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# ON CONSTRAINED ANNEALED BOUNDS FOR PINNING AND WETTING MODELS

FRANCESCO CARAVENNA
*Università di Milano-Bicocca, Dipartimento di Matematica e Applicazioni, Edificio U5, via Cozzi 53, 20125 Milano, Italy*
*and*
*Laboratoire de Probabilités de P 6 & 7 (CNRS U.M.R. 7599) and Université Paris 7 – Denis Diderot, U.F.R. Mathematiques, Case 7012, 2 place Jussieu, 75251 Paris cedex 05, France*
email: f.caravenna@sns.it

GIAMBATTISTA GIACOMIN
*Laboratoire de Probabilités de P 6 & 7 (CNRS U.M.R. 7599) and Université Paris 7 – Denis Diderot, U.F.R. Mathematiques, Case 7012, 2 place Jussieu, 75251 Paris cedex 05, France*
email: giacomin@math.jussieu.fr

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*Abstract*

The free energy of quenched disordered systems is bounded above by the free energy of the corresponding annealed system. This bound may be improved by applying the annealing procedure, which is just Jensen inequality, after having modified the Hamiltonian in a way that the quenched expressions are left unchanged. This procedure is often viewed as a partial annealing or as a constrained annealing, in the sense that the term that is added may be interpreted as a Lagrange multiplier on the disorder variables.
In this note we point out that, for a family of models, some of which have attracted much attention, the multipliers of the form of empirical averages of local functions cannot improve on the basic annealed bound from the viewpoint of characterizing the phase diagram. This class of multipliers is the one that is suitable for computations and it is often believed that in this class one can approximate arbitrarily well the quenched free energy.

## 1 The framework and the main result

### 1.1 The set–up (I): linear chain models

A number of disordered models of linear chains undergoing localization or pinning effects can be put into the following general framework. Let $S := \{S_n\}_{n=0,1,\ldots}$ be a process with $S_n$ taking values in $\mathbb{Z}^d$, $d \in \mathbb{N} := \{1, 2, \ldots\}$ and law $\mathbf{P}$.

The *disorder* in the system is given by a sequence $\omega := \{\omega_n\}_n$ of IID random variables of law $\mathbb{P}$, with $\omega_n$ taking values in $\Gamma \subseteq \mathbb{R}$. As a matter of fact we could simply set $\Gamma = \mathbb{R}$, however several examples that we will present deal with the case in which $\Gamma$ is a finite set and in this situation our results require no measurability conditions. The disorder acts on the paths of $S$ via an Hamiltonian that, for a system of size $N$, is a function $H_{N,\omega}$ of the trajectory $S$, but depending only on $S_0, S_1, \dots, S_N$. One is interested in the properties of the probability measures $\mathbf{P}_{N,\omega}$ defined by giving the density with respect to $\mathbf{P}$:

$$\frac{\mathrm{d}\mathbf{P}_{N,\omega}}{\mathrm{d}\mathbf{P}}(S) \;=\; \frac{1}{Z_{N,\omega}} \exp\left(H_{N,\omega}(S)\right), \tag{1.1}$$

where $Z_{N,\omega} := \mathbf{E}\left[\exp\left(H_{N,\omega}(S)\right)\right]$ is the normalization constant. Our attention focuses on the asymptotic behavior of $\log Z_{N,\omega}$.

In the sequel we will assume:

**Basic Hypothesis.** There exists a sequence $\{D_n\}_n$ of subsets of $\mathbb{Z}^d$ such that $\mathbf{P}(S_n \in D_n \text{ for } n = 1, 2, \dots, N) \overset{N\to\infty}{\asymp} 1$, namely

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \dots, N\right) = 0, \tag{1.2}$$

and such that $H_{N,\omega}(S) = 0$ if $S_n \in D_n$ for $n = 1, 2, \dots, N$.

