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MARTINGALE SELECTION PROBLEM AND ASSET PRICING IN FINITE DISCRETE TIME

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Abstract
Given a set-valued stochastic process $(V_t)_{t=0}^T$, we say that the martingale selection problem is solvable if there exists an adapted sequence of selectors $\xi_t \in V_t$, admitting an equivalent martingale measure. The aim of this note is to underline the connection between this problem and the problems of asset pricing in general discrete-time market models with portfolio constraints and transaction costs. For the case of relatively open convex sets $V_t(\omega)$ we present effective necessary and sufficient conditions for the solvability of a suitably generalized martingale selection problem. We show that this result allows to obtain computationally feasible formulas for the price bounds of contingent claims. For the case of currency markets we also sketch a new proof of the first fundamental theorem of asset pricing.

1 Introduction

This paper is motivated by the problems of arbitrage theory. We deal with discrete-time stochastic securities market models over general probability spaces. Recall that in the context of the market model considered in [3], the absence of arbitrage opportunities is equivalent to the existence of an equivalent martingale measure for the discounted asset price process. Moreover, any arbitrage-free price of a contingent claim is given by the expectation with respect to some of these measures. Various generalizations of these results are available.

In spite of their theoretical importance, the purely existence results of this form are not quite convenient for the calculation of the price bounds of contingent claims. The present note suggests an approach suitable for this purpose.

In Section 2 we present our main tool: the martingale selection theorem. In Section 3 we give two examples, showing that this result allows to obtain computationally feasible formulas for the price bounds of contingent claims in general discrete-time market models with portfolio constraints and transaction costs. For the case of currency markets with friction we also present a new proof of the first fundamental theorem of asset pricing.

2 Martingale selection theorem

Consider a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a sub-σ-algebra $\mathcal{H}$ of $\mathcal{F}$. A set valued map G, assigning some set $G(\omega) \subset \mathbb{R}^d$ to each $\omega \in \Omega$, is called $\mathcal{H}$-measurable if $\{\omega : G(\omega) \cap U \neq \emptyset\} \in \mathcal{H}$ for any open set $U \subset \mathbb{R}^d$. A function $f : \Omega \mapsto \mathbb{R}^d$ is called a selector of G if $f(\omega) \in G(\omega)$ for all $\omega \in \operatorname{dom} G = \{\omega : G(\omega) \neq \emptyset\}$. These definitions can be found e.g. in [5]. All σ-algebras $\mathcal{H} \subset \mathcal{F}$, considered below, are assumed to be complete with respect to $\mathbf{P}$, that is if $A \in \mathcal{H}$, $\mathbf{P}(A) = 0$ and $B \subset A$, then $B \in \mathcal{H}$.
Given a set $A \subset \mathbb{R}^d$, denote by $\operatorname{cl} A$, $\operatorname{ri} A$, $\operatorname{conv} A$ the closure, the relative interior, and the convex hull of A. If A is a cone, then A°, A^* are the polar and the conjugate cones: $-A^* = A^\circ = \{y : \langle x, y\rangle \le 0, \quad x \in A\}$. Here $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^d$. We also put $A - B = \{x - y : x \in A, \ y \in B\}$. For a sub-$\sigma$-algebra $\mathcal{H} \subset \mathcal{F}$ and a d-dimensional $\mathcal{F}$-measurable random vector η denote by $\mathcal{K}(\eta, \mathcal{H}; \omega)$ the support of the regular conditional distribution of η with respect to $\mathcal{H}$.
Assume that the probability space is endowed with the discrete-time filtration $(\mathcal{F}_t)_{t=0}^T$, $\mathcal{F}_0 = \{\emptyset, \Omega\}$, $\mathcal{F}_T = \mathcal{F}$. Let $V = (V_t)_{t=0}^T$ be an adapted sequence of set-valued maps with nonempty relatively open convex values $V_t(\omega) \subset \mathbb{R}^d$. Furthermore, let $(C_t)_{t=0}^{T-1}$ be an adapted sequence of random convex cones and let C_t° be the polar of C_t. We say that the *C-martingale selection problem* for $(V_t)_{t=0}^T$ is solvable if there exist an adapted stochastic process $\xi = (\xi_t)_{t=0}^T$ and a probability measure $\mathbf{Q}$, equivalent to $\mathbf{P}$, such that $\xi_t \in V_t$ and

