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A NOTE ON THE INVARIANCE PRINCIPLE OF THE PRODUCT OF SUMS OF RANDOM VARIABLES[1]

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Abstract
The central limit theorem for the product of sums of various random variables has been studied in a variety of settings. The purpose of this note is to show that this kind of result is a corollary of the invariance principle.

Let $\{X_k; k \geq 1\}$ be a sequence of i.i.d exponential random variables with mean 1, $S_n = \sum_{k=1}^n X_k$, $n \geq 1$. Arnold and Villaseñor (1998) proved that

$$\left(\prod_{k=1}^{n} \frac{S_k}{k} \right)^{1/\sqrt{n}} \xrightarrow{\mathscr{D}} e^{\sqrt{2}N(0,1)}, \quad \text{as} \quad n \to \infty, \tag{1}$$

where $N(0,1)$ is a standard normal random variable. Later Rempala and Wesolowski (2002) extended such a central limit theorem to general i.i.d. positive random variables. Recently, the central limit theorem for product of sums has also been studied for dependent random variables (c.f., Gonchigdanzan and Rempala (2006)). In this note, we will show that this kind of result follows from the invariance principle.
Let $\{S_n; n \geq 1\}$ be a sequence of positive random variables. To present our main idea, we assume that (possibly in an enlarged probability space in which the sequence $\{S_n; n \geq 1\}$ is redefined without changing its distribution) there exists a standard Wiener process $\{W(t) : t \geq 0\}$ and two positive constants μ and σ such that

$$S_n - n\mu - \sigma W(n) = o(\sqrt{n}) \quad a.s. \tag{2}$$

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Then

$$\begin{aligned}\log\prod_{k=1}^n\frac{S_k}{k\mu} &= \sum_{k=1}^n\log\frac{S_k}{k\mu} = \sum_{k=1}^n\log\left(1+\frac{\sigma}{\mu}\frac{W(k)}{k}+o(k^{-1/2})\right)\\ &= \sum_{k=1}^n\left(\frac{\sigma}{\mu}\frac{W(k)}{k}+o(k^{-1/2})\right) = \frac{\sigma}{\mu}\sum_{k=1}^n\frac{W(k)}{k}+o(\sqrt{n})\\ &= \frac{\sigma}{\mu}\int_0^n\frac{W(x)}{x}dx+o(\sqrt{n}) \quad a.s.,\end{aligned} \tag{3}$$

where $\log x = ln(x\vee e)$. It follows that

$$\frac{\mu}{\sigma}\frac{1}{\sqrt{n}}\log\prod_{k=1}^n\frac{S_k}{k\mu}\overset{\mathscr{D}}{\to}\int_0^1\frac{W(x)}{x}dx, \quad \text{as } n\to\infty.$$

It is easily seen that the random variable on the right hand side is a normal random variable with

$$\mathsf{E}\int_0^1\frac{W(x)}{x}dx=\int_0^1\frac{\mathsf{E}W(x)}{x}dx=0$$

and

$$\mathsf{E}\left(\int_0^1\frac{W(x)}{x}dx\right)^2=\int_0^1\int_0^1\frac{\mathsf{E}W(x)W(y)}{xy}dxdy=\int_0^1\int_0^1\frac{\min(x,y)}{xy}dxdy=2.$$

So

$$\left(\prod_{k=1}^n\frac{S_k}{k\mu}\right)^{\gamma/\sqrt{n}}\overset{\mathscr{D}}{\to}e^{\sqrt{2}N(0,1)}, \quad \text{as } n\to\infty, \tag{4}$$

where $\gamma=\mu/\sigma$. If S_n is the partial sum of a sequence $\{X_k;k\geq 1\}$ of i.i.d. random variables, then (2) is satisfied when $\mathsf{E}|X_k|^2\log\log|X_k|<\infty$. (2) is known as the strong invariance principle. To show (4) holds for sums of i.i.d. random variables only with the finite second moments, we replace the condition (2) by a weaker one. The following is our main result.

