Elect. Comm. in Probab. **13** (2008), 194–197

ELECTRONIC COMMUNICATIONS in PROBABILITY

SYMMETRIZATION OF BERNOULLI

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Submitted March 11, 2008, *accepted in final form* March 13, 2008

AMS 2000 Subject classification: 60E10
Keywords: Symmetrization, symmetry resistant, Skorokhod embedding

Abstract
We show that an asymmetric Bernoulli random variable is symmetry resistant in the sense that any independent random variable, which when added to it produces a symmetric sum, must have a variance at least as much as itself. The main instrument is to use Skorokhod embedding to transfer the discrete problem to the realm of stochastic calculus.

1 Introduction

Let X be a random variable. We call an independent random variable Y to be a *symmetrizer* for X, if $X+Y$ has a symmetric distribution around zero. Two simple cases immediately come to mind. One is when X itself is symmetric. In that case the constant $-E(X)$ is a symmetrizer for X. On the other hand, for a general X, an independent random variable Y, which has the same law as $-X$, is obviously a symmetrizer. The difference in these two cases is the fact that the symmetrizer in the former case has zero variance while in the latter case it has the same variance as that of X. Thus, we are led to the question whether given a random variable X which is not symmetric, can one find a symmetrizer which has a variance less than that of X ? If such a symmetrizer cannot be found, the random variable X is said to be *symmetry resistant*.

Symmetry resistance is an interesting property which seems to be surprisingly difficult to prove, even in simplest of cases. For example, let X be a Bernoulli(p) random variable. If $p = 1/2$, it is immediate that the degenerate random variable, $Y \equiv -1/2$, is a symmetrizer for X. Hence, X is not symmetry resistant. However, we shall show that if $p \neq 1/2$, for any symmetrizer Y, we have

$$\mathrm{Var}(Y) \geq pq, \tag{1}$$

where $q = (1-p)$. It is immediate from this inequality that X is symmetry resistant and the minimum variance symmetrizer has the same variance as $-X$.

The last result is the main content of a paper by Kagan, Mallows, Shepp, Vanderbei and Vardi (see [Kagan et al., 1999]), where the reader can look for the motivation and the history of this

[1] RESEARCH PARTIALLY SUPPORTED BY NSF GRANT DMS-0306194 TO THE PROBABILITY GROUP AT CORNELL.

problem. We merely reprove the result here. The previous authors discuss why characteristic functions are not helpful in determining symmetry resistance and how symmetry resistance is independent of decomposability of the random variable into symmetric components. Ultimately the authors use duality theory of linear programing to prove inequality (1). In fact, the correct solution of the linear programing problem had to be first guessed from the output of a linear programing software.

The novelty in this paper is that we have used purely probabilitic techniques, as ubiquitous as Ito's rule, to prove (1). This avoids the technicalities of linear programing and adds to the collection of weaponry of attacking discrete problems through stochastic calculus.

I would like to thank Prof. Larry Shepp for communicating the problem to me and his encouragement during the work.

2 Proof

Proof of inequality (1). Let X be a Bernoulli(p) random variable for $p \neq 1/2$. Let Y be any symmetrizer for X with finite variance. By Skorokhod embedding of mean zero, finite variance random variables in Brownian motion, there is a stopping time τ such that for any standard Brownian motion (i.e., starting from zero), the stopped process has the distribution of $Y - \mathrm{E}(Y)$. That is to say, if W is a standard Brownian motion

$$W_\tau \stackrel{d}{=} Y - \mathrm{E}(Y). \tag{2}$$

Here and throughout $\stackrel{d}{=}$ refers to equality in distribution.

On a suitable probability space construct a process $\{B_t,\ t \in [0,\infty)\}$ such that

$$B_0 \stackrel{d}{=} X - \mathrm{E}(X) \tag{3}$$

and

$$B_t = B_0 + W_t,$$

where B_0 is independent of the standard Brownian motion W_t. Then clearly, B_t is a Brownian motion which has the initial distribution of $X - \mathrm{E}(X)$. Also, by equation (2), we have

$$B_\tau \stackrel{d}{=} X + Y - \mathrm{E}(X+Y).$$

But, since Y is a symmetrizer of X, we should have $\mathrm{E}(X+Y) = 0$, and hence

$$B_\tau \stackrel{d}{=} X + Y. \tag{4}$$

Now, let ρ be any smooth odd function with bounded derivatives on the real line. By Ito's rule, we have

$$\rho(B_t) - \rho(B_0) = M_t + \frac{1}{2}\int_0^t \rho''(B_s)\mathrm{d}s, \tag{5}$$

where M_t is a martingale. Thus, by the optional sampling theorem, we have

$$\mathrm{E}(\rho(B_\tau)) - \mathrm{E}(\rho(B_0)) = \frac{1}{2}\mathrm{E}\left(\int_0^\tau \rho''(B_s)\mathrm{d}s\right). \tag{6}$$

