Elect. Comm. in Probab. **10** (2005), 105–112

ELECTRONIC COMMUNICATIONS in PROBABILITY

A UNIVERSALITY PROPERTY FOR LAST-PASSAGE PERCOLATION PATHS CLOSE TO THE AXIS

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Submitted 4 Oct 2004, *accepted in final form* 2 May 2005

AMS 2000 Subject classification: 60K35
Keywords: Last-passage percolation, universality, Brownian directed percolation.

Abstract

We consider a last-passage directed percolation model in $\mathbb{Z}_+^2$, with i.i.d. weights whose common distribution has a finite $(2+p)$th moment. We study the fluctuations of the passage time from the origin to the point $\big(n, \lfloor n^a \rfloor\big)$. We show that, for suitable a (depending on p), this quantity, appropriately scaled, converges in distribution as $n \to \infty$ to the Tracy-Widom distribution, irrespective of the underlying weight distribution. The argument uses a coupling to a Brownian directed percolation problem and the strong approximation of Komlós, Major and Tusnády.

1 Introduction

The concept of *universality class* plays a key role in statistical mechanics, making it possible to classify a huge variety of models and phenomena by means of well chosen scaling exponents. For instance, many growth models are expected to share similar properties which fall into the framework of the KPZ universality class – see for example the survey by Krug and Spohn [12].

In this note, we focus on the particular example of *directed last-passage percolation.* Let $\omega_i^{(r)}$, $i \geq 0, r \geq 1$ be i.i.d. random variables. We consider directed paths in the lattice $\mathbb{Z}_+^2$, each step of which increases one of the coordinates by 1. For $n \geq 0$, $k \geq 1$, the *(last-)passage time* to the point (n, k) is defined by

$$T(n,k) = \max_{\pi \in \Pi(n,k)} \left\{ \sum_{(i,r) \in \pi} \omega_i^{(r)} \right\}, \tag{1.1}$$

where $\Pi(n,k)$ is the set of directed paths from $(0,1)$ to (n,k). More precisely,

$$\Pi(n,k) = \Big\{ (z_1, z_2, \dots, z_{n+k}) \in \left(\mathbb{Z}_+^2\right)^{n+k} : z_1 = (0,1), \quad z_{n+k} = (n,k),$$
$$z_{j+1} - z_j \in \{(0,1),(1,0)\} \text{ for } 1 \le j \le n+k-1 \Big\}.$$

When the underlying weight distribution is exponential or geometric, scaling exponents for this model are rigorously known. The deviation from a straight line of the optimal path to the point (n,n) is of the order $n^{2/3}$ (corresponding to the exponent $\xi = 2/3$) [3, 10], while the fluctuations of the passage time $T(n,n)$ are of the order $n^{1/3}$ (corresponding to the exponent $\chi = 1/3$). In fact, one can give much more precise information: for the exponential distribution with mean 1, say, it is shown in [9] that

$$n^{-1/3}\big[T(n,n) - 4n\big] \to F_{\mathrm{TW}}, \tag{1.2}$$

where F_{TW} is the "Tracy-Widom" distribution, which also appears as the asymptotic distribution of the largest eigenvalue of a GUE random matrix. It is expected (but not yet proved) that the same scaling exponents, and indeed the same asymptotic distribution in (1.2), should hold for a wide class of underlying weight distributions.

In this note we give a universality result for the quantities $T\left(n, \lfloor n^a \rfloor\right)$ for $a < 1$. Thus, we are concerned with passage times to points which are asymptotically rather close to the horizontal axis, for a general class of underlying weight distribution. Our result is the following:

Theorem 1 *Suppose that* $\mathbb{E}\,|\omega_i^{(r)}|^p < \infty$ *for some* $p > 2$, *with* $\mu = \mathbb{E}\,\omega_i^{(r)}$, $\sigma^2 = \mathrm{Var}(\omega_i^{(r)})$. *Then for all* a *such that* $0 < a < \frac{6}{7}\left(\frac{1}{2} - \frac{1}{p}\right)$,

$$\frac{T\left(n, \lfloor n^a \rfloor\right) - n\mu - 2\sigma\, n^{\frac{1+a}{2}}}{\sigma n^{\frac{1}{2}-\frac{a}{6}}} \to F_{\mathrm{TW}} \tag{1.3}$$

in distribution. In particular, if the weight distribution has finite moments of all orders, then (1.3) holds for all $a \in (0, 3/7)$.