One sees directly that this hypothesis implies

$$\liminf_{N\to\infty} \frac{1}{N} \log Z_{N,\omega} \;\geq\; \lim_{N\to\infty} \frac{1}{N} \log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \dots, N\right) \;=\; 0, \tag{1.3}$$

$\mathbb{P}(\mathrm{d}\omega)$–a.s.. We will assume that $\{(1/N)\log Z_{N,\omega}\}_N$ is a sequence of integrable random variables that converges in the $L^1\left(\mathbb{P}(\mathrm{d}\omega)\right)$ sense and $\mathbb{P}(\mathrm{d}\omega)$–almost surely to a constant, the *free energy*, that we will call $f$. These assumptions are verified in the large majority of the interesting situations, for example whenever super/sub–additivity tools are applicable.

Of course (1.3) says that $f \geq 0$ and one is lead to the natural question of whether $f = 0$ or $f > 0$. In the instances that we are going to consider the free energy may be zero or positive according to some parameters from which $H_{N,\omega}(S)$ depends: $f = 0$ and $f > 0$ are associated to sharply different behaviors of the system.

In order to establish upper bounds on $f$ one may apply directly Jensen inequality (*annealed bound*) obtaining

$$\begin{aligned} f \;&=\; \lim_{N\to\infty} \frac{1}{N}\, \mathbb{E}\big[\log Z_{N,\omega}\big] \\ &\leq\; \liminf_{N\to\infty} \frac{1}{N}\, \log \mathbb{E}\big[Z_{N,\omega}\big] \;=:\; \widetilde{f} \in [0, \infty], \end{aligned} \tag{1.4}$$

and, in our context, if $\widetilde{f} = 0$ then $f = 0$. The annealed bound may be improved by adding to $H_{N,\omega}(S)$ an integrable function $A_N : \Gamma^{\mathbb{N}} \to \mathbb{R}$ such that $\mathbb{E}\left[A_N(\omega)\right] = 0$: in fact $f$ as defined in the first line of (1.4) is unchanged by such transformation, while the second line of (1.4) may depend on the choice of $\{A_N\}_N$. We stress that not only $f$ is left unchanged by $H_{N,\omega}(S) \to H_{N,\omega}(S) + A_N(\omega)$, but $\mathbf{P}_{N,\omega}$ itself is left unchanged (for every $N$). Notice moreover that the *optimal choice* $A_N(\omega) = -\log Z_{N,\omega} + \mathbb{E}\left[\log Z_{N,\omega}\right]$ yields the equality in (1.4).

In the sequel when we refer to $\widetilde{f}$ we mean that $Z_{N,\omega}$ is defined with respect to $H_{N,\omega}$ satisfying the Basic Hypothesis (no $A_N$ term added).

## 1.2 The result

What we prove in this note is that

**Proposition 1.1.** *If $\widetilde{f} > 0$ then for every local bounded measurable function $F : \Gamma^{\mathbb{N}} \longrightarrow \mathbb{R}$ such that $\mathbb{E}[F(\omega)] = 0$ one has*

$$\liminf_{N\to\infty} \frac{1}{N} \log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S) + \sum_{n=0}^{N} F(\theta_n\omega)\right)\right] > 0, \tag{1.5}$$

*where $(\theta_n\omega)_m = \omega_{n+m}$.*

We can sum up this result by saying that when $f = 0$ but $\widetilde{f} > 0$ it is of no use modifying the Hamiltonian by adding the empirical average of a (centered) local (bounded measurable) function.

Notice that requiring $F(\cdot)$ to be bounded and measurable is superfluous if $\Gamma$ is a finite set. From now on the reader should read *local* as a short–cut for *local, measurable and bounded*. We take this occasion also to observe that in principle one should be able to extend the result in the direction of unbounded $F(\cdot)$ or of non IID disorder: this however requires additional assumptions and leads far from the spirit of this note.

