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AN EASY PROOF OF THE $\zeta(2)$ LIMIT IN THE RANDOM ASSIGNMENT PROBLEM

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Abstract
The edges of the complete bipartite graph $K_{n,n}$ are given independent exponentially distributed costs. Let C_n be the minimum total cost of a perfect matching. It was conjectured by M. Mézard and G. Parisi in 1985, and proved by D. Aldous in 2000, that C_n converges in probability to $\pi^2/6$. We give a short proof of this fact, consisting of a proof of the exact formula $1+1/4+1/9+\cdots+1/n^2$ for the expectation of C_n, and a $O(1/n)$ bound on the variance.

1 Introduction

We consider the following random model of the assignment problem: The edges of an m by n complete bipartite graph are assigned independent exponentially distributed costs. A *k-assignment* is a set of k edges of which no two have a vertex in common. The cost of an assignment is the sum of the costs of its edges. Equivalently, if the costs are represented by an m by n matrix, a k-assignment is a set of k matrix entries, no two in the same row or column. We let $C_{k,m,n}$ denote the minimum cost of a k-assignment. We are primarily interested in the case $k = m = n$, where we write $C_n = C_{n,n,n}$.
The distribution of C_n has been investigated for several decades. In 1979, D. Walkup [27] showed that $E(C_n)$ is bounded as $n \to \infty$, a result which was anticipated already in [8]. Further experimental results and improved bounds were obtained in [4, 6, 10, 12, 13, 14, 15, 16, 23, 24]. In a series of papers [19, 20, 21] from 1985–1987, Marc Mézard and Giorgio Parisi gave strong evidence for the conjecture that as $n \to \infty$,

$$E(C_n) \to \frac{\pi^2}{6}. \tag{1}$$

The first proof of (1) was found by David Aldous in 2000 [1, 2].

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In 1998, Parisi conjectured [25] that

$$E\left(C_n\right) = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2}. \tag{2}$$

This suggested a proof by induction on n. The hope of finding such a proof increased further when Don Coppersmith and Gregory Sorkin [6] extended the conjecture (2) to general k, m and n. They suggested that

$$E\left(C_{k,m,n}\right) = \sum_{\substack{i,j\geq 0\\ i+j<k}} \frac{1}{(m-i)(n-j)}, \tag{3}$$

and showed that this reduces to (2) in the case $k = m = n$. In order to establish (3) inductively it would suffice to prove that

$$E\left(C_{k,m,n}\right) - E\left(C_{k-1,m,n-1}\right) = \frac{1}{mn} + \frac{1}{(m-1)n} + \cdots + \frac{1}{(m-k+1)n}. \tag{4}$$

Further generalizations and verifications of special cases were given in [3, 5, 7, 9, 17]. Of particular interest is the paper [5] by Marshall Buck, Clara Chan and David Robbins. They considered a model where each vertex is given a nonnegative weight, and the cost of an edge is exponential with rate equal to the product of the weights of its endpoints. In the next section we consider a special case of this model.
The formulas (2) and (3) were proved in 2003 independently by Chandra Nair, Balaji Prabhakar and Mayank Sharma [22] and by Svante Linusson and the author [18]. These proofs are quite complicated, relying on the verification of more detailed induction hypotheses. Here we give a short proof of (4) based on some of the ideas of Buck, Chan and Robbins. Finally in Section 4 we give a simple proof that $\operatorname{var}(C_n) \to 0$, thereby establishing that $C_n \to \pi^2/6$ in probability.

2 Some results of Buck, Chan and Robbins

In this section we describe some results of the paper [5] by Buck, Chan and Robbins. We include proofs for completeness. Lemma 2.1 follows from Lemma 2 of [5]. For convenience we assume that the edge costs are *generic*, meaning that no two distinct assignments have the same cost. In the random model, this holds with probability 1. We say that a vertex *participates* in an assignment if there is an edge incident to it in the assignment. For $0 \leq r \leq k$, we let σ_r be the minimum cost r-assignment.