Now, since ρ is an odd function, and $X+Y$ is symmetric around zero, by equation (4), $\mathrm{E}(\rho(B_\tau))=0$. Thus (6) reduces to

$$-\mathrm{E}(\rho(B_0)) = \frac{1}{2}\mathrm{E}\left(\int_0^\tau \rho''(B_s)\mathrm{d}s\right). \tag{7}$$

Let us now look at the RHS of (7). By conditioning on B_0, we have

$$\begin{aligned}\mathrm{E}\left(\int_0^\tau \rho''(B_s)\mathrm{d}s\right) &= p\mathrm{E}\left(\int_0^\tau \rho''(q+W_s)\mathrm{d}s\right) \\ &+ q\mathrm{E}\left(\int_0^\tau \rho''(-p+W_s)\mathrm{d}s\right).\end{aligned} \tag{8}$$

Let now impose the following restrictions on ρ (an example will soon follow):

1. $|\rho''| \le 1$.
2. $\rho(1+x) = -\rho(x), \quad \forall x \in \mathbf{R}$.

in addition to the fact that ρ is odd. Thus for any x, we have

$$\rho''(-p+x) = -\rho''(1-p+x) = -\rho''(q+x).$$

Then, by equation (8) we get

$$\mathrm{E}\left(\int_0^\tau \rho''(B_s)\mathrm{d}s\right) = (p-q)\mathrm{E}\left(\int_0^\tau \rho''(q+W_s)\mathrm{d}s\right).$$

Also, by equation (3),

$$\mathrm{E}(\rho(B_0)) = p\rho(q) + q\rho(-p) = p\rho(q) - q\rho(q) = (p-q)\rho(q).$$

Substituting these values in equation (6) gives

$$(q-p)\rho(q) = -\frac{1}{2}(q-p)\mathrm{E}\left(\int_0^\tau \rho''(q+W_s)\mathrm{d}s\right). \tag{9}$$

We now assume that $q \neq p$. The other case of $q=p$ has already been discussed in the previous section. Now, use the fact that $|\rho''| \le 1$, to conclude

$$-\mathrm{E}\left(\int_0^\tau \rho''(q+W_s)\mathrm{d}s\right) \le \mathrm{E}(\tau).$$

Thus, from (9), we get

$$\mathrm{E}(\tau) \ge 2\rho(q).$$

However, by Skorokhod embedding, $\mathrm{E}(\tau) = \mathrm{Var}(Y)$. Hence

$$\mathrm{Var}(Y) \ge 2\rho(q). \tag{10}$$

Finally, we have to exhibit such a ρ. For $0 \leq x \leq 1$, define

$$\rho(x) = \frac{x(1-x)}{2}$$

Extend it to the entire positive axis by the property $\rho(1+x) = -\rho(x)$. That is to say,

$$\rho(x) := (-1)^k \rho(x-k), \ \ 0 \leq k \leq x \leq k+1.$$

And extend to the entire negative axis by the oddness of ρ. That is

$$\rho(-x) := -\rho(x), \ \ x \in \mathbf{R}^+.$$

The function ρ does not have a continuous second derivative. However, the set of discontinuity is just the countable set of integers, and this is sufficient for the usual Ito's rule to go through. See, for example, [Karatzas and Shreve, 1991, page 219]. More importantly,

$$|\rho''(x)| \leq 1,$$

whenever x is not an integer. Hence by equation (10), we get

$$\mathrm{Var}(Y) \geq pq,$$

which is what we claimed. This proves the result. $\square$

It is interesting to see how every inequality above becomes equality only when τ is the Skorokhod embedding for $Y = -X$. This proves the uniqueness of the minimum variance symmetrizer.

References

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