On a mathematical level it is not obvious that the free energy may be approximated via empirical averages of a local function of the disorder, because we are playing with an exchange of limits (recall the optimal choice of $A_N$ above). But we remark that in the physical literature the approach of approximating the free energy via what can be viewed as a constrained annealed computation, the term $\sum_{n=0}^{N} F(\theta_n\omega)$ being interpreted as a Lagrange multiplier, is often considered as an effective way of approximating the quenched free energy. Here we mention in particular [20] and [16] in which this point of view is taken up in a systematic way: the aim is to approach the quenched free energy by constrained annealing via local functions $F$ that are more and more complex, the most natural example being linear combinations of correlations of higher and higher order.

The proof of Proposition 1.1 is based on the simple observation that whenever $A_N$ is centered

$$\frac{1}{N}\log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S) + A_N(\omega)\right)\right] \geq \\ \frac{1}{N}\log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right] + \frac{1}{N}\log \mathbf{P}\left(S_n \in D_n \text{ for } n = 1, 2, \ldots, N\right) =: Q_N + P_N. \tag{1.6}$$

By hypothesis $P_N = o(1)$ so one has to consider the asymptotic behavior of $Q_N$. If $\liminf_N Q_N > 0$ there is nothing to prove. So let us assume that $\liminf_N Q_N = 0$: in this case the inferior limit of the left–hand side of (1.6) may be zero and we want to exclude this possibility when $\widetilde{f} > 0$ and $A_N(\omega) = \sum_{n=0}^{N} F(\theta_n\omega)$, $F$ local and centered (of course in this case $\lim_N Q_N$ does exist). And in Proposition 2.1 below in fact we show that if $\log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right] = o(N)$, then $\sup_\omega |A_N(\omega)| = o(N)$ and therefore the corresponding constrained annealing is just the standard annealing.

**Remark 1.2.** We stress that our Basic Hypothesis is more general than it may look at first. As already observed, one has the freedom of adding to the Hamiltonian $H_{N,\omega}(S)$ any term that does not depend on $S$ (but possibly does depend on $\omega$ and $N$) without changing the model

$\mathbf{P}_{N,\omega}$. It may therefore happen that the *natural* formulation of the Hamiltonian does not satisfy our Basic Hypothesis, but it does after a suitable additive correction. This happens for example in §1.2.3 below: the additive correction in that case is linear in $\omega$ and it corresponds to what in [21] is called *first order* Morita approximation. In these terms, Proposition 1.1 is saying that *higher order* Morita approximations cannot improve the bound on the critical curve found with the first order computation.

**Remark 1.3.** In the Morita approach of [16, 20], when applied to spin systems, it was also taken for granted that the infinite volume measure describing the joint distribution of disorder variables and spin variables can be described as Gibbs measure with a proper (absolutely summable) Hamiltonian. This was shown to be false in general, and potentials with weaker summability properties are needed [7, 17]. This phenomenon underlines from a different perspective that local dependence of the *Morita potential* on the disorder variables *is not enough*.

Let us now look at applications of Proposition 1.1.

### 1.2.1 Random rewards or penalties at the origin

Let $S$, $S_0 = 0 \in \mathbb{Z}^d$, be a random walk with centered IID non degenerate increments $\{X_n\}_n$, $(X_n)_j \in \{-1, 0, 1\}$ for $j = 1, 2, \ldots, d$, and

$$H_{N,\omega} = \beta \sum_{n=1}^{N} (1 + \varepsilon\omega_n)\, \mathbf{1}_{\{S_n=0\}}. \tag{1.7}$$