Heuristically the theorem can be understood as follows. As the optimal path goes from the origin to $(n, \lfloor n^a \rfloor)$, one can imagine that between each step upwards, the path typically takes on the order of n^{1-a} steps to the right. Thus it should behave as the optimal path from the origin to (n^a, n^a) in a last percolation model with Gaussian weights of variance n^{1-a}. On the renormalized scale the expected fluctuations are of order $(n^a)^{1/3}$. In this way, we recover the fluctuation exponent

$$\hat{\chi} = \frac{1-a}{2} + \frac{a}{3} = \frac{1}{2} - \frac{a}{6}\,.$$

This heuristic is made precise by coupling the discrete model with a Brownian directed percolation model for which the fluctuations have been explicitly computed. This is done using the strong approximation of a random walk by a Brownian motion due to Komlós, Major and Tusnády. (We note that this strong approximation has already been applied to similar last-passage percolation models by Glynn and Whitt [7]).

A different sort of universality result for paths near the axis in directed percolation models is given in [15]. By subadditivity, one has the convergence $n^{-1}T(n, \lfloor xn \rfloor) \to \gamma(x)$ a.s. and

in $\mathcal{L}_1$, for some function γ. Under the hypothesis of Theorem 1, it's shown that $\gamma(x) = \mu + 2\sigma\sqrt{x} + o(\sqrt{x})$ as $x \downarrow 0$.
The proof of Theorem 1 is given in the next section. In Section 3, we make some comments on related models and possible extensions.

2 Fluctuations of the passage-time

We first introduce the *Brownian directed percolation model*. Let $B_t^{(r)}$, $t \geq 0$, $r \geq 1$ be a sequence of independent standard Brownian motions. For $t > 0$, $k \geq 1$, define

$$\mathcal{U}(t,k) = \left\{(u_0, u_1, \ldots, u_k) \in \mathbb{R}^{k+1} : 0 = u_0 \leq u_1 \leq \cdots \leq u_k = t\right\},$$

and then let

$$L(t,k) = \sup_{\mathbf{u} \in \mathcal{U}(t,k)} \sum_{r=1}^{k} \left[B_{u_r}^{(r)} - B_{u_{r-1}}^{(r)}\right]. \tag{2.1}$$

One can rewrite the definition of $T(n,k)$ at (1.1) in an analogous way:

$$T(n,k) = \sup_{\mathbf{u} \in \mathcal{U}(n,k)} \sum_{r=1}^{k} \left[S_{\lfloor u_r \rfloor + 1}^{(r)} - S_{\lfloor u_{r-1} \rfloor}^{(r)}\right], \tag{2.2}$$

where $S_m^{(r)} = \sum_{i=0}^{m-1} \omega_i^{(r)}$.
The random variable $L(1,k)$ has the same distribution as the largest eigenvalue of a $k \times k$ GUE random matrix [5], [8], [17]. Hence in particular (e.g. [18])

$$k^{1/6}\left[L(1,k) - 2\sqrt{k}\right] \to F_{\mathrm{TW}}$$

in distribution, where F_{TW} is the Tracy-Widom distribution.
By Brownian scaling, $L(t,k)$ has the same distribution as $\sqrt{t}L(1,k)$. Using this we get, for any $0 < a \leq 1$,

$$\frac{L\left(n, \lfloor n^a \rfloor\right) - 2n^{\frac{1+a}{2}}}{n^{\frac{1}{2} - \frac{a}{6}}} \to F_{\mathrm{TW}} \tag{2.3}$$

in distribution.
Theorem 1 says that the same distributional limit as in (2.3) (in particular, with the same order of fluctuations) occurs for the law of $T(n, \lfloor n^a \rfloor)$, for a general underlying distribution of the weights $\omega_i^{(r)}$, if a is sufficiently small. We will use the following strong approximation result, which combines Theorem 2 of Major [13] and Theorem 4 of Komlós, Major and Tusnády [11]:

Proposition 2 *Suppose $\omega_i, i = 1, 2, \ldots$ are i.i.d. with $\mathbb{E}\,|\omega_i|^p < \infty$ for some $p > 2$, and with $\mathbb{E}\,\omega_i = 0$, $\mathrm{Var}(\omega_i) = 1$. Let $S_m = \sum_{i=0}^{m-1} \omega_i$, $m \geq 1$.*
Then there is a constant C such that for all $n > 0$, there is a coupling of the distribution of $(\omega_1, \ldots, \omega_n)$ and a standard Brownian motion B_t, $0 \leq t \leq n+1$ such that, for all $x \in [n^{1/p}, n^{1/2}]$,

$$\mathbb{P}\left(\max_{m=1,2,\ldots,n+1} |B_m - S_m| > x\right) \leq Cnx^{-p}. \tag{2.4}$$

Proof of Theorem 1:
We may assume that $\mu = 0$ and $\sigma^2 = 1$, so that we need to prove that

$$\frac{T\left(n, \lfloor n^a \rfloor\right) - 2n^{\frac{1+a}{2}}}{n^{\frac{1}{2}-\frac{a}{6}}} \to F_{\mathrm{TW}} \tag{2.5}$$

in distribution (for general μ and σ^2, one can obtain (1.3) from (2.5) after replacing ω by $(\omega - \mu)/\sigma$).
If $(\omega_i^{(r)})_{i,r}$ and $\left(B_t^{(r)}\right)_{t,r}$ are all defined on the same probability space, then from (2.1) and (2.2) we get

$$\begin{aligned}
&|T(n, \lfloor n^a \rfloor) - L(n, \lfloor n^a \rfloor)| \\
&\quad= \left| \sup_{\mathbf{u} \in \mathcal{U}(n, \lfloor n^a \rfloor)} \sum_{r=1}^{\lfloor n^a \rfloor} \left(S^{(r)}_{\lfloor u_r \rfloor + 1} - S^{(r)}_{\lfloor u_{r-1} \rfloor} \right) - \sup_{\mathbf{u}' \in \mathcal{U}(n, \lfloor n^a \rfloor)} \sum_{r=1}^{\lfloor n^a \rfloor} \left(B^{(r)}_{u'_r} - B^{(r)}_{u'_{r-1}} \right) \right| \\
&\quad\le \sup_{\mathbf{u} \in \mathcal{U}(n, \lfloor n^a \rfloor)} \left\{ \sum_{r=1}^{\lfloor n^a \rfloor} \left| S^{(r)}_{\lfloor u_r \rfloor + 1} - B^{(r)}_{\lfloor u_r \rfloor + 1} \right| + \left| S^{(r)}_{\lfloor u_{r-1} \rfloor} - B^{(r)}_{\lfloor u_{r-1} \rfloor} \right| \right. \\
&\qquad\qquad \left. + \left| B^{(r)}_{\lfloor u_r \rfloor + 1} - B^{(r)}_{u_r} \right| + \left| B^{(r)}_{\lfloor u_{r-1} \rfloor} - B^{(r)}_{u_{r-1}} \right| \right\} \\
&\quad\le 2 \sum_{r=1}^{\lfloor n^a \rfloor} \left\{ \max_{i=1,2,\ldots,n+1} \left| S_i^{(r)} - B_i^{(r)} \right| + \sup_{\substack{0 \le s,t \le n+1 \\ |s-t|<2}} \left| B_s^{(r)} - B_t^{(r)} \right| \right\} \\
&\quad= 2 \sum_{r=1}^{\lfloor n^a \rfloor} \left\{ V_n^{(r)} + W_n^{(r)} \right\},
\end{aligned} \tag{2.6}$$

where we have defined

$$V_n^{(r)} = \max_{i=1,2,\ldots,n+1} \left| S_i^{(r)} - B_i^{(r)} \right| \text{ and } W_n^{(r)} = \sup_{\substack{0 \le s,t \le n+1 \\ |s-t|<2}} \left| B_s^{(r)} - B_t^{(r)} \right|.$$

