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ERGODICITY OF PCA : EQUIVALENCE BETWEEN SPATIAL AND TEMPORAL MIXING CONDITIONS

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Abstract

For a general attractive Probabilistic Cellular Automata on $S^{\mathbb{Z}^d}$, we prove that the (time-) convergence towards equilibrium of this Markovian parallel dynamics, exponentially fast in the uniform norm, is equivalent to a condition $(\mathcal{A})$. This condition means the exponential decay of the influence from the boundary for the invariant measures of the system restricted to finite boxes. For a class of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$, with a naturally associated Gibbsian potential φ, we prove that a (spatial-) weak mixing condition $(\mathcal{WM})$ for φ implies the validity of the assumption $(\mathcal{A})$; thus *exponential (time-) ergodicity* of these dynamics towards the unique Gibbs measure associated to φ holds. On some particular examples we state that exponential ergodicity holds as soon as there is no phase transition.

1 Introduction

The main feature of Probabilistic Cellular Automata dynamics (usually abbreviated in PCA) is the parallel, or synchronous, evolution of all interacting elementary components. They are precisely discrete-time Markov chains on a product space S^Λ (*configuration space*) whose transition probability is a *product measure*. In this paper, S (*spin space*) is assumed to be a finite set with total order denoted by $\leqslant$ and Λ (set of *sites*) a subset, finite or infinite, of $\mathbb{Z}^d$. The fact that the transition probability kernel $P(d\sigma|\sigma')$ $(\sigma,\sigma' \in S^\Lambda)$ is a product measure means that all spins $\{\sigma_k : k \in \Lambda\}$ are simultaneously and independently updated. This transition mechanism differs from the one in the most common Gibbs samplers, where only one site is updated at each time step. In opposition to these dynamics with sequential

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updating, it is simple to define PCA's on the infinite set $S^{\mathbb{Z}^d}$ without passing to continuous time.

The main purpose of this article is to study relation between different types of conditions which insure the fastest convergence towards an equilibrium state ($\nu P = \nu$) of PCA dynamics on $S^{\mathbb{Z}^d}$. Let us emphasise that the non-degeneracy hypothesis we will assume implies that the asymptotic behaviour of PCA dynamics on S^Λ where Λ is a finite subset of $\mathbb{Z}^d$ (called *finite volume PCA dynamics*) is well-known. It is a classical result from the theory of finite state space aperiodic irreducible Markov Chains. Such discrete time processes admit a unique stationary probability measure, and are ergodic. However, if the PCA dynamics is considered on $S^{\mathbb{Z}^d}$ (*infinite volume dynamics*), some non-ergodic behaviour may arise (see for instance example 2 section *III in* [8]). The most famous condition which insures ergodicity of the PCA dynamics on $S^{\mathbb{Z}^d}$ is due to Dobrushin and Vasershtein's work (see [15]), and applies in the high-temperature regime. Others conditions of ergodicity for general PCA can be found in the following works: [4, 7, 9, 12, 13]. See for instance Sections 6.1.2 and 6.1.3 in [10] for details. They all are effective only when some high-temperature condition holds or in perturbative cases.

We will here adopt another approach, partially inspired by Martinelli and Olivieri's work for a class of continuous time Interacting Particle Systems called Glauber dynamics (see [14]), and based on a famous statement of Holley about rate of convergence ([6]). We introduce a condition ($\mathcal{A}$) which means the *exponential decay of the influence from the boundary for the invariant measure of the system restricted to any finite box*, which will be here proved to be equivalent to the exponentially fast ergodicity (Theorem 1). The condition ($\mathcal{A}$) we use is not a constructive criterion like the Dobrushin-Vasershtein condition, or its generalised version developed in [12] and numerically studied in [2]. But, theoretically, comparison of spatial and time mixing are always interesting (cf. [14, 3]). Furthermore we present different examples in which ($\mathcal{A}$) is satisfied on a larger domain than Dobrushin-Vasershtein condition, and is moreover optimal for these models.

In section 2 we state our main results. The first and more general one (Theorem 1) is the following: convergence towards equilibrium in the uniform norm with an *exponential rate* is *equivalent* to the condition ($\mathcal{A}$). In other words *exponential mixing in space is equivalent to exponential mixing in time.* It will then be applied to a class of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$, associated in a natural way to a Gibbsian potential φ. We prove that the usual weak mixing condition for φ implies the validity of ($\mathcal{A}$), thus the *exponential ergodicity* of the dynamics towards the unique Gibbs measure associated to φ holds (Theorem 2). For some particular PCA of this class, we also prove that ($\mathcal{A}$) is weaker than the Dobrushin-Vasershtein ergodicity condition and note that the exponential ergodicity holds as soon as there is no phase transition. Our result are then the first optimal ones in this context. Sections 3 and 4 are respectively devoted to the proof of the Theorems and useful Lemmas.

2 Main results

Let P denotes a PCA dynamics on $S^{\mathbb{Z}^d}$. This means a Markov Chain on $S^{\mathbb{Z}^d}$whose transition probability kernel P verifies for all *configuration* $\eta \in S^{\mathbb{Z}^d}$, $\sigma = (\sigma_k)_{k\in\mathbb{Z}^d} \in S^{\mathbb{Z}^d}$, $P(\ d\sigma \mid \eta\ ) = \underset{k\in\mathbb{Z}^d}{\otimes}\ p_k(\ d\sigma_k \mid \eta\ )$, where for all *site* $k \in \mathbb{Z}^d$, for all η, $p_k(\ .\ |\eta)$ is a probability measure on S, called *updating rule*. For any subset Δ of $\mathbb{Z}^d$, and for all configurations σ and η of $S^{\mathbb{Z}^d}$, the configuration $\sigma_\Delta\eta_{\Delta^c}$ is defined by σ_k if $k \in \Delta$, else η_k. Let the notation σ_Δ design $(\sigma_k)_{k\in\Delta}$ too. Let Λ be a finite subset of $\mathbb{Z}^d$ (denoted by $\Lambda \Subset \mathbb{Z}^d$). We call *finite volume PCA dynamics* with *boundary condition* τ ($\tau \in S^{\mathbb{Z}^d}$ or $\tau \in S^{\Lambda^c}$), the Markov Chain on S^Λ whose transition probability P^τ_Λ is defined by: $P^\tau_\Lambda(d\sigma_\Lambda \mid \eta_\Lambda) = \bigotimes_{k\in\Lambda} p_k(\, d\sigma_k \mid \eta_\Lambda\tau_{\Lambda^c}\,)$. It may be identified with the following infinite volume PCA dynamics on $S^{\mathbb{Z}^d}$: $P^\tau_\Lambda(d\sigma \mid \eta_\Lambda) = \bigotimes_{k\in\Lambda} p_k(\, d\sigma_k \mid \eta_\Lambda\tau_{\Lambda^c}\,) \otimes \delta_{\tau_{\Lambda^c}}(d\sigma_{\Lambda^c})$. Let ν^τ_Λ denote the stationary measure associated to the finite volume dynamics P^τ_Λ. For ν probability measure on $S^{\mathbb{Z}^d}$ (equipped with the Borel σ-field associated to the product topology), νP refers to $\nu P(d\sigma) = \int P(d\sigma|\eta)\nu(d\eta)$. Recursively $\nu P^{(n)} = (\nu P^{(n-1)})P$. For each function f on $S^{\mathbb{Z}^d}$, $P(f)$ is the function defined by $P(f)(\eta) = \int f(\sigma)P(d\sigma|\eta)$. All the measures considered in this paper are probability measures.