for $\beta \geq 0$ and $\varepsilon \geq 0$. The random variable $\omega_1$ is chosen such that $\mathbb{E}[\exp(\lambda\omega_1)] < \infty$ for every $\lambda \in \mathbb{R}$, and centered. We write $f(\beta, \varepsilon)$ for $f$: by super–additive arguments $f$ exists and it is self–averaging (this observation is valid for all the models we consider and will not be repeated). We note that for $\varepsilon = 0$ the model can be solved, see e.g. [12], and in particular $f(\beta, 0) = 0$ if and only if $\beta \leq \beta_c(d) := -\log(1 - \mathbf{P}(S \text{ never comes back to } 0))$. Adding the disorder makes this model much more complex: the annealed bound yields $f(\beta, \varepsilon) = 0$ if $\beta \leq \beta_c(d) - \log \mathbb{E}[\exp(\varepsilon\omega_1)] =: \widetilde{\beta}_c$. It is an open question whether $\widetilde{\beta}_c$ coincides with the quenched critical value or not, that is whether $f(\beta, \varepsilon) = 0$ implies $\beta \leq \widetilde{\beta}_c$ or not. For references about this issue we refer to [2] and [23], see however also the next paragraph: the model we are considering can in fact be mapped to the wetting problem ([2, 12]). Proposition 1.1 applies to this context with $D_n = \{0\}^{\complement}$ for every $n$ [8, Ch. 3] and says that one cannot answer this question via constrained annealed bounds.

### 1.2.2 Wetting models in $1 + d$ dimensions

Let $S$ and $\omega$ be as in the previous example and

$$H_{N,\omega} = \begin{cases} \beta \sum_{n=1}^{N} (1 + \varepsilon\omega_n)\, \mathbf{1}_{\{(S_n)_d=0\}} & \text{if } (S_n)_d \geq 0 \text{ for } n = 1, 2, \ldots, N \\ -\infty & \text{otherwise.} \end{cases} \tag{1.8}$$

with $\beta \geq 0$ and $\varepsilon \geq 0$. If one takes the directed walk viewpoint, that is if one considers the walk $\{(n, S_n)\}_n$, then this is a model of a walk constrained above the (hyper–)plane $x_d = 0$ and rewarded $\beta$, on the average, when touching this plane. If $d = 1$ then this is an effective model for a (1+1)–dimensional interface above a wall which mostly attracts it. As a matter of fact in this case there is essentially no loss of generality in considering $d = 1$, since localization is measured in terms of orthogonal displacements of the walk with respect to the wall and we may restrict ourselves to this coordinate. Once again if $\varepsilon = 0$ the model can be solved in detail, see e.g. [12]. Computing the critical $\beta$ and deciding whether the annealed bound is sharp, at least for small $\varepsilon$, is an unresolved and disputed question in the physical literature, see e.g. [9, 6, 26]. Proposition 1.1 applies with the choice $D_n = \mathbb{Z}^{d-1} \times \mathbb{N}$.

### 1.2.3 Copolymer with adsorption models

For definiteness choose $S$ to be a one dimensional simple random walk and take the directed walk viewpoint. Imagine that the space above the horizontal axis is filled with a solvent $A$, while below there is a solvent $B$. We choose $\omega_1 \in \{A, B\}$ and for example

$$H^{AB}_{N,\omega}(S) \,=\, \sum_{n=1}^{N} \left( a\mathbf{1}_{\{\mathrm{sign}(S_n)=+1,\, \omega_n=A\}} + b\mathbf{1}_{\{\mathrm{sign}(S_n)=-1,\, \omega_n=B\}} + c\mathbf{1}_{\{S_n=0\}} \right) \tag{1.9}$$

with $a$, $b$ and $c$ real parameters and $\mathrm{sign}(S_n) = \mathrm{sign}(S_{n-1})$ if $S_n = 0$ (this is just a trick to reward the bonds rather than the sites). In order to apply Proposition 1.1 one has to subtract a disorder dependent term, cf. Remark 1.2: if $a \geq b$ we change the Hamiltonian

$$H_{N,\omega}(S) \,:=\, H^{AB}_{N,\omega}(S) - \sum_{n=1}^{N} a\mathbf{1}_{\{\omega_n=A\}}. \tag{1.10}$$

without changing the measure $\mathbf{P}_{N,\omega}$ while the free energy has the trivial shift from $f$ to $f - a\mathbb{P}(\omega_1 = A)$. One can therefore choose $D_n = \mathbb{Z}^{d-1} \times \mathbb{N}$ and Proposition 1.1 applies. This model has been considered for example in [21].
Note that if $c = 0$ the model can be cast in a form that has been considered by a variety of authors (see e.g. [15, 24, 1, 4, 25, 27, 19, 3]):