For each $r = 1, \ldots, \lfloor n^a \rfloor$ we will couple $\left(\omega_0^{(r)}, \omega_1^{(r)}, \ldots, \omega_n^{(r)}\right)$ and $B_t^{(r)}$, $0 \le t \le n+1$ as in Proposition 2, maintaining the independence for different r, so that the $V_n^{(r)}$, $1 \le r \le \lfloor n^a \rfloor$ are i.i.d. with

$$\mathbb{P}\left(V_n^{(r)} > x\right) \le Cnx^{-p} \tag{2.7}$$

for all $x \in \left[n^{1/p}, n^{1/2}\right]$.
Let A_1 be the event $\left\{\max_{1 \le r \le \lfloor n^a \rfloor} V_n^{(r)} > n^{1/2}\right\}$. Then from (2.7),

$$\mathbb{P}(A_1) \le n^a Cn(n^{1/2})^{-p} = Cn^{a+1-p/2} \to 0 \text{ as } n \to \infty, \tag{2.8}$$

since by assumption $a < \frac{6}{7}\left(\frac{1}{2} - \frac{1}{p}\right) < p\left(\frac{1}{2} - \frac{1}{p}\right) = \frac{p}{2} - 1$.

Also let A_2 be the event $\left\{\max_{1\le r\le\lfloor n^a\rfloor} W_n^{(r)} > n^{1/p}\right\}$. Using the reflection principle and standard estimates on the normal distribution,

$$
\begin{aligned}
\mathbb{P}(A_2) &\le n^a \mathbb{P}\left(\sup_{\substack{0\le s,t\le n+1\\ |s-t|<2}} \left|B_s^{(1)} - B_t^{(1)}\right| > n^{1/p}\right)\\
&\le n^a \sum_{i=0}^{n-2} \mathbb{P}\left(\sup_{i\le t\le i+3} B_t - \inf_{i\le t\le i+3} B_t > n^{1/p}\right)\\
&\le n^{a+1}\mathbb{P}\left(\sup_{0\le t\le 3} |B_t| > n^{1/p}/2\right)\\
&= 4n^{a+1}\mathbb{P}\left(B_3 > n^{1/p}/2\right)\\
&\le c_1 n^{a+1}\exp\left(-c_2 n^{2/p}\right)\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}
\tag{2.9}
$$

From (2.6), (2.7) and the definitions of the events A_1 and A_2, we have

$$
\begin{aligned}
&\mathbb{E}\left[\, |T(n,\lfloor n^a\rfloor) - L(n,\lfloor n^a\rfloor)|\,; A_1^C \cup A_2^C\right]\\
&\qquad\le 2n^a \mathbb{E}\left(V_n^{(1)} + W_n^{(1)}; A_1^C\cup A_2^C\right)\\
&\qquad\le 2n^a \left[n^{1/p} + \mathbb{E}\left(V_n^{(1)} - n^{1/p}; n^{1/p} \le V_n^{(1)} \le n^{1/2}\right) + n^{1/p}\right]\\
&\qquad\le 2n^a\left(2n^{1/p} + \int_{n^{1/p}}^{n^{1/2}} \mathbb{P}\left(V_n^{(1)} > x\right)dx\right)\\
&\qquad\le 2n^a\left(n^{1/p} + \int_{n^{1/p}}^{n^{1/2}} Cnx^{-p}dx\right)\\
&\qquad= 2n^a\left(n^{1/p} + C_2 n[-x^{-p+1}]_{n^{1/p}}^{n^{1/2}}\right)\\
&\qquad\le C_3 n^a n^{1/p},
\end{aligned}
\tag{2.10}
$$

where C_2 and C_3 are constants independent of n.
Together with (2.8) and (2.9), this gives that, for any $\epsilon > 0$,

$$
\mathbb{P}\left(|T(n,\lfloor n^a\rfloor) - L(n,\lfloor n^a\rfloor)| > n^{a+1/p+\epsilon}\right) \to 0 \text{ as } n\to\infty. \tag{2.11}
$$

The assumption $a < \frac{6}{7}\left(\frac{1}{2}-\frac{1}{p}\right)$ implies that, for ϵ sufficiently small, $a + \frac{1}{p} + \epsilon < \frac{1}{2} - \frac{a}{6}$. Thus

$$
\frac{|T(n,\lfloor n^a\rfloor) - L(n,\lfloor n^a\rfloor)|}{n^{\frac{1}{2}-\frac{a}{6}}} \to 0 \text{ in distribution, as } n\to\infty. \tag{2.12}
$$