PCA dynamics considered here are assumed to be *non degenerate*: $\forall k \in \mathbb{Z}^d$, $\forall \eta \in S^{\mathbb{Z}^d}$, $\forall s \in S$, $p_k(\, s \mid \eta\,) > 0$; they are also *local*, which means: $\forall k \in \mathbb{Z}^d, \exists\, V_k \Subset \mathbb{Z}^d, p_k(\,.\,|\eta) = p_k(\,.\,|\eta_{V_k})$ and they are also translation invariant: $\forall k \in \mathbb{Z}^d$, $\forall s \in S$, $\forall \eta \in S^{\mathbb{Z}^d}$, $\quad p_k(\, s \mid \eta\,) = p_0(\, s \mid \theta_{-k}\eta\,)$, where $\theta_{k_0}(\sigma)$ defines the translation of a configuration σ of $S^{\mathbb{Z}^d}$ with $\theta_{k_0}(\sigma) = (\sigma_{k-k_0})_{k\in\mathbb{Z}^d}$.

Attractivity of PCA dynamics is moreover assumed here: One can order two configurations by defining $\sigma \preccurlyeq \eta$ if $\forall k \in \Lambda, \sigma_k \leqslant \eta_k$. A real function f on S^Λ will then be said to be increasing if $\sigma \preccurlyeq \eta$ implies $f(\sigma) \leqslant f(\eta)$. Thus two probability measures ν_1 and ν_2 satisfy the stochastic ordering $\nu_1 \preccurlyeq \nu_2$ if, for all increasing functions f on S^Λ, $\nu_1(f) \leqslant \nu_2(f)$, with the notation $\nu_i(f) = \int f(\sigma)\nu_i(d\sigma)$. As Markov chain, a PCA dynamics P on S^Λ ($\Lambda \subset \mathbb{Z}^d$) is *attractive* if for all increasing function f, $P(f)$ is still increasing. Let us define too, for $s \in S, \sigma \in S^\Lambda$, the function $G_k(s,\sigma)$ by:

$$G_k(s,\ .\ ) = \sum_{s'\geqslant s} p_k(s'|\ .\ ). \tag{1}$$

Recall that a PCA dynamics is attractive if, and only if, for all k in Λ, and all value $s \in S$, the function $G_k(s,.)$ is increasing (in σ).

A real valued function f on $S^{\mathbb{Z}^d}$ is said *local* if $\exists \Lambda_f \Subset \mathbb{Z}^d$, $\forall \sigma \in S^{\mathbb{Z}^d}$, $f(\sigma) = f(\sigma_{\Lambda_f})$. We define, for each f continuous function on the compact $S^{\mathbb{Z}^d}$ and for all k in $\mathbb{Z}^d$,

$$\Delta_f(k) = \sup\Big\{\Big|f(\sigma) - f(\eta)\Big| \ :\ (\sigma,\eta) \in (S^{\mathbb{Z}^d})^2, \sigma_{\{k\}^c} \equiv \eta_{\{k\}^c}\Big\},$$

and the semi-norm $|||\ f\ ||| = \sum_{k\in\mathbb{Z}^d} \Delta_f(k)$. For L integer, $\mathcal{B}(L)$ is the ball $\mathcal{B}(0,L)$ with respect to the norm $\|k\|_1 = \sum_{i=1}^d |k_i|$, $k = (k_1,k_2,\ldots,k_d) \in \mathbb{Z}^d$.

Theorem 1 *Let S be a totally ordered finite set with maximal (resp. minimal) element denoted by $+$(resp. $-$). $\mathbf{+}$ (resp. $\mathbf{-}$) denotes configurations equal to $+$ (resp. $-$) in all sites. Let P be an attractive, translation invariant, non degenerate, local PCA dynamics on $S^{\mathbb{Z}^d}$. Let $\nu^{\mathbf{+}}_{\mathcal{B}(L)}$ (resp. $\nu^{\mathbf{-}}_{\mathcal{B}(L)}$) be the stationary measure of $P^{\mathbf{+}}_{\mathcal{B}(L)}$ (resp. $P^{\mathbf{-}}_{\mathcal{B}(L)}$). The following spatial mixing condition:* $\exists C > 0,\ \exists M > 0,\ \exists L_1 \in \mathbb{N}^*, \forall L \in \mathbb{N}^*, L \geqslant L_1$,

$$\int \sigma_0\ d\nu^{\mathbf{+}}_{\mathcal{B}(L)} - \int \sigma_0\ d\nu^{\mathbf{-}}_{\mathcal{B}(L)} \leqslant Ce^{-ML} \tag{$\mathcal{A}$}$$

is equivalent to the convergence of the dynamics P towards the unique equilibrium state ν with exponential rate: $\exists\lambda > 0$, $\exists n_1$, $\forall n \geqslant n_1$, $\forall f$ *local function on* $S^{\mathbb{Z}^d}$:

$$\sup_{\sigma} \left| \delta_\sigma P^{(n)}(f) - \nu(f) \right| \leqslant 2 ||| \, f \, ||| \, e^{-\lambda n}. \tag{2}$$

In order to better interpret the meaning of condition $(\mathcal{A})$ and the relevance of Theorem 1, we then apply it to a wide class of reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$. First, let us recall some known facts about *reversible PCA dynamics* (that is to say PCA dynamics whose set of reversible measures $\mathcal{R}$ is not empty). The study of the qualitative nature of their equilibrium states as Gibbs measures was initiated by Kozlov and Vasilyev (see [8, 16]). Gibbs measures with respect to some dynamics' naturally associated potential, are indeed natural candidates as stationary states. In [1, 10], precise relations were established between the sets of stationary measures, reversible measures and some Gibbs measures (see Proposition 3.3 in [1]). Moreover, unlike what is done (or expected to hold) for continuous time Interacting Particle Systems like Glauber dynamics or gradient diffusions, it is shown that Gibbs measures may be non stationary for PCA's dynamics, which is a characteristic manifestation of the discrete time case.