$$H_{N,\omega}(S) \,=\, \lambda \sum_{n=1}^{N} (\omega_n + h)\,\mathrm{sign}(S_n), \tag{1.11}$$

with $\omega$ taking values in $\mathbb{R}$. Once again the Hamiltonian has to be corrected by subtracting the term $\lambda \sum_n(\omega_n + h)$ in order to apply Proposition 1.1. One readily sees that (1.10) and (1.11) are the same model when in the second case $\omega$ takes only the values $\pm 1$, $A = +1$ and $B = -1$, and $h = (a - b)/(a + b)$, $\lambda = (a + b)/4$.
Proposition 1.1 acquires some interest in this context given the fact that the physical literature is rather split on the precise value of the critical curve and on whether the annealed bound is sharp or not, see [3] for details on this issue. In [5] we present numerical evidence on the fact that the annealed curve does not coincide with the quenched one, and in view of Proposition 1.1 this would mean that constrained annealing via local functions cannot capture the phase diagram of the quenched system.

### 1.2.4 Further linear chain models and observations

In spite of substantial numerical evidence that in several instances $f = 0$ but $\widetilde{f} > 0$, we are unaware of an *interesting* model for which this situation is rigorously known to happen.

Consider however the case $\mathbb{P}(\omega_1 = +1) = \mathbb{P}(\omega_1 = -1) = 1/2$ and

$$H_{N,\omega}(S) = \beta \sum_{n=1}^{N} (1 + \varepsilon\omega_n)\, \mathbf{1}_{\{S_n = n\}}, \tag{1.12}$$

with $\beta$ and $\varepsilon$ real numbers and $S$ the standard simple symmetric random walk on $\mathbb{Z}$. We observe that Proposition 1.1 applies to this case with $D_n = \{n\}^{\complement}$ and that the model is solvable in detail. In particular $f(\beta, \varepsilon) = (\beta - \log 2) \vee 0$, regardless of the value of $\varepsilon$. The annealed computation instead yields $\widetilde{f}(\beta, \varepsilon) = (\beta + \log\cosh(\varepsilon) - \log 2) \vee 0$. Notice in particular that the critical values of $\beta$, respectively $\log 2$ and $\log 2 - \log\cosh(\varepsilon)$, differ as long as there is disorder in the system ($\varepsilon \neq 0$). It is interesting to see in this toy model how the *optimal choice* of $A_N$, mentioned at the end of § 1.1, is rather far from being the empirical average of a local function, when $N$ is large.

**Remark 1.4.** We point out that we restricted our examples only to cases in which $S$ is a simple random walk, but in principle our approach goes through for much more general models, like walks with correlated increments or self–interacting walks, see [22] for an example. And of course $S_n$ takes values in $\mathbb{Z}^d$ only for ease of exposition and can be easily generalized. Another important class of models to which our arguments apply is the disordered Poland–Scheraga one [10].

## 1.3 The set–up (II): interface pinning models

It is natural to wonder whether one can go beyond the linear chain set–up. The answer is positive and we give the example of $(d+1)$–dimensional effective interface models, $d > 1$, natural generalization of the $(1+1)$–dimensional interfaces considered in the previous section. By this we mean for example the case of $S := \{S_n\}_{n\in\mathbb{Z}^d}$ with $S_n \in \mathbb{R}$ and the law of $S$ is $\mathbf{P} = \mathbf{P}_N$:

$$\mathbf{P}(\mathrm{d}\varphi) \propto \exp\left( -\frac{1}{2} \sum_{n,n' : |n-n'|=1} U(\varphi_n - \varphi_{n'}) \right) \prod_{n\in V_N} \mathrm{d}\varphi_n \prod_{n \in V_N^{\complement}} \delta_0(\mathrm{d}\varphi_n), \tag{1.13}$$

where $V_N = [-N/2, N/2]^d \cap \mathbb{Z}^d$ and $U(\cdot)$ is a measurable function such $\lim_{r\to\pm\infty} U(r) = +\infty$ sufficiently rapidly to make the right–hand side of (1.13) integrable (note that we may assume $U(\cdot)$ to be even). As a matter of fact, in order to have a treatable model one has to restrict rather strongly the choice of $U(\cdot)$: interface models are extremely challenging even without introducing pinning potentials (or, of course, disorder). Connected to that is also the reason why we have chosen the continuous set–up for interface models: discrete models are even more challenging [13].