Lemma 2.1. *Suppose that $r < \min(m, n)$. Then every vertex that participates in σ_r also participates in σ_{r+1}.*

Proof. Let H be the symmetric difference $\sigma_r \triangle \sigma_{r+1}$ of σ_r and σ_{r+1}, in other words the set of edges that belong to one of them but not to the other. Since no vertex has degree more than 2, H consists of paths and cycles. We claim that H consists of a single path. If this would not be the case, then it would be possible to find a subset $H_1 \subseteq H$ consisting of one or two components of H (a cycle or two paths) such that H_1 contains equally many edges from σ_r and σ_{r+1}. By genericity, the edge sets $H_1 \cap \sigma_r$ and $H_1 \cap \sigma_{r+1}$ cannot have equal total cost. Therefore either $H_1 \triangle \sigma_r$ has smaller cost than σ_r, or $H_1 \triangle \sigma_{r+1}$ has smaller cost than σ_{r+1}, a contradiction. The fact that H is a path implies the statement of the lemma. $\square$

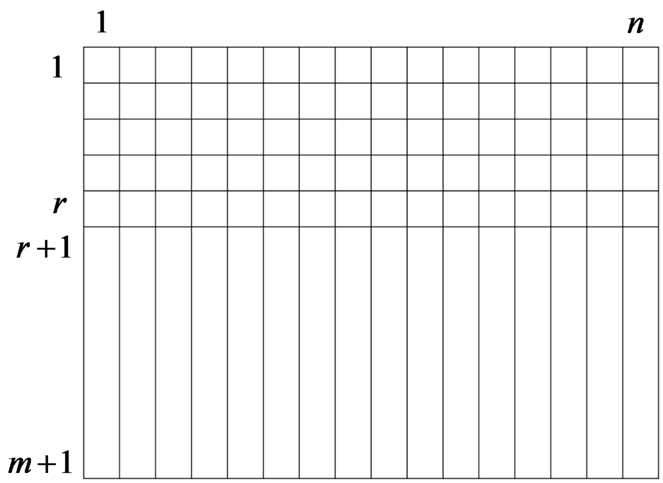

Figure 1: The matrix divided into blocks.

Here we consider a special case of the Buck-Chan-Robbins setting. We let the vertex sets be $A = \{a_1, \dots, a_{m+1}\}$ and $B = \{b_1, \dots, b_n\}$. The vertex a_{m+1} is special: The edges from a_{m+1} are exponentially distributed of rate $\lambda > 0$, and all other edges are exponential of rate 1. This corresponds in the Buck-Chan-Robbins model to letting a_{m+1} have weight λ, and all other vertices have weight 1. The following lemma is a special case of Lemma 5 of [5], where the authors speculate that "This result may be the reason that simple formulas exist...". We believe that they were right.

Lemma 2.2. *Condition on the event that a_{m+1} does not participate in σ_r. Then the probability that it participates in σ_{r+1} is*

$$\frac{\lambda}{m - r + \lambda}. \tag{5}$$

Proof. Suppose without loss of generality that the vertices of A participating in σ_r are $a_1, \dots, a_r$. Now form a "contraction" K' of the original graph K by identifying the vertices $a_{r+1}, \dots, a_{m+1}$ to a vertex a'_{r+1} (so that in K' there are multiple edges from a'_{r+1}).

We condition on the cost of the minimum edge between each pair of vertices in K'. This can easily be visualized in the matrix setting. The matrix entries are divided into *blocks* consisting either of a single matrix entry $M_{i,j}$ for $i \le r$, or of the set of matrix entries $M_{r+1,j}, \dots, M_{m+1,j}$, see Figure 1. We know the minimum cost of the edges within each block, but not the location of the edge having this minimum cost.

It follows from Lemma 2.1 that σ_{r+1} cannot contain two edges from $a_{r+1}, \dots, a_{m+1}$. Therefore σ_{r+1} is essentially determined by the minimum $(r+1)$-assignment σ'_{r+1} in K'. Once we know the edge from a'_{r+1} that belongs to σ'_{r+1}, we know that it corresponds to the unique edge from $\{a_{r+1}, \dots, a_{m+1}\}$ that belongs to σ_{r+1}. It follows from the "memorylessness" of the exponential distribution that the unique vertex of $a_{r+1}, \dots, a_{m+1}$ that participates in σ_{r+1} is distributed with probabilities proportional to the rates of the edge costs. This gives probability equal to (5) for the vertex a_{m+1}. ∎

Corollary 2.3. The probability that a_{m+1} participates in σ_k is

$$1-\frac{m}{m+\lambda}\cdot\frac{m-1}{m-1+\lambda}\cdots\frac{m-k+1}{m-k+1+\lambda}= \\ 1-\left(1+\frac{\lambda}{m}\right)^{-1}\cdots\left(1+\frac{\lambda}{m-k+1}\right)^{-1}= \\ \left(\frac{1}{m}+\frac{1}{m-1}+\cdots+\frac{1}{m-k+1}\right)\lambda+O(\lambda^2),$$

as $\lambda\to 0$.

