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PATHWISE UNIQUENESS FOR REFLECTING BROWNIAN MOTION IN CERTAIN PLANAR LIPSCHITZ DOMAINS

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Abstract

We give a simple proof that in a Lipschitz domain in two dimensions with Lipschitz constant one, there is pathwise uniqueness for the Skorokhod equation governing reflecting Brownian motion.

Suppose that $D \subset \mathbb{R}^2$ is a Lipschitz domain and let $\mathbf{n}(x)$ denote the inward-pointing unit normal vector at those points $x \in \partial D$ for which such a vector can be uniquely defined (such x form a subset of ∂D of full surface measure). Suppose $(\Omega, \mathcal{F}, \mathbf{P})$ is a probability space. Consider the following equation for reflecting Brownian motion with normal reflection taking values in $\overline{D}$, known as the (stochastic) Skorokhod equation:

$$X_t = x_0 + W_t + \int_0^t \mathbf{n}(X_s)\, dL_s \qquad t \geq 0. \tag{1}$$

We suppose there is a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and $W = \{W_t, t \geq 0\}$ is a 2-dimensional Brownian motion with respect to $\{\mathcal{F}_t\}$. In particular, if $s < t$, we have $W_t - W_s$ independent of $\mathcal{F}_s$. Also $L = \{L_t,\ t \geq 0\}$ is the local time of $X = \{X_t,\ t \geq 0\}$ on ∂D, that is, a continuous nondecreasing process that increases only when X is on the boundary ∂D and such that L does not charge any set of zero surface measure. Moreover we require X to be adapted to $\{\mathcal{F}_t\}$.

We say that pathwise uniqueness holds for (1) if whenever X and X' are two solutions to (1) with possibly two different filtrations $\{\mathcal{F}_t\}$ and $\{\mathcal{F}'_t\}$, resp., then $\mathbf{P}(X_t = X'_t \text{ for all } t \geq 0) = 1$. In this note we give a short proof of the following theorem.

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Theorem 1 *Suppose $D \subset \mathbb{R}^2$ is a Lipschitz domain whose boundary is represented locally by Lipschitz functions with Lipschitz constant 1. Then we have pathwise uniqueness for the solution of (1).*

We remark that there are varying definitions of pathwise uniqueness in the literature. Some references, e.g., [4], allow different filtrations for X and X', while others, e.g., [5], do not. We prove pathwise uniqueness with the definition used by [4], which yields the strongest theorem. Theorem 1 was first proved in [2], with a vastly more complicated proof. Moreover, in that proof, it was required that the Lipschitz constant be strictly less than one. Strong existence was also proved in [2]; it will be apparent from our proof that we also establish strong existence. In $C^{1+\alpha}$ domains with $\alpha > 0$, the assumption that L not charge any sets of zero surface measure is superfluous; see [3], Theorem 4.2. (There is an error in the proof of Theorem 3.5 of that paper, but this does not affect Theorem 4.2.)
In a forthcoming paper, the authors plan to prove that pathwise uniqueness holds in $C^{1+\alpha}$ domains in $\mathbf{R}^d$ for $d \geq 3$ and $\alpha > 1/2$, but that pathwise uniqueness fails for some $C^{1+\alpha}$ domains in $\mathbf{R}^3$ with $\alpha > 0$. We do not have a conjecture as to whether pathwise uniqueness holds in all two-dimensional Lipschitz domains.