$$\mathbf{E}_{\mathbf{Q}}(\xi_t - \xi_{t-1} | \mathcal{F}_{t-1}) \in C_{t-1}^\circ \text{ a.s.}$$

for all $t \in \{1, \dots, T\}$. Let us call ξ a $(\mathbf{Q}, C)$-martingale selector of V. We omit C in all notation if $C_t = \mathbb{R}^d$.
Consider an $\mathcal{F}$-measurable set-valued map G with the closed values $G(\omega) \neq \emptyset$ a.s. Given a sequence $\{f_i\}_{i=1}^\infty$ of $\mathcal{F}$-measurable selectors of G such that the sets $\{f_i(\omega)\}_{i=1}^\infty$ are dense in $G(\omega)$ a.s. (such a sequence always exists [5]), we put

$$\mathcal{K}(G, \mathcal{H}; \omega) = \operatorname{cl}\left(\bigcup_{i=1}^\infty \mathcal{K}(f_i, \mathcal{H}; \omega)\right).$$

We refer to [15], [16], [17] for another, but essentially the same definition of $\mathcal{K}(G, \mathcal{H})$, which does not involve the sequence $\{f_i\}_{i=1}^\infty$ and is expressed directly in terms of G. If $G(\omega) = \emptyset$ on a set of positive measure, then we put $\mathcal{K}(G, \mathcal{H}) = \emptyset$.

Theorem 1 *The C-martingale selection problem for $(V_t)_{t=0}^T$ is solvable iff the set-valued maps, defined recursively by $W_T = \operatorname{cl} V_T$;*

$$W_t = \operatorname{cl}(V_t \cap Y_t), \quad Y_t = \operatorname{ri}(\operatorname{conv} \mathcal{K}(W_{t+1}, \mathcal{F}_t)) - C_t^\circ, \ 0 \le t \le T-1 \tag{2.1}$$

have nonempty values a.s. Every $(\mathbf{Q}, C)$-martingale selector ξ of V take values in W a.s. Moreover, $\xi \in \operatorname{ri} W$ if $C = \mathbb{R}^d$.

Theorem 1 is an improvement of the main result of [15], where the sets $V_t(\omega)$ are assumed to be open and $C_t = \mathbb{R}^d$. It is shown in [16] that the theorem can be extended to the relatively open sets $V_t(\omega)$.
Sufficiency is the "difficult" part of Theorem 1. To sketch the proof, suppose the sets W_t are nonempty and take some selector $\xi_0 \in \operatorname{ri} W_0$. We claim that there exist adapted sequences $(\xi_t)_{t=0}^T$, $\xi_t \in \operatorname{ri} W_t$; $(\delta_t)_{t=1}^T$, $\delta_t > 0$ such that

$$\mathbf{E}_{\mathbf{P}}(\delta_{t+1}(\xi_{t+1} - \xi_t) | \mathcal{F}_t) \in C_t^\circ; \quad \mathbf{E}_{\mathbf{P}}(\delta_{t+1} | \mathcal{F}_t) = 1, \quad 0 \le t \le T-1.$$

These sequences are constructed inductively. Given some t, the induction step is described as follows. We take some selector $\xi_t \in \operatorname{ri} W_t$ and represent it in the form

$$\xi_t = \eta_t - \zeta_t, \tag{2.2}$$

where $\eta_t \in \operatorname{ri}(\operatorname{conv} \mathcal{K}(W_{t+1}, \mathcal{F}_t))$, $\zeta_t \in C_t^\circ$ and all elements indexed by t are assumed to be $\mathcal{F}_t$-measurable. It is crucial to prove that there exist an element $\xi_{t+1} \in \operatorname{ri} W_{t+1}$ and a random variable $\delta_{t+1} > 0$ such that

$$\eta_t = \mathbf{E}_{\mathbf{P}}(\delta_{t+1}\xi_{t+1}|\mathcal{F}_t), \quad \mathbf{E}_{\mathbf{P}}(\delta_{t+1}|\mathcal{F}_t) = 1. \tag{2.3}$$