Theorem 1 *Let $\{S_k;k\geq 1\}$ be a nondecreasing sequence of positive random variables. Suppose there exists a standard Wiener process $\{W(t);t\geq 0\}$ and two positive constants μ and σ such that*

$$W_n(t)=:\frac{S_{[nt]}-[nt]\mu}{\sigma\sqrt{n}}\overset{\mathscr{D}}{\to}W(t) \quad in \;\; D[0,1], \quad as \;\; n\to\infty \tag{5}$$

and

$$\sup_n\frac{E|S_n-n\mu|}{\sqrt{n}}<\infty. \tag{6}$$

Then

$$\left(\prod_{k=1}^{[nt]}\frac{S_k}{k\mu}\right)^{\gamma/\sqrt{n}}\overset{\mathscr{D}}{\to}\exp\left\{\int_0^t\frac{W(x)}{x}dx\right\} \quad in \;\; D[0,1], \quad as \;\; n\to\infty, \tag{7}$$

where $\gamma=\mu/\sigma$.

Remark 1 *(5) is known as the weak invariance principle. The conditions (5) and (6) are satisfied for many random variables sequences. For example, if* $\{X_k; k \geq 1\}$ *are i.i.d. positive random variables with mean* μ *and variance* σ^2 *and* $S_n = \sum_{i=1}^n X_k$*, then (5) is satisfied by the invariance principle (c.f., Theorem 14.1 of Billingsley (1999)). Also, for any* $n \geq 1$*,*

$$\mathsf{E}\left[\frac{|S_n - n\mu|}{\sqrt{n}}\right] \leq \left\{\mathsf{Var}\left[\frac{S_n - n\mu}{\sqrt{n}}\right]\right\}^{1/2} = \sigma,$$

by the Cauchy-Schwarz inequality, so Condition (6) is also satisfied. Many dependent random sequences also satisfy these two conditions.

Proof of Theorem 1. For $x > -1$, write $\log(1+x) = x + x\theta(x)$, where $\theta(x) \to 0$, as $x \to 0$. Then for any $t > 0$,

$$\log\left(\prod_{k=1}^{[nt]} \frac{S_k}{k\mu}\right)^{\gamma/\sqrt{n}} = \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k} + \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k} \theta\left(\frac{S_k}{k\mu} - 1\right). \tag{8}$$

Notice that for any $\rho > 1$,

$$\max_{\rho^n \leq k < \rho^{n+1}} \frac{|S_k - k\mu|}{k} \leq \max\left\{\frac{|S_{[\rho^{n+1}]} - [\rho^{n+1}]\mu|}{\rho^n}, \frac{|S_{[\rho^n]} - [\rho^n]\mu|}{\rho^n}\right\} + \mu\big((\rho - 1) + \frac{1}{\rho^n}\big).$$

Together with (6), it follows that, for any $n_0 \geq 1$,

$$\begin{aligned}
&\mathsf{E}\left[\max_{k \geq \rho^{n_0}} \frac{|S_k - k\mu|}{k}\right] \leq \rho \mathsf{E}\left[\max_{n \geq n_0} \frac{|S_{[\rho^n]} - [\rho^n]\mu|}{\rho^n}\right] + \mu\big((\rho - 1) + \frac{1}{\rho^{n_0}}\big) \\
\leq& \rho \sup_k \frac{E|S_k - k\mu|}{\sqrt{k}} \sum_{n=n_0}^{\infty} \rho^{-n/2} + \mu\big((\rho - 1) + \frac{1}{\rho^{n_0}}\big) \to 0,
\end{aligned}$$

as $n_0 \to \infty$ and then $\rho \to 1$. It follows that

$$\max_{k \geq k_0} \left|\frac{S_k}{k\mu} - 1\right| \xrightarrow{P} 0, \quad \text{as} \quad k_0 \to \infty,$$

which implies that

$$\frac{S_k}{k\mu} - 1 \to 0 \ \ a.s., \quad \text{as} \ \ k \to \infty.$$

Hence we conclude that

$$\theta\left(\frac{S_k}{k\mu} - 1\right) \to 0 \ \ a.s., \quad \text{as} \ \ k \to \infty.$$

On the other hand, by (6), we have

$$\frac{1}{\sqrt{n}} \mathsf{E}\left[\sum_{k=1}^n \frac{|S_k - k\mu|}{k}\right] \leq C_0 \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{1}{\sqrt{k}} \leq 2C_0. \tag{9}$$

It follows that

$$\max_{0 \leq t \leq 1} \left|\frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k\mu} \theta\left(\frac{S_k}{k\mu} - 1\right)\right| = \frac{1}{\sqrt{n}} \sum_{k=1}^n \frac{|S_k - k\mu|}{k} o(1) = o_P(1).$$