The disorder in the system this time is given by an IID *field* $\omega := \{\omega_n\}_{n\in\mathbb{Z}^d}$ and $H_{N,\omega}(S)$ depends only upon $S_n$ with $n \in V_N$: $\omega_0$ takes once again values in $\Gamma$. The definition (1.1) of $\mathbf{P}_{N,\omega}$ is unchanged and the Basic Hypothesis varies in the obvious way, that is we assume that there exists $\{D_n\}_{n\in\mathbb{Z}^d}$ such that

$$\lim_{N\to\infty} \frac{1}{N^d} \log \mathbf{P}\left(S_n \in D_n \text{ for } n \in V_N\right) = 0, \tag{1.14}$$

and such that $H_{N,\omega}(S) = 0$ if $S_n \in D_n$ for every $n \in V_N$. Like for linear chains we assume the existence of the quenched free energy, that is of the $L^1(\mathbb{P}(\mathrm{d}\omega))$ and $\mathbb{P}(\mathrm{d}\omega)$–a.s. limit of the sequence $\left\{N^{-d}\log Z_{N,\omega}\right\}_N$ and like in the linear chain case we have $0 \le f \le \widetilde{f}$, where $\widetilde{f}$ is again the annealed free energy defined in analogy with (1.4).

The punch–line of this section is that Proposition 1.1 holds in this new set–up and it is proven exactly in the same way:

**Proposition 1.5.** *If $\widetilde{f} > 0$ then for every local bounded measurable function $F : \Gamma^{\mathbb{Z}^d} \longrightarrow \mathbb{R}$ such that $\mathbb{E}\left[F(\omega)\right] = 0$ one has*

$$\liminf_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}\mathbf{E}\left[\exp\left(H_{N,\omega}(S) + \sum_{n\in\Lambda_N} F(\theta_n\omega)\right)\right] > 0. \tag{1.15}$$

In order to give examples of applications we may consider the $d+1$ dimensional model of random rewards and penalties near the origin, that is the case of

$$H_{N,\omega} = \beta \sum_{n\in V_N} (1+\varepsilon\omega_n)\, \mathbf{1}_{\{S_n\in(-1,1)\}}, \tag{1.16}$$

but one can write natural straightforward generalizations of the wetting models and of the copolymer with adsorption. The Basic Hypothesis in all these cases is a probability estimate on what is known as an *entropic repulsion event*, that is, for example, the event that $S_n \ge 1$ for every $n \in V_N$ and one can for example show that such a probability is bounded below by $\exp\left(-cN^{d-1}\right)$, $c > 0$, if $U(\cdot)$ is $C^2$ and $\inf_r U''(r) > 0$, see [13] and references therein. So in this case one may apply Proposition 1.1 to conclude that one cannot improve on the annealed bound by constraining via local functions.
Two comments, of opposite spirit, are however in order (for details see the lecture notes [13]):

1. The Basic Hypothesis requires a substantially weaker estimate and it is reasonable to expect that one is able to verify it in greater generality.

2. The understanding of the associated deterministic models ($\varepsilon = 0$ for random rewards and wetting models and the annealed models in general) is still extremely partial. Somewhat satisfactory results are available for quadratic $U(\cdot)$, that is $\mathbf{P}$ is Gaussian, but even in this case one has to give up the precise estimates available for the linear chain case (like computing exactly $\beta_c$) and basic questions are still open. So the application of Proposition 1.5, while being relevant on a conceptual level, yields a result that has little quantitative content.