As soon as this is verified (see [16, Lemma 1]), we get

$$\mathbf{E}_{\mathbf{P}}(\delta_{t+1}(\xi_{t+1} - \xi_t)|\mathcal{F}_t) = \zeta_t \in C_t^\circ.$$

It remains to introduce the positive $\mathbf{P}$-martingale

$$(z_t)_{t=0}^T; \quad z_0 = 1, \ z_t = \prod_{k=1}^t \delta_k, \ t \geq 1$$

and to check that ξ is a C-martingale under the measure $\mathbf{Q}$ with the density $d\mathbf{Q}/d\mathbf{P} = z_T$.
Let us also justify that any $(\mathbf{Q}, C)$-martingale selector ξ of V take values in W. Evidently, $\xi_T \in V_T = \operatorname{ri} W_T$. Assume that $\xi_{t+1} \in W_{t+1}$. By Theorem 2 of [14] we have

$$C_t^\circ \cap (\operatorname{ri}(\operatorname{conv} \mathcal{K}(\xi_{t+1}, \mathcal{F}_t)) - \xi_t) \neq \emptyset.$$

It follows that

$$\xi_t = V_t \cap (\operatorname{ri}(\operatorname{conv} \mathcal{K}(\xi_{t+1}, \mathcal{F}_t)) - C_t^\circ) \subset V_t \cap Y_t \subset W_t.$$

The last assertion of Theorem 1 is proved likewise.

3 Applications to mathematical finance

3.1 Frictionless market with portfolio constraints

Assume that the discounted prices of d traded assets are described by a d-dimensional adapted stochastic process $(S_t)_{t=0}^T$ and investor's discounted gain is given by

$$G_t^\gamma = \sum_{n=1}^t \langle \gamma_{n-1}, \Delta S_n \rangle, \quad \Delta S_n = S_n - S_{n-1}.$$

An adapted admissible portfolio process γ is subject to constraints of the form $\gamma_n \in B_n$, where B_n are $\mathcal{F}_n$-measurable random convex cones. See [12], [2], [10], [4], [14] for more information on this model. The market satisfies the no-arbitrage (NA) condition if $G_T^\gamma \geq 0$ a.s. implies that $G_T^\gamma = 0$ a.s. for any admissible investment strategy γ.
Recall that a contingent claim, represented by an $\mathcal{F}_T$-measurable random variable f_T, is called super-hedgeable (resp. sub-hedgeable) at a price $x \in \mathbb{R}$ if there exists an admissible portfolio process γ such that $x + G_T^\gamma \geq f_T$ (resp. $x - G_T^\gamma \leq f_T$) a.s. The upper (resp. the lower) price $\overline{\pi}_0$ (resp. $\underline{\pi}_0$) of f_T is the infimum (resp. the supremum) of all such x (see e.g. [19]).

Theorem 2 *Let the cones B_t be polyhedral and assume that NA condition is satisfied. Then the upper and the lower prices of a contingent claim f_T can be computed recursively by $\overline{\pi}_T = \underline{\pi}_T = f_T$;*

$$\overline{\pi}_t = \sup\{y : (S_t, y) \in \mathrm{ri}\,(\mathrm{conv}\,\mathcal{K}((S_{t+1}, \overline{\pi}_{t+1}), \mathcal{F}_t)) - B_t^\circ \times \{0\}\},\ t \le T-1; \tag{3.1}$$

$$\underline{\pi}_t = \inf\{y : (S_t, y) \in \mathrm{ri}\,(\mathrm{conv}\,\mathcal{K}((S_{t+1}, \underline{\pi}_{t+1}), \mathcal{F}_t)) - B_t^\circ \times \{0\}\},\ t \le T-1. \tag{3.2}$$