Proof. This follows from Lemmas 2.1 and 2.2. ◻

3 Proof of the Coppersmith-Sorkin formula

We show that the Coppersmith-Sorkin formula (3) can easily be deduced from Corollary 2.3. The reason that this was overlooked for several years is probably that it seems that by letting $\lambda\to 0$, we eliminate the extra vertex a_{m+1} and just get the original problem back.

We let X be the cost of the minimum k-assignment in the m by n graph $\{a_1,\dots,a_m\}\times\{b_1,\dots,b_n\}$ and let Y be the cost of the minimum $(k-1)$-assignment in the m by $n-1$ graph $\{a_1,\dots,a_m\}\times\{b_1,\dots,b_{n-1}\}$. Clearly X and Y are essentially the same as $C_{k,m,n}$ and $C_{k-1,m,n-1}$ respectively, but in this model, X and Y are also coupled in a specific way.

We let w denote the cost of the edge (a_{m+1},b_n), and let I be the indicator variable for the event that the cost of the cheapest k-assignment that contains this edge is smaller than the cost of the cheapest k-assignment that does not use a_{m+1}. In other words, I is the indicator variable for the event that $Y+w<X$.

Lemma 3.1. *In the limit $\lambda\to 0$,*

$$E(I)=\left(\frac{1}{mn}+\frac{1}{(m-1)n}+\cdots+\frac{1}{(m-k+1)n}\right)\lambda+O(\lambda^2). \tag{6}$$

Proof. It follows from Corollary 2.3 that the probability that (a_{m+1},b_n) participates in the minimum k-assignment is given by (6). If it does, then $w<X-Y$. Conversely, if $w<X-Y$ and no other edge from a_{m+1} has cost smaller than X, then (a_{m+1},b_n) participates in the minimum k-assignment, and when $\lambda\to 0$, the probability that there are two distinct edges from a_{m+1} of cost smaller than X is of order $O(\lambda^2)$. ◻

On the other hand, the fact that w is exponentially distributed of rate λ means that

$$E(I)=P(w<X-Y)=E\left(1-e^{-\lambda(X-Y)}\right)=1-E\left(e^{-\lambda(X-Y)}\right).$$

Hence $E(I)$, regarded as a function of λ, is essentially the Laplace transform of $X-Y$. In particular $E(X-Y)$ is the derivative of $E(I)$ evaluated at $\lambda=0$:

$$E(X-Y)=\frac{d}{d\lambda}E(I)|_{\lambda=0}=\frac{1}{mn}+\frac{1}{(m-1)n}+\cdots+\frac{1}{(m-k+1)n}.$$

This establishes (4) and thereby (3), (2) and (1).

4 A bound on the variance

The Parisi formula (2) shows that as $n \to \infty$, $E(C_n)$ converges to $\zeta(2) = \pi^2/6$. To establish $\zeta(2)$ as a "universal constant" for the assignment problem, it is also of interest to prove convergence in probability. This can be done by showing that $\mathrm{var}(C_n) \to 0$. The upper bound

$$\mathrm{var}(C_n) = O\left(\frac{(\log n)^4}{n(\log\log n)^2}\right)$$

was obtained by Michel Talagrand [26] in 1995 by an application of his isoperimetric inequality. In [28] it was shown that

$$\mathrm{var}(C_n) \sim \frac{4\zeta(2) - 4\zeta(3)}{n}. \tag{7}$$

These proofs are both quite complicated, and our purpose here is to present a relatively simple argument demonstrating that $\mathrm{var}(C_n) = O(1/n)$.