# 2 On zero free energy and null potentials

In this Section $d \ge 1$. Let $\{\omega_n\}_{n\in\mathbb{Z}^d}$ be an IID family of random variables under the probability measure $\mathbb{P}$, taking values in $\Gamma = \mathbb{R}$. The law of $\omega_1$ is denoted by $\nu$.
We are interested in the family $A = \{A_N\}_{N\in\mathbb{N}}$ of empirical averages of a local function $F$, that is

$$A_N(\omega) = \sum_{n\in V_N} F\left(\theta_n\omega\right), \tag{2.1}$$

where $F : \Gamma^{\mathbb{Z}^d} \to \mathbb{R}$ depends only on the variables indexed by a finite set $\Lambda \subset \mathbb{Z}^d$, that is $F(\omega) = F(\omega')$ if $\omega_n = \omega'_n$ for every $n \in \Lambda$. Notice that, by standard (super–additivity) arguments, the limit

$$L(F) := \lim_{N\to\infty} \frac{1}{N^d} \log \mathbb{E}\left[\exp\left(A_N(\omega)\right)\right], \tag{2.2}$$

exists. Moreover, by Jensen's inequality, $L(F) \geq \mathbb{E}\left[F(\omega)\right]$.

We will prove the following:

**Proposition 2.1.** *Assume that* $\mathbb{E}\left[F(\omega)\right] = 0$. *If* $L(F) = 0$, *then*

$$\lim_{N\to\infty} \frac{1}{N^d} \sup_{\omega} |A_N(\omega)| = 0\,. \tag{2.3}$$

Of course, since the result is uniform in $\omega$, the proposition covers also the linear chain set–up, where one considers $\theta_{\lfloor N/2\rfloor+1}V_N$ rather than $V_N$.

*Proof.* We consider the *potential*, in the sense of [11, Def. (2.2)], $\Phi := \{\Phi_B\}_{B\subset\mathbb{Z}^d}$ defined by

$$\Phi_B(\omega) \;=\; \begin{cases} F\left(\theta_{-n}\omega\right) & \text{if there exists } n \text{ such that } \theta_n B = \Lambda, \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

Let $\nu$ be the *single spin* reference measure [11, Def. (2.9)] and let us set

$$Z_N^\Phi(\omega) \;:=\; \int \exp\left(H_N^\Phi(\sigma)\right) \prod_{n\in V_N} \nu(\mathrm{d}\sigma_n) \prod_{n\in V_N^{\complement}} \delta_{\omega_n}(\mathrm{d}\sigma_n), \tag{2.5}$$

with $H_N^\Phi(\sigma) := \sum_{B: B\cap V_N\neq\emptyset} \Phi_B(\sigma)$. Note that $A_N(\cdot)$ differs from $H_N^\Phi(\cdot)$ only by boundary terms so that $\sup_\omega |A_N(\omega) - H_N^\Phi(\omega)| \leq CN^{d-1}$ for some $C > 0$ (we recall that $F(\cdot)$ is bounded). Therefore it suffices to show that (2.3) holds with $A_N(\cdot)$ replaced by $H_N^\Phi(\cdot)$.

Let us consider the $\theta$–invariant Gibbs measure $\mu$ associated to the potential $\Phi$, the existence of which is established in a standard way by taking infinite volume limits with periodic boundary conditions (if $\nu$ has unbounded support tightness follows from the fact that $F(\cdot)$ is bounded). By [11, Theorem (15.30)] the *relative entropy density* of $\nu^\infty$ ($\nu^\infty(\mathrm{d}\omega) := \prod_{n\in\mathbb{Z}^d} \nu(\mathrm{d}\omega_n)$) with respect to $\mu$ exists and can be written as