Proof. Assume that the contingent claim f_T is assigned with a price process $(f_t)_{t=0}^T$. In addition, we allow it to be traded together with S without additional constraints. Since the cones B_t are polyhedral, it follows from Lemma 3.1 of [10] (see also [4], [14]) that the extended market with the assets $(S_t, f_t)_{t=0}^T$ and the portfolio constraints $C_t = B_t \times \mathbb{R}$ is arbitrage-free iff (S, f) is a $(\mathbf{Q}, C)$-martingale under some equivalent measure $\mathbf{Q}$. This condition can be restated as follows:

$$f_t = \mathbf{E}_{\mathbf{Q}}(f_T|\mathcal{F}_t) \quad \text{for some} \quad \mathbf{Q} \in \mathcal{P}(B; f_T), \tag{3.3}$$

where $\mathcal{P}(B; f_T) = \{\mathbf{Q} \sim \mathbf{P} : \Delta S_t \in L^1(\mathbf{Q}),\ \mathbf{E}_{\mathbf{Q}}(\Delta S_t|\mathcal{F}_{t-1}) \in B_{t-1}^\circ,\ 1 \le t \le T,\ f_T \in L^1(\mathbf{Q})\}$. Consider the cone of random variables y, dominated by G_T^γ for some admissible portfolio process γ. Lemma 3.1 of [10] implies that this cone is closed in the topology of convergence in probability. By an appropriate equivalent change of measure, we may assume without loss of generality that $f_T \in L^1(\mathbf{P})$. Then the separation arguments (see [11] (Appendix A)) imply that

$$\underline{b} = \inf\{\mathbf{E}_{\mathbf{Q}} f_T : \mathbf{Q} \in \mathcal{P}(B; f_T)\}, \quad \overline{b} = \sup\{\mathbf{E}_{\mathbf{Q}} f_T : \mathbf{Q} \in \mathcal{P}(B; f_T)\}$$

are the lower and the upper prices of f_T. We are to show that they coincide with $\underline{\pi}_0$ and $\overline{\pi}_0$ respectively.

Note that by (3.3) the existence of an arbitrage-free market extension (S, f) is equivalent to the solvability of the C-martingale selection problem for the sequence

$$V_t = \{S_t\} \times \mathbb{R},\ t \le T-1; \quad V_T = \{(S_T, f_T)\}.$$

We claim that the sequence $(W_t)_{t=0}^T$, defined in (2.1), is of the form $W_t = \{S_t\} \times [\underline{\pi}_t, \overline{\pi}_t]$. For $t = T$ this assertion is true by the definition of W_T. Assume that $W_{t+1} = \{S_{t+1}\} \times [\underline{\pi}_{t+1}, \overline{\pi}_{t+1}]$. Then

$$W_t = \mathrm{cl}\ ((\{S_t\} \times \mathbb{R}) \cap Y_t) = \{S_t\} \times [\underline{h}_t, \overline{h}_t],$$

where

$$\underline{h}_t = \inf\{y : (S_t, y) \in Y_t\}, \quad \overline{h}_t = \sup\{y : (S_t, y) \in Y_t\}$$

and

$$Y_t = \mathrm{ri}\,(\mathrm{conv}\,\mathcal{K}(W_{t+1}, \mathcal{F}_t)) - C_t^\circ = \mathrm{ri}\,(\mathrm{conv}\,\mathcal{K}(\{S_{t+1}\} \times [\underline{\pi}_{t+1}, \overline{\pi}_{t+1}], \mathcal{F}_t)) - B_t^\circ \times \{0\}.$$

Supremum and infimum in (3.1), (3.2) are taken over the sets, which are contained in Y_t. Thus, $[\underline{\pi}_t, \overline{\pi}_t] \subset [\underline{h}_t, \overline{h}_t]$.

To prove the reverse inclusion consider a selector (S_t, g_t) of $\mathrm{ri}\, W_t = \{S_t\} \times \mathrm{ri}\,[\underline{h}_t, \overline{h}_t]$. This selector admits a representation of the form (2.2), (2.3):

$$(S_t, g_t) = \mathbf{E}(\delta_{t+1}(S_{t+1}, g_{t+1})|\mathcal{F}_t) - (\zeta_t', 0),$$

where $\delta_{t+1} > 0$, $\mathbf{E}(\delta_{t+1}|\mathcal{F}_t) = 1$, $\zeta_t' \in B_t^\circ$, and $g_{t+1} \in \mathrm{ri}\,[\underline{\pi}_{t+1}, \overline{\pi}_{t+1}]$. Hence,

$$\underline{v}_t = \mathbf{E}(\delta_{t+1}\underline{\pi}_{t+1}|\mathcal{F}_t) \le g_t \le \mathbf{E}(\delta_{t+1}\overline{\pi}_{t+1}|\mathcal{F}_t) = \overline{v}_t.$$