Proof. Standard arguments allow us to limit ourselves to domains of the following form

$$D = \{(x_1, x_2) : f(x_1) < x_2\},$$

where $f : \mathbb{R} \to \mathbb{R}$ satisfies the following conditions: $f(0) = 0$ and $|f(x_1) - f(y_1)| \leq |x_1 - y_1|$. Consider any $x_0 \in \overline{D}$ and processes X and Y taking values in $\overline{D}$ such that a.s.,

$$\begin{aligned} X_t &= x_0 + W_t + \int_0^t \mathbf{n}(X_s)\, dL_s^X, \qquad t \geq 0, \\ Y_t &= x_0 + W_t + \int_0^t \mathbf{n}(Y_s)\, dL_s^Y, \qquad t \geq 0. \end{aligned} \tag{2}$$

We will first assume that the filtrations for X and Y are the same, and then remove that assumption at the end of the proof. Here L^X is the local time of X on ∂D, that is, a continuous nondecreasing process that increases only when X is on the boundary ∂D and that does not charge any set of zero surface measure. The processes L^Y is defined in an analogous way relative to Y.
We will write $X_t = (X_t^1, X_t^2)$ and similarly for Y. Let

$$\begin{aligned} V_t &= \begin{cases} X_t & \text{if } X_t^1 < Y_t^1, \\ Y_t & \text{otherwise,} \end{cases} \\ L_t^V &= \int_0^t \mathbf{1}_{\{X_s^1 < Y_s^1\}} dL_s^X + \int_0^t \mathbf{1}_{\{X_s^1 \geq Y_s^1\}} dL_s^Y. \end{aligned}$$

Next we will show that, a.s.,

$$V_t = x_0 + W_t + \int_0^t \mathbf{n}(V_s)\, dL_s^V, \qquad t \geq 0. \tag{3}$$

The following proof of (3) applies to almost all trajectories because it refers to properties that hold a.s. We will define below times t_1 and t_2. They are random in the sense that they depend on ω in the sample space but we do not make any claims about their measurability. In particular, we do not claim that they are stopping times.

Let K be the open cone $\{(x_1, x_2) : x_2 > |x_1|\}$. First we will show that there are no $t > 0$ such that $X_t - Y_t \in K$ or $Y_t - X_t \in K$. Suppose that there exists $t_1 > 0$ such that $X_{t_1} - Y_{t_1} \in K$. Note that $X_0 - Y_0 = 0 \notin K$. Let $t_2 = \sup\{t \in (0, t_1) : X_t - Y_t \notin K\}$ and note that $X_{t_2} - Y_{t_2} \notin K$ because K is open. Hence t_2 is strictly less than t_1. For $t \in (t_2, t_1)$, $X_t - Y_t \in K$, so $X_t \in D$, because for any $x \in \partial D$ and $y \in \mathbb{R}^2$ such that $x - y \in K$, we have $y \notin \overline{D}$. We see that $L^X_{t_1} - L^X_{t_2} = 0$. We have

$$X_t - Y_t = \int_0^t \mathbf{n}(X_s)\, dL^X_s - \int_0^t \mathbf{n}(Y_s)\, dL^Y_s.$$

Since $L^X_{t_1} - L^X_{t_2} = 0$,

$$(X_{t_1} - Y_{t_1}) - (X_{t_2} - Y_{t_2}) = -\int_{t_2}^{t_1} \mathbf{n}(Y_s)\, dL^Y_s. \tag{4}$$

We have $\mathbf{n}(x) \in \overline{K}$ for every $x \in \partial D$ where $\mathbf{n}(x)$ is well defined. Hence $\int_{t_2}^{t_1} \mathbf{n}(X_s) dL^X_s \in \overline{K}$. For all $x, y, z \in \mathbb{R}^2$ such that $x \in K$, $y \notin K$ and $-z \in \overline{K}$, we have $x - y \neq z$. We apply this to $x = X_{t_1} - Y_{t_1}$, $y = X_{t_2} - Y_{t_2}$ and $z = -\int_{t_2}^{t_1} \mathbf{n}(X_s) dL^X_s$ to obtain a contradiction with (4). This contradiction shows that there does not exist t with $X_t - Y_t \in K$. By the same argument with X and Y reversed, there does not exist t with $Y_t - X_t \in K$.