Assume until the end of this section and in section 4 that $S = \{-1,+1\}$. We call class $\mathcal{C}_0$ the family of PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$ whose updating rule $(p_k)_{k\in\mathbb{Z}^d}$ is given by: $\forall k \in \mathbb{Z}^d$, $\forall \eta \in S^{\mathbb{Z}^d}$, $\forall s \in S$

$$p_k(s \mid \eta) = \frac{1}{2}\Big(1 + s\tanh(\beta \sum_{k'\in\mathbb{Z}^d} \mathcal{K}(k'-k)\eta_{k'})\Big), \tag{3}$$

where β is a positive real parameter and $\mathcal{K} \,:\, \mathbb{Z}^d \to \mathbb{R}$ is an interaction function between sites which is symmetric and has finite range $R > 0$ (*i.e.* for all k of $\mathbb{Z}^d$ such that $\|k\|_1 > R$ then $\mathcal{K}(k) = 0$). Remark that $\beta = 0$ is the independent case (sites don't interact), and that when β increases, the dynamics becomes less and less random. So β may be thought as a kind of inverse temperature parameter. See subsection 4.1.1 in [10] for the generality of the class $\mathcal{C}_0$ among reversible PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$. Due to their definition, PCA dynamics in $\mathcal{C}_0$ are local, translation invariant, non degenerate. It is known (see [8, 1]) that any PCA dynamics P in $\mathcal{C}_0$ admits at least one reversible measure which is a Gibbs measure associated to the following translation invariant multibody potential φ:

$$\begin{array}{lcll} \varphi_{U_k}(\sigma_{U_k}) & = & -\log\cosh\Big(\beta\sum_j \mathcal{K}(k-j)\sigma_j\Big) & \text{where } U_k = \{j : \mathcal{K}(k-j) \neq 0\} \\ \varphi_\Lambda(\sigma_\Lambda) & = & 0 \ \text{ otherwise.} & \end{array} \tag{4}$$

Moreover Proposition 3.3 in [1] stated the precise relations $\mathcal{R} = \mathcal{S} \cap \mathcal{G}(\varphi)$ and $\mathcal{R}_s = \mathcal{S}_s$, where $\mathcal{S}$ (resp. $\mathcal{R}$) denotes the set of P-stationary (resp. P-reversible) measures, $\mathcal{S}_s$ and $\mathcal{R}_s$ their respective space-translation invariant measures' parts, and $\mathcal{G}(\varphi)$ the set of Gibbs measures on $S^{\mathbb{Z}^d}$ associated to the potential φ.
One also checks that such a PCA dynamics P is attractive, if and only if function $\mathcal{K}(.)$ is non-negative (see Property 4.1.2 in [10]). From now on, let us assume that $\mathcal{K}$ is non negative.

Mixing conditions for a potential φ define different regions in the domain of absence of phase transition for the associated Gibbs measures. Strong mixing conditions are usually related to the domain where Dobrushin's uniqueness holds, and weak mixing conditions are expected to be valid in the main part of the uniqueness domain: See [14] for a review on these conditions. Here, we call weak mixing condition for the potential φ, the condition: $\exists C > 0,\ \exists M > 0,\ \forall L \geqslant 2$,

$$\int \sigma_0\ \mu(d\sigma_{\mathcal{B}(L)}|\sigma_{\mathcal{B}(L)^c} = +1) - \int \sigma_0\ \mu(d\sigma_{\mathcal{B}(L)}|\sigma_{\mathcal{B}(L)^c} = -1) \leqslant Ce^{-ML} \qquad (\mathcal{WM})$$

where μ is the unique Gibbs measure associated to φ. For ferromagnetic potentials, it is indeed the equivalent form of more general weak mixing condition.

Theorem 2 *Let P be an attractive PCA dynamics on $\{-1,+1\}^{\mathbb{Z}^d}$ of the class $\mathcal{C}_0$ defined by (3), let φ denote the potential canonically associated defined in (4), and $\mathcal{G}(\varphi)$ the set of Gibbs measures w.r.t φ.*

- *If there is phase transition (i.e. $\#\mathcal{G}(\varphi) > 1$) then the dynamics P is non-ergodic.*
- *Otherwise, when there is no phase transition (i.e. $\mathcal{G}(\varphi) = \{\mu\}$) the dynamics P is ergodic towards the unique Gibbs measure μ.*
  Moreover if we assume the potential φ satisfies the weak mixing condition $(\mathcal{WM})$, then the convergence towards μ holds with exponential rate.

In [1], we established that, for nearest neighbour interaction function $\mathcal{K}$, phase transition holds for β large. For instance, when $d = 2$, let P_J be the PCA dynamics of the class $\mathcal{C}_0$ obtained taking: $\mathcal{K}(\pm e_1) = \mathcal{K}(\pm e_2) = J > 0,\ \mathcal{K}(k) = 0$ otherwise, where (e_1, e_2) is a basis of $\mathbb{R}^2$ and J a positive constant. The canonically associated potential φ_J (*cf.* (4)) is the following four-body potential: $\varphi_{J,V_k}(\sigma_{V_k}) = -\log\cosh(\beta J \sum_{j\in U_k} \sigma_j)$ where $U_k = \{k - e_1, k + e_1, k - e_2, k + e_2\}$. From Theorem 2 we conclude here that for β large, the PCA P_J is non-ergodic since it has at least two different stationary states ν^- and ν^+.

Let us now discuss how large is the domain where condition $(\mathcal{WM})$ holds. One conjectures Weak Mixing condition for Gibbs measure is valid up to the critical temperature, that is, as soon as there is no phase transition. In that sense, our main result would give ergodicity with exponential rate on a much larger region as the region where the Dobrushin-Vasershtein criterion holds. In fact, let us mention the reference [5], where, using percolation techniques, it is proved that in dimension $d = 2$, for a ferromagnetic nearest neighbour Ising model without extremal magnetic field, the associated Gibbs measure is weak mixing as soon as it is unique (*i.e.* $\forall\beta, \beta < \beta_c$). In order to precise this assertion, let us consider the dynamics P_J. A projection argument relates the potential φ_J associated to P_J with the usual Ising ferromagnetic pair potential with intensity coefficient J (see [16]). Due to Higuchi's result, we know that the Gibbs state associated to this potential φ_J is weak mixing as soon as there is no phase transition, which happens for β lower than the critical value β_c, which coincides with the Ising critical inverse temperature $\beta_c = \frac{\log(1+\sqrt{2})}{2J}$. In other words, we obtain that the PCA dynamics P_J is ergodic with exponential rate for $\beta < \beta_c$ and non-ergodic for $\beta > \beta_c$. Taking $J = 1$, $\beta_c \simeq 0.441$; since Dobrushin-Vasershtein criterion applies only for $\beta < \frac{1}{2}\mathrm{Argth}(\frac{1}{2}) \simeq 0.275$ (*cf.* part 6.1.2 *in* [10]), ours is better.

3 Proof of the Theorem 1

The proof of Theorem 1 is based on the existence of some coupling of PCA dynamics preserving the stochastic ordering. Let $(P^1, P^2, \ldots, P^N)$ be an increasing N-uple of PCA dynamics which means PCA related by the following monotonicity property $\forall k \in \mathbb{Z}^d$, $\forall \zeta^1 \preccurlyeq \zeta^2 \preccurlyeq \ldots \preccurlyeq \zeta^N \in S^{\mathbb{Z}^d}, \forall s \in S$, $G^1_k(s \mid \zeta^1) \leqslant G^2_k(s \mid \zeta^2) \leqslant \ldots \leqslant G^N_k(s \mid \zeta^N)$ where G^i is the function associated to P^i by (1). There exists (cf. [11]) a monotone synchronous coupling on $(S^{\mathbb{Z}^d})^N$ denoted by $P^1 \circledast P^2 \circledast \ldots \circledast P^N$ with the following property: for all initial configuration $\sigma^1 \preccurlyeq \sigma^2 \preccurlyeq \ldots \preccurlyeq \sigma^N$ and for all times n,

$$P^1 \circledast \ldots \circledast P^N \ \left(\ \omega^1(n) \preccurlyeq \ldots \preccurlyeq \omega^N(n) \ \middle| \ (\omega^1, \ldots, \omega^N)(0) = (\sigma^1, \ldots, \sigma^N) \ \right) = 1.$$

Such a coupling will be called *increasing synchronous coupling*. The notation $\mathbb{P}$ denotes the coupling $P \circledast P \circledast \ldots \circledast P$ of N times the same PCA dynamics P, where N will be a finite large enough number.

