vector space; it is a monotone space since if $(g_n)_n$ is a sequence of functions in $\mathcal{H}$, $0 \le g_n \le 1$, and g_n increase to g, then, by monotone convergence, g is an element of $\mathcal{H}$. Since $\mathcal{H}$ contains constants and the class $\mathcal{C}$ of bounded functions depending only on a finite number of coordinates, then, by the monotone class theorem, $\mathcal{H}$ contains every bounded function measurable with respect to the sigma field $\sigma(\mathcal{C}) = \mathcal{E}$.

□

Lemma 4. *Let μ be a sigma-finite measure, and ϕ be a strictly positive ($\phi > 0$), integrable function. Then, for every $M \ge 0$, $p, q > 1$ such that $1 = 1/p + 1/q$,*

$$K = \left\{ f : \int \frac{|f|^p}{\phi^{p/q}}\, d\mu \le M \right\}$$

is a relatively compact set of $L^1(\mu)$ endowed with the weak topology $\sigma(L^1, L^\infty)$.

Proof. Following Theorem IV.8.9 of Dunford–Schwarz [9] we have to prove that

1. K is a bounded subset of L^1 .
2. If E_n is a sequence of measurable sets that decreases to $\emptyset$, then $\sup_{f\in K} \left| \int_{E_n} f\, d\mu \right| \underset{n\to\infty}{\longmapsto} 0$.

By Hölder's inequality, for f in K

$$\begin{aligned}\int 1_{E_n} |f|\, d\mu &\le \left\| \frac{f}{\phi^{1/q}} \right\|_{L^p(\mu)} \left\| 1_{E_n} \phi^{1/q} \right\|_{L^q(\mu)} \\ &\le M^{1/p} \left(\int 1_{E_n} \phi d\mu \right)^{1/q} .\end{aligned}$$

Taking $E_0 = \Omega$ the whole space, we see that K is bounded in L^1. The dominated convergence theorem gives that $\int 1_{E_n} \phi\, d\mu \underset{n\to\infty}{\longmapsto} 0$, so the convergence is uniform for f in K. □

Proposition 5. *Suppose that*

1. *μ is a sigma-finite measure on $(0, +\infty)$ such that*

$$\int \mu(dx)\, x^{-1/2} < +\infty\,.$$

2.

$$V_h(t) = \int_0^t h(t-u)\, dB_u\,,$$

where $(B_u, u \in \mathbb{R})$ is a standard Brownian motion, and h is the Laplace transform

$$h(u) = \int e^{-ux}\, d\mu(x) \qquad (u > 0)\,.$$

3. ϕ is a Lipschitz continuous function such that $\mathbb{E}[|\phi(N)|] < +\infty$ where N denotes a standard Gaussian random variable.

Then, h is a square integrable function on $(0, +\infty)$, and if $A^2 = \int_0^\infty h^2(u)\,du$, then, we have the almost sure convergence

$$\frac{1}{t}\int_0^t \phi\left(\frac{1}{A}V_h(s)\right) ds \underset{t\to\infty}{\longmapsto} \mathbb{E}[\phi(N)] .$$

Proof. By Fubini–Tonelli's theorem

$$\begin{aligned}\int_0^\infty h(u)^2\,du &= \int_0^\infty du\Big(\int e^{-xu}\,d\mu(x)\Big)\Big(\int e^{-yu}d\mu(y)\Big)\\ &= \int\int d\mu(x)\,d\mu(y)\int_0^\infty e^{-(x+y)u}\,du\\ &= \int\int d\mu(x)d\mu(y)\,\frac{1}{x+y} = \mathbb{E}\big[(\langle 1, \mathcal{Y}_\infty\rangle)^2\big] < +\infty .\end{aligned}$$

Without any loss in generality, we may assume that $\int_0^\infty h^2(u)\,du = 1$.
It is not very hard to use Kolmogorov's continuity criterion to construct, up to a null set, a bicontinuous version of the process $(Y_t^x, x > 0\,; t \geq 0)$. Therefore, the law $\mathbb{P}_y$ may be considered as a probability measure on the canonical space of continuous functions $w : \mathbb{R}_+ \to (L^1(\mu), \sigma(L^1, L^\infty)$. Since ν_∞ is an invariant measure for the semi group $(P_t\,; t \geq 0)$, we may consider the probability measure

$$\mathbb{P}_{\nu_\infty}(\bullet) = \int \nu_\infty(dy)\mathbb{P}_y(\bullet) .$$