Simple geometry shows that if $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$, $x_1 = y_1$, $x - y \notin K$ and $y - x \notin K$ then $x = y$. We apply this observation to $x = X_t$ and $y = Y_t$ to conclude that if $X^1_t = Y^1_t$, then $X_t = Y_t$. This implies that if $V^1_t = X^1_t$ then $V_t = X_t$.

Fix some $t_0 > 0$, let $J = [0, t_0]$ and $u_1 = \sup\{u \le t_0 : X^1_u = Y^1_u\}$. By the continuity of X and Y, the set $I = \{t \in (0, u_1) : X^1_t < Y^1_t\}$ is open. Thus it consists of a finite or countable union of disjoint intervals $\{I_n\}$. For any $I_n = (s_1, s_2)$ we have $X^1_{s_1} = Y^1_{s_1}$ and, therefore, $X_{s_1} = Y_{s_1}$. Similarly, $X_{s_2} = Y_{s_2}$. It follows that

$$\int_{I_n} \mathbf{n}(X_s) dL^X_s = \int_{I_n} \mathbf{n}(Y_s)\, dL^Y_s. \tag{5}$$

Suppose without loss of generality that $V_{t_0} = Y_{t_0}$. Then by (2)

$$V_{t_0} = x_0 + W_{t_0} + \int_0^{t_0} \mathbf{n}(Y_s)\, dL^Y_s.$$

By (5),

$$V_{t_0} = x_0 + W_{t_0} + \int_{I_1} \mathbf{n}(X_s)\, dL^X_s + \int_{J \setminus I_1} \mathbf{n}(Y_s)\, dL^Y_s.$$

By induction, for any n,

$$V_{t_0} = x_0 + W_{t_0} + \int_{\bigcup_{k \le n} I_k} \mathbf{n}(X_s)\, dL^X_s + \int_{J \setminus \bigcup_{k \le n} I_k} \mathbf{n}(Y_s)\, dL^Y_s.$$

We can pass to the limit by the bounded convergence theorem applied to each component of the 2-dimensional vectors on the measure spaces defined by dL^X and dL^Y on the interval J. We obtain in the limit

$$\begin{aligned} V_{t_0} &= x_0 + W_{t_0} + \int_{\bigcup_{k\geq 0} I_k} \mathbf{n}(X_s)\, dL_s^X + \int_{J\setminus \bigcup_{k\geq 0} I_k} \mathbf{n}(Y_s)\, dL_s^Y \\ &= x_0 + W_{t_0} + \int_{\bigcup_{k\geq 0} I_k} \mathbf{n}(V_s)\, dL_s^X + \int_{J\setminus \bigcup_{k\geq 0} I_k} \mathbf{n}(V_s)\, dL_s^Y \\ &= x_0 + W_{t_0} + \int_0^{t_0} \mathbf{n}(V_s)\, dL_s^V. \end{aligned}$$

This proves (3).
It follows from (3) and Theorem 1.1 (i) of [2] that V has the distribution of reflecting Brownian motion in $\overline{D}$ as defined in [2]. Since X and V have identical distributions and $V_t^1 \leq X_t^1$ for every $t \geq 0$, a.s., we conclude that $V_t^1 = X_t^1$ for every $t \geq 0$, a.s. The same is true with X replaced by Y. Therefore we have that $X_t = V_t = Y_t$ for every $t \geq 0$, a.s.
We have therefore proved pathwise uniqueness in the sense of [5], p. 339. Then by Theorem IX.1.7(ii) of [5], a strong solution to (1) exists. (The context of that theorem is a bit different, but the proof applies to the present situation almost without change.) Finally, by the proof of Theorem 5.8 of [2], we have pathwise uniqueness even when the filtrations of X and Y are not the same. □

The overall structure of our proof is similar to that of the proof of Theorem 3.1 in [1]. Martin Barlow pointed out to us that an alternate way of avoiding consideration of the two different definitions of pathwise uniqueness is to pass to the Loeb space. We would like to thank Dave White for pointing out a mistake in the original version of the proof.

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