This coupling allows us to develop some monotonicity argument and to state the following result, whose proof is in [11]:

Proposition 3 *The measure ν^+_Λ (resp. ν^-_Λ) is the maximal (resp. minimal) measure of the set $\{\nu^\tau_\Lambda : \ \tau \in S^{\Lambda^c}\}$ of stationary measures associated to the PCA dynamics P^τ_Λ on the fixed finite volume Λ and with boundary condition τ. Let ν^+ and ν^- denote the maximal and the minimal elements of the set $\mathcal{S}$ of stationary measures associated to the PCA dynamics P. Following relations hold:*

$$\nu^+ = \lim_{L\to\infty} \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}} = \lim_{n\to\infty} \delta_+ P^{(n)} \tag{5}$$

$$\nu^- = \lim_{L\to\infty} \nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} = \lim_{n\to\infty} \delta_- P^{(n)}. \tag{6}$$

In particular, P admits a unique stationary measure ν if and only if $\nu^- = \nu^+$.

Note that $P^{(n)}$ denotes $P \circ P \circ \ldots \circ P$, and so is for instance $\delta_+ P^{(n)}$ the law at time n of the Markov Chain with transition kernel P and initial distribution δ_+.

Remark 4 Note the following range of dependence w.r.t. the past for local PCA. Let us define $\overline{\Lambda} = \cup_{k\in\Lambda} V_k = \overline{\Lambda}^{(1)}$, and $\overline{\Lambda}^{(n)} = \cup_{k\in\overline{\Lambda}^{(n-1)}} V_k$. Then: $\forall n, \forall \Lambda \Subset \mathbb{Z}^d$, $\forall (\sigma,\eta) \in (S^{\mathbb{Z}^d})^2$ with $\sigma_{\overline{\Lambda}^{(n)}} \equiv \eta_{\overline{\Lambda}^{(n)}}$, $\mathbb{P}\left(\omega^1_\Lambda(n) \equiv \omega^2_\Lambda(n) \middle| (\omega^1,\omega^2)(0) = (\sigma,\eta)\right) = 1$.

Proof. ((2) implies $(\mathcal{A})$ in Theorem 1)
It uses a usual strategy and takes advantage of the coupling $P \circledast P^+_{\mathcal{B}(L)}$. Let L be a fixed integer, larger than $L_1 = n_1$ where n_1 is defined in (2). Using the relation (stated in [11])

$$\nu^-_{\mathcal{B}(L)} \otimes \delta_{(-)_{\mathcal{B}(L)^c}} \preccurlyeq \nu \preccurlyeq \nu^+_{\mathcal{B}(L)} \otimes \delta_{(+)_{\mathcal{B}(L)^c}}; \tag{7}$$

the positivity of each following term is stated. We have:

$$0 \leqslant \int \sigma_0 \ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0 \ d\nu^-_{\mathcal{B}(L)} = \Big(\int \sigma_0 \ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0 \ d\nu \Big) + \Big(\int \sigma_0 \ d\nu - \int \sigma_0 \ d\nu^-_{\mathcal{B}(L)} \Big),$$

and we will state that each part is lower than $2\||\, f_0 \,\|| \, e^{-\lambda L}$ (where $f_0(\sigma) = \sigma_0$). We only give the proof for $\int \ \sigma_0 \ d\nu^+_{\mathcal{B}(L)} - \int \ \sigma_0 \ d\nu$ since the proof for the minimal $-$ boundary condition is analogous. For any $n \in \mathbb{N}^*$,

$$\nu^+_{\mathcal{B}(L)}(\sigma_0) - \nu(\sigma_0) = \Big(\nu^+_{\mathcal{B}(L)}(\sigma_0) - \delta_+ {P^+_{\mathcal{B}(L)}}^{(n)}(f_0) \Big) + \Big(\delta_+ {P^+_{\mathcal{B}(L)}}^{(n)}(f_0) - \delta_+ P^{(n)}(f_0) \Big) + \Big(\delta_+ P^{(n)}(f_0) - \nu(\sigma_0) \Big).$$

Using the monotonicity of $P^{+}_{\mathcal{B}(L)} \circledast P^{+}_{\mathcal{B}(L)}$ the first term is non positive. Using the assumption (2) the third term is bounded from above by $2 ||| f_0 ||| e^{-\lambda n}$ ($\forall n \geqslant n_1$). Choose now $n = L$. Rewrite the second term as $\mathbb{Q}^{+,+}(\omega^2_0(n) - \omega^1_0(n))$ where

$$\mathbb{Q}^{+,+}(\ .\ ) = P \circledast P^{+}_{\mathcal{B}(L)}\big(\ .\ |(\omega^1, \omega^2)(0) = (+, +)\big).$$

Using Lemma 5, we bound the second term from above with $\kappa' \, \mathbb{Q}^{+,+}(\omega^2_0(n) \neq \omega^1_0(n))$. According to the construction of the coupling and using Remark 4, note that with respect to $\mathbb{Q}^{+,+}(.)$, $\omega^2_0(n) \neq \omega^1_0(n)$ is possible only if it exists a previous time n' ($0 < n' < n$) and a site k in $\mathcal{B}(L)^c \cap \overline{\{0\}}^{(n')}$ such that $\omega^2_k(n') = \omega^1_k(n') \neq +$. By taking $n = L$, we have $\overline{\{0\}}^{(n')} \subset \mathcal{B}(L)$; so is this event empty, which ensures $\mathbb{Q}^{+,+}(\omega^2_0(n) \neq \omega^1_0(n)) = 0$. Thus is $(\mathcal{A})$ proved. □

Proof. (($\mathcal{A}$) implies (2) in Theorem 1)
The most delicate part is to establish the exponential rate of convergence towards equilibrium. Our proof is inspired by Martinelli and Olivieri proof of exponential ergodicity for continuous time Glauber dynamics on $\{-1, +1\}^{\mathbb{Z}^d}$ (see [14]). For any time $n \in \mathbb{N}$, let us define a coefficient which controls the ergodicity:

$$\rho(n) = \mathbb{P}\Big(\omega^1_0(n) \neq \omega^2_0(n) \Big| (\omega^1, \omega^2)(0) = (-, +)\Big). \tag{8}$$

If we assume the exponential bound ($\mathcal{A}$), thanks to forthcoming Lemma 8, we deduce that $\lim_{n\to\infty} \rho(n) = 0$. Reporting assumption ($\mathcal{A}$) in the inequality (10), we can use forthcoming Lemma 11 to deduce that $(\rho(n))_{n\in\mathbb{N}^*}$ converge to 0 faster than $\frac{1}{n^d}$. Finally, using inequality (9) and Lemma 12, we conclude that $\rho(n)$ converges to 0 exponentially fast; thus, thanks to Lemma 7, conclusion holds. □

Technical lemmas: First remark the easy fact:

Lemma 5 *Let $(\Omega, \mathfrak{A}, \mathcal{P})$ be a probability space, and Z a random variable with values in a finite set $\{z_1 < \ldots < z_m\}$ of $\mathbb{R}$, such that $\mathcal{P}(Z \geqslant 0) = 1$. Then, if $\kappa = \max\{\frac{1}{z_i}, z_i > 0, 1 \leqslant i \leqslant m\}$ and $\kappa' = max\{z_i, 1 \leqslant i \leqslant m\}$ (which do not depend on the law of Z under $\mathcal{P}$) we have: $\mathcal{P}(Z \neq 0) \leqslant \kappa \int Z d\mathcal{P}$ and $\int Z d\mathcal{P} \leqslant \kappa' \mathcal{P}(Z \neq 0)$.*

Using the monotonicity property of the coupling, the two following Lemmas are easily proved.