From Theorem 3 of [7] we deduce that

$$(S_t, \overline{v}_t) = \mathbf{E}(\delta_{t+1}(S_{t+1}, \overline{\pi}_{t+1})|\mathcal{F}_t) - (\zeta'_t, 0) \in \mathrm{ri}\,(\mathrm{conv}\,\mathcal{K}((S_{t+1}, \overline{\pi}_{t+1}), \mathcal{F}_t)) - B_t^\circ \times \{0\}.$$

Comparing this result with the definition of $\overline{\pi}_t$, we get the inequality $\overline{v}_t \le \overline{\pi}_t$. The related inequality $\underline{\pi}_t \le \underline{v}_t$ is obtained in the same way.
Consequently for any selector g_t of $\mathrm{ri}\,[\underline{h}_t, \overline{h}_t]$ we have $\underline{\pi}_t \le g_t \le \overline{\pi}_t$. This yields the desired inclusion $[\underline{h}_t, \overline{h}_t] \subset [\underline{\pi}_t, \overline{\pi}_t]$.
If $\mathbf{Q} \in \mathcal{P}(B; f_T)$ then $(S_t, f_t = \mathbf{E_Q}(f_T|\mathcal{F}_t)_{t=0}^T)$ is a $(\mathbf{Q}, C)$-martingale selector of $(V_t)_{t=0}^T$. By Theorem 1 we have $(S_t, f_t) \in W_t$. In particular, $f_0 = \mathbf{E_Q} f_T \in [\underline{\pi}_0, \overline{\pi}_0]$. Thus, $[\underline{b}, \overline{b}] \subset [\underline{\pi}_0, \overline{\pi}_0]$.
On the other hand, for any $f_0 \in \mathrm{ri}\,[\underline{\pi}_0, \overline{\pi}_0]$ we can construct a $(\mathbf{Q}, C)$-martingale selector (S, f) of V as in the proof of Theorem 1 since $(S_0, f_0) \in \mathrm{ri}\, W_0$. It follows that $f_0 = \mathbf{E_Q} f_T \in [\underline{b}, \overline{b}]$ for some $\mathbf{Q} \in \mathcal{P}(B; f_T)$ and $[\underline{\pi}_0, \overline{\pi}_0] \subset [\underline{b}, \overline{b}]$. The proof is complete.
Theorem 2 is not entirely new: for a rather general case of path-dependent options $f = f(S_0, \dots, S_T)$ after some calculations it gives the same results as in [1] and [18]. Certainly, $(\overline{\pi}_t)_{t=0}^T$ coincides with the minimal hedging strategy [1].

3.2 Currency market with friction

Our second example concerns Kabanov's model of currency market with transaction costs [8], [20] (generalizing the model of [6] in the discrete-time case).
Following [20] assume that there are d traded currencies. Their mutual bid and ask prices are specified by an adapted $d \times d$ matrix process $(\Pi_t)_{t=0}^T$, $\Pi_t = (\pi_t^{ij})_{1 \le i,j \le d}$ such that $\pi^{ij} > 0$, $\pi^{ii} = 1$, $\pi^{ij} \le \pi^{ik}\pi^{kj}$. The solvency cone K_t is spanned by the vectors $\{e_i\}_{i=1}^d$ of the standard basis in $\mathbb{R}^d$ and by the elements $\pi^{ij} e_i - e_j$. The elements of investor's time-t portfolio $\theta_t = (\theta_t^i)_{i=1}^d$ represent the amount of each currency, expressed in physical units. An adapted portfolio process $\theta = (\theta_t)_{t=0}^T$ is called self-financing if $\theta_t - \theta_{t-1} \in -K_t$ a.s., $t = 0, \dots, T$, where $\theta_{-1} = 0$.
Let $L^0(G, \mathcal{H})$ be the set of $\mathcal{H}$-measurable elements η such that $\eta \in G$ a.s. Denote by $A_t(\Pi)$ the convex cone in $L^0(\mathbb{R}^d, \mathcal{F}_t)$ formed by the elements θ_t of all self-financing portfolio processes θ.
According to the definition of [20], a bid-ask process $(\Pi_t)_{t=0}^T$ satisfies the robust no-arbitrage condition (NA^r) if there exists a bid-ask process $(\widetilde{\Pi}_t)_{t=0}^T$ such that