We first establish a simple correlation inequality which is closely related to the Harris inequality [11]. Let $X_1, \dots, X_N$ be random variables (not necessarily independent), and let f and g be two real valued functions of $X_1, \dots, X_N$. For $0 \le i \le N$, let $f_i = E(f | X_1, \dots, X_i)$, and similarly $g_i = E(g|X_1, \dots, X_i)$. In particular $f_0 = E(f)$, $f_N = f$, and similarly for g. The following lemma requires that these and certain other expectations are well-defined. Let us simply assume that regardless of $X_1, \dots, X_i$, all the conditional moments of f and g are finite, since this will clearly hold in the application we have in mind.

Lemma 4.1. *Suppose that for every i and every outcome of $X_1, \dots, X_N$,*

$$(f_{i+1} - f_i)(g_{i+1} - g_i) \ge 0. \tag{8}$$

Then f and g are positively correlated, in other words,

$$E(fg) \ge E(f)E(g). \tag{9}$$

Proof. Equation (8) can be written

$$f_{i+1}g_{i+1} \ge (f_{i+1} - f_i)g_i + (g_{i+1} - g_i)f_i + f_i g_i.$$

Notice that $f_{i+1} - f_i$, although not in general independent of g_i, has zero expectation conditioning on $X_1, \dots, X_i$ and thereby on g_i. It follows that $E\left((f_{i+1} - f_i)g_i\right) = 0$, and similarly for the second term. We conclude that $E(f_{i+1}g_{i+1}) \ge E(f_i g_i)$, and by induction that

$$E(fg) = E(f_N g_N) \ge E(f_0 g_0) = f_0 g_0 = E(f)E(g).$$

□

The random graph model that we use is the same as in the previous section, but we modify the concept of "assignment" by allowing an arbitrary number of edges from the special vertex a_{m+1} (but still at most one edge from each other vertex). This is not essential for the argument, but simplifies some details. Lemmas 2.1 and 2.2 as well as Corollary 2.3 are still valid in this setting.

We let C be the cost of the minimum k-assignment σ_k (with the modified definition), and we let J be the indicator variable for the event that a_{m+1} participates in σ_k.

Lemma 4.2.

$$E(C \cdot J) \leq E(C) \cdot E(J). \tag{10}$$

Proof. Let $f = C$, and let $g = 1 - J$ be the indicator variable for the event that a_{m+1} does not participate in σ_k. As the notation indicates, we are going to design a random process $X_1, \dots, X_N$ such that Lemma 4.1 applies. This process is governed by the edge costs, and $X_1, \dots, X_N$ will give us successively more information about the edge costs, until σ_k and its cost are determined. A generic step of the process is similar to the situation in the proof of Lemma 2.2.

We let $M(r)$ be the matrix of "blocks" when σ_r is known, that is, the $r+1$ by n matrix of block minima as in Figure 1. Moreover we let $\theta_1, \dots, \theta_k$ be vertices in A such that for $r \leq k$, σ_r uses the vertices $\theta_1, \dots, \theta_r$.

When we apply Lemma 4.1, the sequence $X_1, \dots, X_N$ is taken to be the sequence

$$M(0), \theta_1, M(1), \theta_2, M(2), \dots, \theta_k.$$

Notice first that the cost f of the minimum k-assignment is determined by $M(k-1)$, and that $\theta_1, \dots, \theta_k$ determine g, that is, they determine whether or not a_{m+1} participates in σ_k.

In order to apply the lemma, we have to verify that each time we get a new piece of information, the conditional expectations of f and g change in the same direction, if they change. By the argument in the proof of Lemma 2.2, we have

$$\begin{aligned} E\left(g|M(0), \theta_1, \dots, M(r-1), \theta_r\right) &= E\left(g|M(0), \theta_1, \dots, M(r-1), \theta_r, M(r)\right) \\ &= \frac{m-r}{m-r+\lambda} \cdot \frac{m-r-1}{m-r-1+\lambda} \cdots \frac{m-k+1}{m-k+1+\lambda}, \end{aligned}$$

unless $v_{n+1} \in \{\theta_1, \dots, \theta_r\}$, but in that case it is already clear that $g = 0$. Therefore when we get to know another row in the matrix, the conditional expectation of g does not change, which means that for this case, the hypothesis of Lemma 4.1 holds.

The other case to consider is when we already know $M(0), \dots, M(r)$ and $\theta_1, \dots, \theta_r$, and are being informed of θ_{r+1}. In this case the conditional expectations of f and g can obviously both change. For g, there are only two possibilities. Either $\theta_{r+1} = a_{m+1}$, which means that $g = 0$, or $\theta_{r+1} \neq a_{m+1}$, which implies that the conditional expectation of g increases.