Lemma 6 ▪ $\forall \sigma, \eta \in S^{\mathbb{Z}^d}$, $\sigma \preccurlyeq \eta$, $\mathbb{P}\Big(\omega^1_0(n) \neq \omega^2_0(n) \Big| (\omega^1, \omega^2)(0) = (\sigma, \eta)\Big) \leqslant \rho(n)$.
▪ $\forall \Lambda \Subset \mathbb{Z}^d$, $\forall n \in \mathbb{N}$, $\forall \xi \in S^{\mathbb{Z}^d}$,
$P^{-}_{\Lambda}\big(\omega(n) \in . \big| \omega(0) = \xi_\Lambda(-)_{\Lambda^c}\big) \preccurlyeq P\big(\omega(n) \in . \big| \omega(0) = \xi\big) \preccurlyeq P^{+}_{\Lambda}\big(\omega(n) \in . \big| \omega(0) = \xi_\Lambda(+)_{\Lambda^c}\big)$.
▪ $\rho(n) \leqslant P^{-}_{\Lambda} \circledast P^{+}_{\Lambda}(\omega^1_0(n) \neq \omega^2_0(n)\ |(\omega^1, \omega^2)(0) = (-, +))$.

Lemma 7 *The sequence $(\rho(n))_{n\in\mathbb{N}^*}$ is decreasing, and $\forall f$, $\forall \sigma, \eta$,*

$$\Big| P(f(\omega(n))|\omega(0) = \sigma) - P(f(\omega(n))|\omega(0) = \eta) \Big| \leqslant\ 2\ |||\ f\ |||\ \rho(n).$$

Thus, if $\lim_{n\to\infty} \rho(n) = 0$, the dynamics P is ergodic, and $\sup_\sigma \Big| P(f(\omega(n))|\omega(0) = \sigma) - \nu(f) \Big|$ $\leqslant 2\ |||\ f\ |||\ \ \rho(n)$, where ν denotes the unique stationary measure.

Note that due to the monotonicity of $\rho(.)$, we can restrict ourselves to the case $\rho(.) > 0$.

Lemma 8 *$\exists \kappa$, $\forall \Lambda \Subset \mathbb{Z}^d$, $\lim_{n\to\infty} \rho(n) \leqslant \kappa\big(\int \sigma_0 \; d\nu_\Lambda^+ - \int \sigma_0 \; d\nu_\Lambda^-\big)$.*

PROOF. Note $P_\Lambda^- \circledast P_\Lambda^+\Big(\omega_0^1(n) \leqslant \omega_0^2(n)) \;\Big|\; (\omega^1(0), \omega^2(0)) = (-,+)\Big) = 1$ since the coupling preserves the order. So, thanks to Lemma 5, applied with $\mathcal{P} = P_\Lambda^- \circledast P_\Lambda^+(\;.\;|(\omega^1(0), \omega^2(0)) = (-,+))$ and $Z = \omega_0^2(n) - \omega_0^1(n)$ we have:

$P_\Lambda^- \circledast P_\Lambda^+\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1(0), \omega^2(0)) = (-,+)\Big) \leqslant \kappa\Big(P_\Lambda^+(\omega_0(n)|\omega(0) = +) - P_\Lambda^-(\omega_0(n)|\omega(0) = -)\Big)$

where $\kappa = (\min\{s - s' : s > s'; s, s' \in S\})^{-1}$. By Lemma 6, $\rho(n)$ is bounded from above by the l.h.s of the previous inequality. We conclude by taking the limit in n, and using the finite volume ergodicity. □

Remark 9 *As an immediate consequence of Lemma 8 we get $\lim_{n\to\infty} \rho(n) = 0$, which implies the ergodicity of P thanks to Lemma 7.*

Let us denote by $R = \max_{k' \in V_0} \|k'\|_1$ the finite range of the local translation invariant PCA dynamics P.

Lemma 10 *The following two inequalities hold:*

$$\forall n \in \mathbb{N}^*, \;\; \rho(2n) \leqslant (2nR+1)^d \rho^2(n) \;; \tag{9}$$

$$\forall n, \forall L \in \mathbb{N}^*, \;\; \rho(2n) \leqslant 2(2L+1)^d \rho^2(n) + 2\kappa\Big(\int \sigma_0 \; d\nu_{\mathcal{B}(L)}^+ - \int \sigma_0 \; d\nu_{\mathcal{B}(L)}^-\Big) \;. \tag{10}$$

PROOF. Let n be a fixed integer.
Proof of inequality (9)
Let $\nu_n^{-,+}(\;.\;) = \mathbb{P}\Big((\omega^1, \omega^2)(n) \in .\; \Big|(\omega^1, \omega^2)(0) = (-,+)\Big)$. Using Markov property of $\mathbb{P}$:

$$\rho(2n) = \int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^2(2n)\Big|(\omega^1, \omega^2)(n) = (\xi^-, \xi^+)\Big)\nu_n^{-,+}(d\xi^-, d\xi^+) \;.$$

Note that $\nu_n^{-,+}$-almost surely, $\xi^- \preccurlyeq \xi^+$. Let $A = \{(\xi^-, \xi^+) : \; \exists k \in \mathbb{Z}^d, \; \|k\|_1 \leqslant nR, \; \xi_k^- \neq \xi_k^+\}$. Thanks to Remark 4 observe that the exact control of interaction information's propagation for PCA implies that the above integral vanishes on A^c because $\mathcal{B}(nR) \supset \overline{\{0\}}^{(n)}$, and so $\xi_{\mathcal{B}(nR)}^- \equiv \xi_{\mathcal{B}(nR)}^+$. Then:

$$\rho(2n) = \int_A \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1, \omega^2)(0) = (\xi^-, \xi^+)\Big) \; \nu_n^{-,+}(d\xi^-, d\xi^+) \;.$$

Using Lemma 6, we obtain $\rho(2n) \leqslant \rho(n) \; \nu_n^{-,+}(A)$.
Writing $A = \cup_{\{k \in \mathbb{Z}^d: \; \|k\|_1 \leqslant nR\}}\{(\xi^-, \xi^+) : \; \xi_k^- \neq \xi_k^+\}$ we deduce:

$$\nu_n^{-,+}(A) \leqslant \sum_{k \in \mathbb{Z}^d, \|k\|_1 \leqslant nR} \mathbb{P}\Big(\omega_k^1(n) \neq \omega_k^2(n)\Big|(\omega^1, \omega^2)(0) = (-,+)\Big).$$

Since P is translation invariant, the conclusion follows from $\nu_n^{-,+}(A) \leqslant \rho(n)\#\mathcal{B}(nR) \leqslant \rho(n)(2nR+1)^d$ where $\#\mathcal{B}(nR)$ denotes the cardinality of $\mathcal{B}(nR)$.