$$[1/\widetilde{\pi}_t^{ji}, \widetilde{\pi}_t^{ij}] \subset \mathrm{ri}\,[1/\pi_t^{ji}, \pi_t^{ij}]$$

for all i, j, t and $A_T(\widetilde{\Pi}) \cap L^0(\mathbb{R}^d, \mathcal{F}_T) = \{0\}$. An adapted stochastic process $Z = (Z_t)_{t=0}^T$ is called a strictly consistent price process if Z is a martingale under $\mathbf{P}$ and $Z_t \in \mathrm{ri}\, K_t^*$ a.s., $t = 0, \dots, T$.
Let $a^- = \max\{-a, 0\}$. Given a random vector $\zeta_T \in L^0(\mathbb{R}^d, \mathcal{F}_T)$ denote by $\mathcal{Z}(\Pi, \zeta_T)$ the set of strictly consistent price processes Z such that $\langle \zeta_T, Z_T \rangle^- \in L^1(\mathbf{P})$ and consider the sets

$$\mathcal{J}(\zeta_T) = \{\zeta_0 \in \mathbb{R}^d : \langle \zeta_0, Z_0 \rangle = \mathbf{E_P} \langle \zeta_T, Z_T \rangle \text{ for some } Z \in \mathcal{Z}(\Pi, \zeta_T)\},$$

$$\mathcal{J}^+(\zeta_T) = \{\zeta_0 \in \mathbb{R}^d : \langle \zeta_0, Z_0 \rangle > \mathbf{E_P} \langle \zeta_T, Z_T \rangle \text{ for all } Z \in \mathcal{Z}(\Pi, \zeta_T)\},$$

$$\mathcal{J}^-(\zeta_T) = \{\zeta_0 \in \mathbb{R}^d : \langle \zeta_0, Z_0 \rangle < \mathbf{E_P} \langle \zeta_T, Z_T \rangle \text{ for all } Z \in \mathcal{Z}(\Pi, \zeta_T)\}.$$

It is easy to check that the sets $\mathcal{J}$, $\mathcal{J}^+$, $\mathcal{J}^-$ are disjoint, the sets $\mathcal{J}^+$, $\mathcal{J}^-$ are convex and

$$\mathcal{J} \cup \mathcal{J}^+ \cup \mathcal{J}^- = \mathbb{R}^d. \tag{3.4}$$

Furthermore, by Theorem 4.1 of [20] under NA^r condition the closure of $\mathcal{J}^+(\zeta_T)$ contains exactly those initial endowments ζ_0, which are needed to superreplicate ζ_T:

$$\mathrm{cl}\,\mathcal{J}^+(\zeta_T) = \{\zeta_0 \in \mathbb{R}^d : \zeta_T - \zeta_0 \in A_T(\Pi)\}.$$

Theorem 3 *Suppose NA^r condition is satisfied. An initial endowment $\zeta_0 \in \mathbb{R}^d$ belongs to $\mathcal{J}(\zeta_T)$ iff the martingale selection problem for the set-valued stochastic sequence*

$$\begin{aligned} V_0 &= \{(x,y) : x \in \mathrm{ri}\, K_0^*,\ y = \langle \zeta_0, x\rangle\}; \quad V_t = \mathrm{ri}\, K_t^* \times \mathbb{R},\ 1 \le t \le T-1; \\ V_T &= \{(x,y) : x \in \mathrm{ri}\, K_T^*,\ y = \langle \zeta_T, x\rangle\}. \end{aligned}$$

is solvable. Moreover, $\mathcal{J}(\zeta_T) = \{\zeta_0 \in \mathbb{R}^d : (x, \langle \zeta_0, x\rangle) \in \mathrm{ri}\, W_0 \text{ for some } x \in \mathbb{R}^d\}$, where W_0 is defined in Theorem 1.