Using (2.3) we obtain (2.5) as desired. $\square$

3 Further remarks

3.1 Larger values of a

It seems unlikely that the value 3/7 in Theorem 1 represents a real threshold. For $a > 3/7$, consider the typical difference between the weight of the maximal Brownian path and the weight of the discrete approximation; this will be large compared to the order of fluctuations of the maximum. However, the standard deviation of this difference may be smaller; one might expect it to be of order $n^{a/2}$ rather than order n^a, since it is composed of n^a terms of constant order which one expects to become independent as n becomes large. An argument along these lines would effectively allow us to replace n^a by $n^{a/2}$ in (2.10) leading to a bound $a < 3/4$ rather than $a < 3/7$. However, above $a = 3/4$ it seems that the behaviour is genuinely different and more sophisticated arguments would be required: the fluctuations of the error in the discrete approximation to the Brownian path are likely to be larger than the fluctuations of the maximal weight itself, and so one might no longer expect the maximal discrete path to follow closely the maximal Brownian path (even when the weights are "strongly coupled" to the Brownian motions as above).

3.2 Transverse fluctuations

The exponent $\hat{\chi} = \frac{1}{2} - \frac{a}{6}$ which we obtain should be related to the transversal fluctuations of the optimal path away from the straight line $\{y = n^{a-1}x\}$. Let us introduce the exponent

$$\hat{\xi} = \lim_{n\to\infty} \frac{1}{2\log n} \log \mathbb{E}\left(\left(v_{\lfloor \frac{n}{2} \rfloor} - \frac{n^a}{2}\right)^2\right).$$

where, say, v_i is the smallest value r such that the point (i, r) is contained in the optimal path. Corresponding to the universal value of $\hat{\chi}$ obtained, one would expect that $\hat{\xi}$ should be also a universal exponent equal to $2a/3$. To see this, we follow the heuristics explained in the introduction. On a renormalized level, the optimal path should behave as the optimal path from the origin to (n^a, n^a) in a last percolation model with Gaussian weights. This would imply that the transverse fluctuations should scale like $(n^a)^{2/3}$, where $\xi = 2/3$ is the (predicted) standard fluctuation exponent for directed last-passage percolation.
The strategy used in [10] for the derivation of the transverse fluctuations requires not only the knowledge of the last passage time fluctuations, but also a precise control of the moderate deviations. Our approach does not allow us to derive such sharp estimates (in particular we are missing some uniformity with respect to the direction of the path). For this reason, we do not yet have a proof of the universality of the transversal fluctuation exponent in our framework.

3.3 Related models

The last-passage percolation processes have a natural interpretation in terms of systems of queues in tandem (see for example [1, 7, 14]). Considering paths near the axis corresponds to considering regimes of very high or very low load in the queueing systems. In the case of exponential weight distribution, these queueing systems correspond closely to totally asymmetric exclusion processes or totally asymmetric zero-range processes. There are also close links with systems of non-colliding particles. See for example [16] for a survey.

Of course, there are also strong connections between these models and random matrix theory. We mention one particular direction related to the topic of this paper. Let $A_{n,k}$ be an $n \times k$ random matrix with i.i.d. entries, and let $Y_{n,k} = A_{n,k}(A_{n,k})^*$. In the special case of the Laguerre ensemble, where the common distribution of the entries is complex Gaussian, one has an explicit correspondence between the largest eigenvalue of Y and the passage time to (n, k) in a directed percolation model with exponential weights (see for example Proposition 1.4 of [9], and [6] and Section 6.1 of [2] for extensions). For a general distribution, there may not exist an explicit mapping between the matrix model and the directed percolation model, but we believe that a similar averaging mechanism to that observed in our context will also play a role in the random matrix setting. Thus on the basis of the analysis of the last passage time fluctuations for paths close to the axis, we conjecture that when k and n tend to infinity with suitable rates, the fluctuations of the largest eigenvalue of $Y_{n,k}$ should depend only on the mean and the variance of the coefficients of $A_{n,k}$.

Note

Related results have recently been obtained independently by Baik and Suidan [4]. Their methods also involve a coupling to a Brownian directed percolation problem, but via a Skorohod embedding technique rather than the strong approximation used here.

Acknowledgments

We thank Giambattista Giacomin, Yueyun Hu, Neil O'Connell and Gérard Ben Arous for valuable discussions, and Patrik Ferrari for helpful comments.

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