Proof of inequality (10)

Write $\rho(2n) = \int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^3(2n)\Big|(\omega^1,\omega^2,\omega^3)(0) = (-,\eta,+)\Big)\nu(d\eta)$ where ν is a P-stationary measure. Note that $\omega_0^1(n) \leqslant \omega_0^2(n) \leqslant \omega_0^3(n)$,
$\mathbb{P}\Big((\omega^1,\omega^2,\omega^3) \in \ . \ \Big| \ (\omega^1,\omega^2,\omega^3)(0) = (-,\eta,+)\Big)$-almost surely, so that
$\{\omega_0^1(n) \neq \omega_0^3(n)\} = \{\omega_0^1(n) \neq \omega_0^2(n)\} \ \cup \ \{\omega_0^2(n) \neq \omega_0^3(n)\}$, where the union is non necessarily disjoint (unless cardinality of S is 2). Thus, following decomposition holds:

$$\begin{aligned}\rho(2n) \leqslant & \int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^2(2n)\Big|(\omega^1,\omega^2)(0) = (-,\eta)\Big) \ \nu(d\eta) \\ & + \int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^2(2n)\Big|(\omega^1,\omega^2)(0) = (\eta,+)\Big) \ \nu(d\eta) \ . \end{aligned} \tag{11}$$

It is then enough to prove that each of these quantities are bounded from above by half the quantity wanted. Consider first the second term in the r.h.s. .
Let $\nu_n^{\eta,+} = \mathbb{P}\Big((\omega^1,\omega^2)(n) = \ . \ \Big| \ (\omega^1,\omega^2)(0) = (\eta,+)\Big)$. Let us write:

$$\begin{aligned} & \int \mathbb{P}\Big(\omega_0^1(2n) \neq \omega_0^2(2n)\Big|(\omega^1,\omega^2)(0) = (\eta,+)\Big) \ \nu(d\eta) \\ & = \iint \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1,\omega^2)(0) = (\xi^1,\xi^2)\Big) \ \nu_n^{\eta,+}(d\xi^1,d\xi^2) \ \nu(d\eta) \ . \end{aligned}$$

Let $L \in \mathbb{N}^*$ and $A_L = \{(\xi^1,\xi^2) \in (S^{\mathbb{Z}^d})^2 : \ (\xi^1)_{\mathcal{B}(L)} \equiv (\xi^2)_{\mathcal{B}(L)}\}$. Let decompose the integration with respect to (ξ^1,ξ^2) into an integration on A_L^c and A_L. We will prove that:

$$\begin{aligned}(I) \ = \ & \iint_{A_L^c} \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1,\omega^2)(0) = (\xi^1,\xi^2)\Big) \ \nu_n^{\eta,+}(d\xi^1,d\xi^2) \ \nu(d\eta) \\ & \leqslant (2L+1)^d \rho^2(n), \end{aligned} \tag{12}$$

$$\begin{aligned}(II) \ = \ & \iint_{A_L} \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1,\omega^2)(0) = (\xi^1,\xi^2)\Big) \ \nu_n^{\eta,+}(d\xi^1,d\xi^2) \ \nu(d\eta) \\ & \leqslant \kappa\Big(\int \sigma_0 \ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0 \ d\nu^-_{\mathcal{B}(L)}\Big) \end{aligned} \tag{13}$$

Let us consider part (I). Thanks to $\nu_n^{\eta,+}(\xi^1 \preccurlyeq \xi^2) = 1$ and using Lemma 6, we have
$(I) \leqslant \rho(n) \int \nu_n^{\eta,+}(A_L^c) \ \nu(d\eta)$. Note that A_L^c may also be written $\cup_{k\in\mathcal{B}(L)}\{(\xi^1,\xi^2) : \ (\xi^1)_k \neq (\xi^1)_k\}$.
Thus we have:

$$\nu_n^{\eta,+}(A_L^c) \leqslant \sum_{k\in\mathcal{B}(L)} \nu_n^{\eta,+}\{(\xi^1,\xi^2) : \ (\xi^1)_k \neq (\xi^2)_k\} \ .$$

Using translation invariance of the coupling and Lemma 6, the previous general term is equal to $\mathbb{P}\Big(\omega_k^1(n) \neq \omega_k^2(n)\Big|(\omega^1,\omega^2)(0) = (\eta,+)\Big) \leqslant \rho(n)$. So $\nu_n^{\eta,+}(A_L^c) \leqslant \#\mathcal{B}(L) \ \rho(n)$, and then (12) follows.

Part (II): let $\tau \in S^{\mathcal{B}(L)}$ be fixed, and define $A_{L,\tau} = \{(\xi^1,\xi^2) : \ (\xi^1)_{\mathcal{B}(L)} \equiv (\xi^2)_{\mathcal{B}(L)} \equiv \tau\}$. So $A_L = \bigsqcup_{\tau\in S^{\mathcal{B}(L)}} A_{L,\tau}$ and following decomposition holds:

$$(II) = \int \sum_{\tau\in S^{\mathcal{B}(L)}} \int \mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1,\omega^2)(0) = (\xi_1,\xi_2)\Big) \mathbb{1}_{A_{L,\tau}}(\xi^1,\xi^2) \nu_n^{\eta,+}(d\xi^1,d\xi^2) \ \nu(d\eta). \tag{14}$$

Let us now use the finite volume dynamics. $\nu_n^{\eta,+}$ almost surely, we have $\xi^1 \preccurlyeq \xi^2$, $(\xi^1)_{\mathcal{B}(L)} = (\xi^2)_{\mathcal{B}(L)} = \tau$ and also $\xi^2 = \tau(\xi^2)_{\mathcal{B}(L)^c} \preccurlyeq \tau(+)_{\mathcal{B}(L)^c}$, $\tau(-)_{\mathcal{B}(L)^c} \preccurlyeq \xi^1 = \tau(\xi^1)_{\mathcal{B}(L)^c}$. Then:

$$P^-_{\mathcal{B}(L)} \circledast P \circledast P \circledast P^+_{\mathcal{B}(L)}\big(\omega^1 \preccurlyeq \omega^2 \preccurlyeq \omega^3 \preccurlyeq \omega^4 \,\Big|(\omega^1_{\mathcal{B}(L)}, \omega^2, \omega^3, \omega^4_{\mathcal{B}(L)})(0) = \\ (\tau, \tau(\xi^1)_{\mathcal{B}(L)^c}, \tau(\xi^2)_{\mathcal{B}(L)^c}, \tau)\big) = 1$$

which implies:

$$\begin{aligned} &\mathbb{P}\Big(\omega_0^1(n) \neq \omega_0^2(n)\Big|(\omega^1,\omega^2)(0) = (\tau\xi^1_{\mathcal{B}(L)^c}, \tau\xi^2_{\mathcal{B}(L)^c})\Big) \\ &\leqslant \; P^-_{\mathcal{B}(L)} \circledast P^+_{\mathcal{B}(L)}\big(\omega_0^1(n) \neq \omega_0^2(n) \mid (\omega^1,\omega^2)(0) = (\tau,\tau)\big) \\ &\leqslant \; \kappa\Big(P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau) - P^-_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\Big), \end{aligned} \tag{15}$$

where the last inequality comes from Lemma 5 and from the fact that $P^-_{\mathcal{B}(L)} \circledast P^+_{\mathcal{B}(L)}\Big(\,.\, \Big|\, (\omega^1,\omega^2)(0) = (\tau,\tau)\Big)$-almost surely, we have $\omega_0^1(n) \leqslant \omega_0^2(n)$.
On the other hand, note the following inequality:

$$\begin{aligned} \nu_n^{\eta,+}(A_{L,\tau}) &= \mathbb{P}\Big(\omega^1(n)_{\mathcal{B}(L)} \equiv \omega^2_{\mathcal{B}(L)}(n) \equiv \tau\Big|(\omega^1,\omega^2)(0) = (\eta,+)\Big) \\ &\leqslant \nu_n^{\eta,+}\Big((\xi^1,\xi^2):\ (\xi^1)_{\mathcal{B}(L)} \equiv \tau\Big) = P(\omega_{\mathcal{B}(L)}(n) = \tau \mid \omega_{\mathcal{B}(L)}(0) = \eta)\ . \end{aligned} \tag{16}$$

Reporting (15) and (16) in (14) we find

$$(II) \leqslant \kappa \int \sum_{\tau \in S^{\mathcal{B}(L)}} \Big(P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau) - P^-_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\Big) \\ P(\omega_{\mathcal{B}(L)}(n) = \tau \mid \omega_{\mathcal{B}(L)}(0) = \eta)\ \nu(d\eta) \leqslant \kappa\big((a) - (b)\big)\ .$$

We remark that $(a) = \int P\Big(f_{n,+}(\omega_{\mathcal{B}(L)}(n)) \,\Big|\, \omega_{\mathcal{B}(L)}(0) = \eta\Big)\ \nu(d\eta)$ with $f_{n,+}(\tau) = P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)$. Using the fact that the function $f_{n,+}(.)$ is increasing, and Lemma 6 we state:

$$(a) \leqslant \int \sum_{\tau \in S^{\mathcal{B}(L)}} P^+_{\mathcal{B}(L)}(\omega_0(n) \mid \omega_{\mathcal{B}(L)}(0) = \tau)\ P^+_{\mathcal{B}(L)}(\omega_{\mathcal{B}(L)}(n) = \tau \mid \omega_{\mathcal{B}(L)}(0) = \eta_{\mathcal{B}(L)})\ \nu(d\eta)\ .$$

Using Markov property for the $P^+_{\mathcal{B}(L)}$ finite volume dynamics, we find: $(a) \leqslant \nu(f_{2n,+})$. The function $f_{2n,+}$ is increasing; thanks to inequality (7), we thus have $(a) \leqslant \nu^+_{\mathcal{B}(L)}(f_{2n,+})$. We can now write:

$$(a) \leqslant \int P^+_{\mathcal{B}(L)}(\omega_0(2n) | \omega_{\mathcal{B}(L)}(0) = \eta_{\mathcal{B}(L)})\ \nu^+_{\mathcal{B}(L)}(d\eta_{\mathcal{B}(L)}) = \int \sigma_0\ d\nu^+_{\mathcal{B}(L)},$$

where the last equality comes from the stationarity of $\nu^+_{\mathcal{B}(L)}$ with respect to $P^+_{\mathcal{B}(L)}$.

Analogously we prove $(b) \geqslant \int \sigma_0\ d\nu^-_{\mathcal{B}(L)}$. Thus, the following inequality holds:

$$(II) \leqslant \kappa\big((a) - (b)\big) \leqslant \kappa\left(\int \sigma_0\ d\nu^+_{\mathcal{B}(L)} - \int \sigma_0\ d\nu^-_{\mathcal{B}(L)}\right),$$

which gives the estimate of the second term in inequality (11). The first term is treated in the same way. So the recursive inequality (10) is established. □

We now state some general analytic lemmas; for proofs see [10, 14].

Lemma 11 *If* $\lim_{n\to\infty}\rho(n)=0$ *and if* $\exists\ (\tilde{C},M)\in(\mathbb{R}_*^+)^2,\ \exists L_1\in\mathbb{N}^*,\forall L\in\mathbb{N}^*,L\geqslant L_1,\forall n\in\mathbb{N}^*$

$$\rho(2n)\leqslant 2(2L+1)^d\rho(n)^2+2\tilde{C}e^{-ML}$$

then $\lim_{n\to\infty}n^d\rho(n)=0$.

Lemma 12 *If* $\lim_{n\to\infty}n^d\rho(n)=0$, *and if inequality (9) holds then, for all* n_1 *such that* $(2^d\hat{C})\ n_1^d\rho(n_1)<1$, *we have:*

$$\forall n\geqslant n_1,\ \rho(n)\leqslant e^{-\lambda n}$$

where $\lambda=-\frac{1}{2n_1}\log(2^d\hat{C}n_1^d\rho(n_1))>0$.

4 Proof of the Theorem 2

For general PCA in finite volume, invariant measures are not explicitly known; but for the class $\mathcal{C}_0$ here considered, we computed them (*cf.* Proposition 3.1 *in* [1]). The unique reversible measure for the PCA dynamics P_Λ^τ is defined by

$$\nu_\Lambda^\tau(\sigma)=\frac{1}{\mathcal{W}_\Lambda^\tau}\prod_{k\in\Lambda}\cosh\left(\beta\sum_{j\in\mathbb{Z}^d}\mathcal{K}(k-j)\tilde{\sigma}_j\right)e^{\beta\sigma_k\sum_{j\in\Lambda^c}\mathcal{K}(k-j)\tau_j},\tag{17}$$

where $\tilde{\sigma}=\sigma_\Lambda\tau_{\Lambda^c}$, and $\mathcal{W}_\Lambda^\tau$ is the normalisation factor. Such measure does not coincide with the finite volume Gibbs measures $\mu_\Lambda^\tau(\sigma)=\frac{1}{\mathcal{Z}_\Lambda^\tau}\exp(-\sum_{A\subset\mathbb{Z}^d,A\cap\Lambda\neq\varnothing}\varphi_A(\sigma_\Lambda\tau_{\Lambda^c}))$ contrary to what happens for Glauber dynamics when detailed balance holds. Nevertheless, they are related as relation (18) attempts. We will not write down all technical computations which prove relations (18), (19). Interested reader may refer respectively to Proposition 4.1.8 and Property 4.1.12 *in* [10].