$$\lim_{N\to\infty} \frac{1}{N^d} \mathcal{H}_{V_N}\left(\nu^\infty \,|\, \mu\right) \;=\; \lim_{N\to\infty} \frac{1}{N^d} \log Z_N^\Phi(\omega) \;-\; \int F(\omega)\, \nu^\infty(\mathrm{d}\omega), \tag{2.6}$$

where $\mathcal{H}_{V_N}(\nu^\infty|\mu)$ is the relative entropy of $\nu^\infty$ with respect to $\mu$, when both measures are restricted to the $\sigma$–algebra generated by the variables $\{\omega_n\}_{n\in V_N}$. We have of course used the standard definition of relative entropy, $\mathcal{H}(\mu_1|\mu_2) = \int \log(\mathrm{d}\mu_1/\mathrm{d}\mu_2)\mathrm{d}\mu_1$ for $\mu_1$ and $\mu_2$ two probability measures with $\mu_1$ absolutely continuous with respect to $\mu_2$. A last remark on formula (2.6) is that it holds for any choice of $\omega$: this is just the independence of the free energy on boundary conditions. This independence may be seen directly since $\log(Z_N^\Phi(\omega)/Z_N^\Phi(\omega')) = O(N^{d-1})$ uniformly in $\omega$ and $\omega'$ and this implies also that the first term in the right–hand side of (2.6) may be replaced by $L(F)$.

Notice now that both terms in the right–hand side of (2.6) are zero, respectively by the hypotheses $L(F) = 0$ and $\mathbb{E}[F(\omega)] = 0$, and therefore, as a consequence of the Gibbs variational principle [11, Theorem (15.37)], $\nu^\infty$ is a Gibbs measure with the same specification of $\mu$, but of course $\nu^\infty$ is the Gibbs measure with potential $\Phi^{(0)}$ identically equal to zero and single spin measure $\nu$. This means that $\Phi - \Phi^{(0)}(= \Phi)$ is a *negligible* potential, that is [11, Theorem (2.34)] the function

$$\sum_{B: B\cap V_N \neq \emptyset} \left(\Phi_B(\omega) - \Phi_B^{(0)}(\omega)\right) \tag{2.7}$$

does not depend on the variables $\omega_n$ for $n \in V_N$. We can write

$$\begin{aligned} \mathcal{H}_N^\Phi(\omega) \,=\, \sum_{B: B\cap V_N \neq \emptyset} \Phi_B(\omega) \,&=\, \sum_{B: B\subset V_N} \Phi_B(\omega) + \sum_{B: B\cap V_N \neq \emptyset,\, B\not\subset V_N} \Phi_B(\omega) \\ &=: \, I_N(\omega) + R_N(\omega), \end{aligned} \tag{2.8}$$

and since $\mathcal{H}_N^\Phi(\omega)$ does not depend on the $\omega_n$'s for $n \in V_N$ we may change in the right–hand side the configuration $\omega$ with $\widetilde{\omega}$ defined by setting $\widetilde{\omega}_n = \omega_n$ for $n \in V_N^{\complement}$ and $\omega_n = c$, $c$ an arbitrary fixed constant, for $n \in V_N$. Therefore, in random variable terms, we have

$$\mathcal{H}_N^\Phi(\omega) \,=\, c_N + R_N(\widetilde{\omega}), \tag{2.9}$$

with $c_N = I_N(\widetilde{\omega})$ (notice that it is not random and it depends only on the choice of $c$). From the immediate estimate $\sup_\omega |R_N(\omega)| \leq CN^{d-1}$ for some $C = C(F) > 0$ it follows that for all $\omega$

$$c_N - CN^{d-1} \,\leq\, \mathcal{H}_N^\Phi(\omega) \,\leq\, c_N + CN^{d-1}, \tag{2.10}$$

and the hypothesis $L(F) = 0$ yields immediately $\lim_{N\to\infty} c_N/N^d = 0$. Therefore

$$\sup_\omega \left|\mathcal{H}_N^\Phi(\omega)\right| \,\leq\, c_N + CN^{d-1} \,=\, o(N^d), \tag{2.11}$$

and the proof is complete.

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