Proof. Suppose $(\xi, \mathbf{Q})$ is a solution of the martingale selection problem for the sequence $(V_t)_{t=0}^T$, i.e. $\xi = (Y, f) \in V$ is a martingale under some probability measure $\mathbf{Q}$, equivalent to $\mathbf{P}$. Denote by $(z_t)_{t=0}^T$ the density process of $\mathbf{Q}$ with respect to $\mathbf{P}$:

$$z_T = d\mathbf{Q}/d\mathbf{P}, \quad z_{t-1} = \mathbf{E_P}(z_t | \mathcal{F}_{t-1}), \quad t \le T.$$

Then $(Z_t, g_t)_{t=0}^T = (z_t Y_t, z_t f_t)_{t=0}^T$ is a martingale under $\mathbf{P}$, Z is a strictly consistent price process, and

$$\langle \zeta_0, Z_0\rangle = \langle \zeta_0, Y_0\rangle = f_0 = \mathbf{E_Q} f_T = \mathbf{E_Q}\langle \zeta_T, Y_T\rangle = \mathbf{E_P}\langle \zeta_T, Z_T\rangle.$$

Thus, $\zeta_0 \in \mathcal{J}(\zeta_T)$.
Conversely, if $\langle \zeta_0, Z_0\rangle = \mathbf{E_P}\langle \zeta_T, Z_T\rangle$ for some $Z \in \mathcal{Z}(\Pi, \zeta_T)$ then we get a solution (Z, g) of the martingale selection problem for V by putting $g_t = \mathbf{E_P}(\langle \zeta_T, Z_T\rangle | \mathcal{F}_t)$.
The last statement of the theorem follows from the fact that the starting points $(Z_0, \langle \zeta_0, Z_0\rangle)$ of $\mathbf{Q}$-martingale selectors of V are exactly the points of $\mathrm{ri}\, W_0$ (see Theorem 1 and the sketch of its proof). The proof is complete.
Note, that the set W_0 can be computed recursively by the formulas given in Theorem 1. By Theorem 3 it contains all information about the set $\mathcal{J}(\zeta_T)$ and hence about the partition (3.4).
Finally, we reproduce an extended version of the first fundamental theorem of asset pricing for currency markets as it is given [17].

Theorem 4 *For a bid-ask process $(\Pi_t)_{t=0}^T$ the following conditions are equivalent:*

(a) *NA^r condition is satisfied;*

(b) *there exists a strictly consistent price process Z;*

(c) *all elements W_t of the sequence of the set-valued maps*

$$W_T = K_T^*,$$

$$W_t = \mathrm{cl}\,(\mathrm{ri}\, K_t^* \cap Y_t), \quad Y_t = \mathrm{ri}\,(\mathrm{conv}\, \mathcal{K}(W_{t+1}, \mathcal{F}_t)), \quad 0 \le t \le T-1$$

take nonempty values a.s.;

(d) *if $\sum_{t=0}^T x_t = 0$, where $x_t \in L^0(-K_t, \mathcal{F}_t)$, then $x_t = L^0(K_t \cap (-K_t), \mathcal{F}_t)$, $t = 0, \dots, T$.*

Conditions (a) and (b) were introduced in the paper [20], where their equivalence was established. Condition (d) was introduced in [8], [9] and its equivalence to (a) and (b) was proved in the latter paper.
Below we sketch a new proof of Theorem 4. In this proof condition (c), introduced in [17], plays a central role. The proof involves only measurable selection arguments and tools from finite-dimensional convex analysis. It avoids the direct justification of the implication (a) $\Longrightarrow$ (b) and does not appeal to the closedness (in probability) of the cone $A_T(\Pi)$ of hedgeable claims. Recently a related approach (under an additional assumption of efficient friction [8]) was independently proposed by Miklós Rásonyi [13].
Proof of Theorem 4. The equivalence of conditions (c) and (d) was established in [17] by direct calculations. The equivalence of (b) and (c) follows from Theorem 1 (or Theorem 1 of [16]) since the existence of a strictly consistent price process is equivalent to the solvability of the martingale selection problem for the sequence $V_t = \mathrm{ri}\, K_t^*$ (see the introductory section of [15] or the proof of Theorem 3 above). The implication (b) $\Longrightarrow$ (a) can be regarded as an "easy" part of the theorem. Its proof can be found in [20] or [9]. The remaining implication (a) $\Longrightarrow$ (d) was proved in [9] (Lemma 5).
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