Let Λ,Λ' two finite subsets of $\mathbb{Z}^d$ such that $\Lambda\subset\Lambda'$ and $\partial_i\Lambda\cap\partial_i\Lambda'=\varnothing$, where $\partial_i\Lambda\triangleq\{k\in\Lambda: U_k\cap\Lambda^c\neq\varnothing\}$. Let τ' be a boundary condition of Λ and $\mu_\Lambda^{\tau'}$ denotes the finite volume Gibbs distribution associated to the potential φ on the volume Λ with boundary condition τ'. We then state:

$$\nu_{\Lambda'}^\tau(d\sigma_\Lambda|\sigma_{\Lambda'\setminus\Lambda})=\mu_\Lambda^{\sigma_{\Lambda'\setminus\Lambda}\tau_{\Lambda'^c}}(d\sigma_\Lambda)\ .\tag{18}$$

Note that the potential φ is not really a ferromagnetic potential in the usual sense. However we can check that associated finite volume Gibbs measures verify a kind of monotone behaviour: $\tau_1\preccurlyeq\tau_2\Rightarrow\mu_\Lambda^{\tau_1}\preccurlyeq\mu_\Lambda^{\tau_2}$ (see Proposition 4.1.9 *in* [10]). In particular, Gibbs measures on $S^{\mathbb{Z}^d}$ obtained as $\mu^+=\lim_{\Lambda\nearrow\mathbb{Z}^d}\mu_\Lambda^{(+)_{\Lambda^c}}$ and $\mu^-=\lim_{\Lambda\nearrow\mathbb{Z}^d}\mu_\Lambda^{(-)_{\Lambda^c}}$ are extremal states in the sense of stochastic ordering of the set $\mathcal{G}(\varphi)$. Recall μ probability measure on $S^{\mathbb{Z}^d}$ is in $\mathcal{G}(\varphi)$ if, *per definitionem*, for any finite volume $\Lambda\subset\mathbb{Z}^d$, a version of the conditioned measure $\mu(d\sigma_\Lambda|\sigma_{\Lambda^c})$ is $\mu_\Lambda^{\sigma_{\Lambda^c}}(d\sigma_\Lambda)$. Finally, let us state the following lemma:

Lemma 13 *If the Weak Mixing Condition* $(\mathcal{WM})$ *holds for the potential* φ *associated to the PCA dynamics* P, *then assumption* $(\mathcal{A})$ *holds for* P.

Proof. According to the finite range R, let $L > R$. It is enough to show $\left(\int \sigma_0 \, d\nu^+_{\mathcal{B}(L)} - \int \sigma_0 \, d\nu^-_{\mathcal{B}(L)}\right) \leqslant \left(\int \sigma_0 \, d\mu^+_{\mathcal{B}(L-R)} - \int \sigma_0 \, d\mu^-_{\mathcal{B}(L-R)}\right)$. Let us first check $\int \sigma_0 \, d\nu^+_{\mathcal{B}(L)} \leqslant \int \sigma_0 \, d\mu^+_{\mathcal{B}(L-R)}$. Let f_0 be the increasing function defined on $S^{\mathbb{Z}^d}$ by $f_0(\sigma) = \sigma_0$. Note $\int \sigma_0 \, d\nu^+_{\mathcal{B}(L)} = \nu^+_{\mathcal{B}(L)}(\ \nu^+_{\mathcal{B}(L)}(\ f_0 \ |\sigma_{\mathcal{B}(L)\setminus\mathcal{B}(L-R)}))$. Using relation (18) with $\Lambda' = \mathcal{B}(L)$ and $\Lambda = \mathcal{B}(L-R)$, we then get $\nu^+_{\mathcal{B}(L)}(f_0) = \nu^+_{\mathcal{B}(L)}(\mu^{\sigma_{\mathcal{B}(L)\setminus\mathcal{B}(L-R)}(+\mathbf{1})_{\mathcal{B}(L)^c}}_{\mathcal{B}(L-R)}(f_0))$. On the other hand, using the monotonicity in the boundary condition of the finite volume Gibbs measures, we find $\mu^{\sigma_{\mathcal{B}(L)\setminus\mathcal{B}(L-R)}(+\mathbf{1})_{\mathcal{B}(L)^c}}_{\mathcal{B}(L-R)}(f_0) \leqslant \mu^{(+\mathbf{1})_{\mathcal{B}(L-R)^c}}_{\mathcal{B}(L-R)}(f_0)$. So desired inequality holds. $\nu^-_{\mathcal{B}(L)}(f_0) \geqslant \mu^-_{\mathcal{B}(L-R)}(f_0)$ can be analogously checked. □

Lemma 14 *For a PCA dynamics P of class $\mathcal{C}_0$ with $\mathcal{K}(.)$ non negative, the extremal stationary measures ν^-, ν^+ coincide respectively with extremal Gibbs measures μ^- and μ^+ of $\mathcal{G}(\varphi)$ (possibly these four measures coincide).*

Proof. Let Λ, Λ' be two finite subsets of $\mathbb{Z}^d$ such that $\Lambda \subset \Lambda'$. Then, for all configurations $\sigma_{\Lambda'\setminus\Lambda} \in S^{\Lambda'\setminus\Lambda}$, finite volume reversible measures with extremal boundary condition are such that:

$$\nu^+_{\Lambda'}\left((.)_\Lambda|\sigma_{\Lambda'\setminus\Lambda}\right) \preccurlyeq \nu^+_\Lambda(.) \ ; \quad \nu^-_{\Lambda'}\left((.)_\Lambda|\sigma_{\Lambda'\setminus\Lambda}\right) \succcurlyeq \nu^-_\Lambda(.) \tag{19}$$

(see Property 4.1.12 *in* [10] for a precise proof). Using relation (18), we can deduce from the previous result the following inequalities between finite volume Gibbs measure and reversible measure, with extremal boundary condition: $\mu^+_\Lambda \preccurlyeq \nu^+_\Lambda$ and $\mu^-_\Lambda \succcurlyeq \nu^-_\Lambda$. Taking now the limit in volume, we find: $\mu^+ \preccurlyeq \nu^+$ and $\mu^- \succcurlyeq \nu^-$.

On the other hand, ν^+_Λ is P^+_Λ-reversible, so taking the limit, ν^+ is P-reversible. Analogously, ν^- is P-reversible. From $\mathcal{R} = \mathcal{S} \cap \mathcal{G}(\varphi)$, we conclude ν^- and ν^+ are Gibbs measures, so thanks to the fact that μ^- and μ^+ are stochastic ordering extremal states for Gibbs measures, we deduce: $\nu^+ \preccurlyeq \mu^+$ and $\mu^- \preccurlyeq \nu^-$. Thus the conclusion follows. □

Here is the proof of Theorem 2:

Proof. When there is phase transition, since μ^- and μ^+ are extremal states for $\mathcal{G}(\varphi)$, we have that $\mu^- \neq \mu^+$. So, using Lemma 14, the two reversible (also stationary) measures ν^- and ν^+ are different. Then, dynamics P can not be ergodic.

When there is no phase transition, then $\mathcal{G}(\varphi) = \{\mu\}$ where $\mu = \mu^- = \mu^+$ is the unique Gibbs state. Thanks to Lemma 14, it holds $\nu^- = \mu^- = \mu^+ = \nu^+$. The Proposition 3 states the uniqueness of the P-stationary measure and the ergodicity of the PCA dynamics P.

Finally, if weak mixing condition $(\mathcal{WM})$ is assumed, then Lemma 13 implies that inequality $(\mathcal{A})$ holds. We conclude using Theorem 1. □

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