Birkhoff's ergodic theorem states then that if F is a measurable function in $L^1(\mathbb{P}_{\nu_\infty})$ and if $G(w) = \mathbb{E}[F \mid \mathcal{I}](w)$ is a version of the conditional expectation of F with respect to the sigma-field of shift-invariant sets, then

$$\frac{1}{t}\int_0^t F(w(s+u), u \geq 0)\,ds \to G(w)$$

$\mathbb{P}_{\nu_\infty}$ almost surely and in $L^1(\mathbb{P}_{\nu_\infty})$. We let $F(w(u), u \geq 0) = \phi(\langle w(o)\rangle)$ to obtain that, for ν_∞ almost every $y \in L^1(\mu)$, almost surely, under $\mathbb{P}_y$,

$$\frac{1}{t}\int_0^t \phi(V_h(s))\,ds \to G(\mathcal{Y}_t\,; t \geq 0\,) .$$

Therefore, there exists $y \in L^1(\mu)$ and a bounded random variable $Z(\omega) = G(\mathcal{Y}_t + y(.)e^{-t.}\,; t \geq 0\,)$, such that, under $\mathbb{P}_0$, almost surely,

$$\frac{1}{t}\int_0^t \phi(V_h(s) + \langle y(.)e^{-s.}\rangle)\,ds \to Z .$$

To conclude our proof, we only have to prove that Z is almost surely constant. We shall do this by establishing that Z is a.s. equal to a random variable in the asymptotic sigma-field of the Brownian motion B.

Observe first that

$$\frac{1}{t}\int_0^t \phi(V_h(s))\,ds \to Z\,.$$

Indeed, since ϕ is Lipschitz continuous with, say, constant c,

$$\begin{aligned}\frac{1}{t}\int_0^t \left|\phi(V_h(s)+\langle y(.)e^{-s.}\rangle)-\phi(V_h(s))\right|ds &\le \frac{1}{t}c\int|y(x)|\int_0^t e^{-sx}\,dx\\ &= c\int d\mu(x)|y(x)|\frac{1-e^{-tx}}{tx}\end{aligned}$$

and, as t goes to $+\infty$, this converges to 0, by dominated convergence, since $1-e^{-z}\le z$ for $z\ge 0$.
We now fix $a>0$, and observe that

$$\frac{1}{t}\int_a^t \phi(V_h(s))\,ds \to Z\,.$$

Let us introduce the random variables, for $t\ge a$,

$$\begin{aligned}X_t &= \int_a^t \phi(V_h(s))\,ds\\ X_t^a &= \int_a^t \phi(\int_a^s h(s-u)\,dB_u)\,ds\,.\end{aligned}$$

Then, ϕ being Lipschitz with constant c,

$$\begin{aligned}\mathbb{E}[|X_t-X_t^a|] &\le c\int_a^t \mathbb{E}\left[\left|\int_0^a h(s-u)\,dB_u\right|\right]ds\\ &\le c'\int_a^t\left(\int_0^a h(s-u)^2\,du\right)^{\frac{1}{2}}ds\\ &\le c'\int_a^t\left(\int_0^a h(s-u)^2\,du\right)^{\frac{1}{2}}ds\\ &\le c'\left((t-a)\int_0^t dvh(v)^2\int ds1_{(s-a<v<s)}\right)^{\frac{1}{2}}\\ &\le c'\sqrt{a(t-a)}\left(\int_0^\infty h(v)^2\,dv\right)^{\frac{1}{2}} \qquad = c''\sqrt{a(t-a)}\,.\end{aligned}$$

Therefore, $\frac{1}{t}(X_t-X_t^a)$ converges in L^1 to 0, and we can find a sequence t_n increasing to $+\infty$ such that $\frac{1}{t_n}(X_{t_n}^a-X_{t_n})$ converges almost surely to 0.
Hence, $Z(\omega)=\lim\frac{1}{t_n}X_{t_n}^a$ a.s., is equal, almost surely, to a random variable which is measurable with respect to the sigma field $\sigma(B_u,u\ge a)$. Since $a>0$ is arbitrary, we obtain that Z is a.s. equal to a random measurable measurable with respect to the tail sigma field $\mathcal{T}=\cap_{a>0}\sigma(B_u,u\ge a)$. Since this sigma field is a.s. trivial, we get that Z is a.s. constant. $\square$

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