To verify the hypothesis of Lemma 4.1, it clearly suffices to assume that $\{\theta_1, \dots, \theta_r\} = \{1, \dots, r\}$, and to show that if $\theta_{r+1} = a_{m+1}$, then the conditional expectation of f decreases. Since we know $M(r)$, we know to which "block" of the matrix the new edge belongs, that is, there is a j such that we know that exactly one of the edges in the set $E' = \{(a_i, b_j) : r+1 \leq i \leq m+1\}$ will belong to σ_{r+1}.

We now condition on the costs of all edges that are not in E'. Since we know $M(r)$, we also know the minimum edge cost, say α, in E'. We now observe that if the minimum cost edge in E' is (a_{m+1}, b_j), then no other edge in E' can participate in σ_k, because in an assignment, any edge in E' can be replaced by (a_{m+1}, b_j). It follows that the value of f given that $M_{m+1,j} = \alpha$ is the same regardless of the costs of the other edges in E'. In particular it is the same as the value of f given that all edges in E' have cost α, which is certainly not greater than the conditional expecation of f given that some edge other than (a_{m+1}, b_j) has the minimum cost α in E'.

It follows that Lemma 4.1 applies, and this completes the proof. □

The inequality (10) allows us to establish an upper bound on $\mathrm{var}(C_n)$ which is of the right order of magnitude (it is easy to see that $\mathrm{var}(C_n) \geq 1/n$, see [3]).

Theorem 4.3.

$$\operatorname{var}(C_n) < \frac{\pi^2}{3n}.$$

Proof. We let X, Y, I and w be as in Section 3, with I being the indicator variable of the event $Y + w < X$. Again C denotes the cost of σ_k, and J is the indicator variable for the event that a_{m+1} participates in σ_k.

Obviously

$$E(C) \le E(X). \tag{11}$$

Again the probability that there are two distinct edges from a_{m+1} of cost smaller than X is of order $O(\lambda^2)$. Therefore

$$E(J) = nE(I) + O(\lambda^2) = n\lambda E(X - Y) + O(\lambda^2). \tag{12}$$

Similarly

$$E(C \cdot J) = nE(I \cdot (Y + w)) + O(\lambda^2). \tag{13}$$

If we condition on X and Y, then

$$\begin{aligned} E(I \cdot (Y + w)) &= \int_0^{X-Y} \lambda e^{-\lambda t}(Y + t)\,dt \\ &= Y(X - Y)\lambda + \frac{(X - Y)^2}{2}\lambda + O(\lambda^2) = \frac{1}{2}\left(X^2 - Y^2\right)\lambda + O(\lambda^2). \end{aligned} \tag{14}$$

If, in the inequality (10), we substitute the results of (11), (12), (13) and (14), then after dividing by $n\lambda$ we obtain

$$\frac{1}{2}E\left(X^2 - Y^2\right) \le E(X)^2 - E(X)E(Y) + O(\lambda).$$

After deleting the error term, this can be rearranged as

$$\operatorname{var}(X) - \operatorname{var}(Y) \le (E(X) - E(Y))^2.$$

But we already know that $E(X) - E(Y)$ is given by (4). Therefore it follows inductively that

$$\operatorname{var}(C_n) \le \sum_{i=1}^{n} \frac{1}{i^2}\left(\frac{1}{n} + \cdots + \frac{1}{n - i + 1}\right)^2 \le \sum_{i=1}^{n} \frac{1}{i^2}\left(\log(n + 1/2) - \log(n + 1/2 - i)\right)^2.$$

If we replace the sum over i by an integral with respect to a continuous variable, then the integrand is convex, and

$$\begin{aligned} \operatorname{var}(C_n) &\le \int_0^{n+1/2} \frac{\left(\log(n + 1/2) - \log(n + 1/2 - x)\right)^2}{x^2}\,dx \\ &= \frac{1}{n + 1/2}\int_0^1 \frac{\log(1 - x)^2}{x^2}\,dx = \frac{2\zeta(2)}{n + 1/2} < \frac{\pi^2}{3n}. \end{aligned}$$

$\square$

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