So, according to (8) it is suffices to show that

$$Y_n(t) =: \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k} \xrightarrow{\mathscr{D}} \int_0^t \frac{W(x)}{x} dx \quad \text{in} \quad D[0,1], \quad \text{as} \quad n \to \infty. \tag{10}$$

Let

$$H_\epsilon(f)(t) = \begin{cases} \displaystyle\int_\epsilon^t \frac{f(x)}{x} dx, & \epsilon < t \le 1, \\ 0, & 0 \le t \le \epsilon \end{cases}$$

and

$$Y_{n,\epsilon}(t) = \begin{cases} \displaystyle\frac{1}{\sigma\sqrt{n}} \sum_{k=[n\epsilon]+1}^{[nt]} \frac{S_k - k\mu}{k}, & \epsilon < t \le 1, \\ 0, & 0 \le t \le \epsilon. \end{cases}$$

It is obvious that

$$\max_{0 \le t \le 1} \left| \int_0^t \frac{W(x)}{x} dx - H_\epsilon(W)(t) \right| = \sup_{0 \le t \le \epsilon} \left| \int_0^t \frac{W(x)}{x} dx \right| \to 0 \quad a.s., \text{ as } \epsilon \to 0 \tag{11}$$

and

$$\begin{aligned} &E \max_{0 \le t \le \epsilon} |Y_n(t) - Y_{n,\epsilon}(t)| = E\left\{ \max_{0 \le t \le \epsilon} E\left| \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[nt]} \frac{S_k - k\mu}{k} \right| \right\} \\ \le& \frac{1}{\sigma\sqrt{n}} \sum_{k=1}^{[n\epsilon]} \frac{\mathsf{E}|S_k - k\mu|}{k} \le \frac{C_0}{\sigma\sqrt{n}} \sum_{k=1}^{[n\epsilon]} \frac{1}{\sqrt{k}} \le \frac{2C_0}{\sigma\sqrt{n}} \sqrt{[n\epsilon]} \le C\sqrt{\epsilon}, \end{aligned} \tag{12}$$

by (6). On the other hand, it is easily seen that, for n large enough such that $n\epsilon \ge 1$,

$$\begin{aligned} &\sup_{\epsilon \le t \le 1} \left| \sum_{k=[n\epsilon]+1}^{[nt]} \frac{S_k - k\mu}{k} - \int_{n\epsilon}^{nt} \frac{S_{[x]} - [x]\mu}{x} dx \right| \\ =& \sup_{\epsilon \le t \le 1} \left| \int_{[n\epsilon]+1 \le x < [nt]+1} \frac{S_{[x]} - [x]\mu}{[x]} dx - \int_{n\epsilon}^{nt} \frac{S_{[x]} - [x]\mu}{x} dx \right| \\ \le& \left| \int_{n\epsilon \le x < [n\epsilon]+1} \frac{S_{[x]} - [x]\mu}{x} dx \right| + \sup_{\epsilon \le t \le 1} \left| \int_{nt \le x < [nt]+1} \frac{S_{[x]} - [x]\mu}{x} dx \right| \\ &+ \sup_{\epsilon \le t \le 1} \left| \int_{[n\epsilon]+1 \le x < [nt]+1} \left(S_{[x]} - [x]\mu\right) \left(\frac{1}{x} - \frac{1}{[x]} \right) dx \right| \\ \le& \max_{k \le n} |S_k - k\mu| \sup_{\epsilon \le t \le 1} \left(\frac{2}{n\epsilon} + \frac{2}{nt} + \frac{1}{n\epsilon} \right) \\ \le& 5 \max_{k \le n} |S_k - k\mu| / (n\epsilon) = O_P(\sqrt{n})/n = o_P(1) \end{aligned}$$

by noticing that $\max_{k \le n} |S_k - k\mu| / \sqrt{n} \xrightarrow{\mathscr{D}} \sigma \sup_{0 \le t \le 1} |W(t)|$ according to (5). So

$$\frac{1}{\sigma\sqrt{n}} \sum_{k=[n\epsilon]+1}^{[nt]} \frac{S_k - k\mu}{k} = \frac{1}{\sigma\sqrt{n}} \int_{n\epsilon}^{nt} \frac{S_{[x]} - [x]\mu}{x} dx + o_P(1) = \int_\epsilon^t \frac{W_n(x)}{x} dx + o_P(1)$$

uniformly in $t \in [\epsilon, 1]$. Notice that $H_\epsilon(\cdot)$ is a continuous mapping on the space $D[0,1]$. Using the continuous mapping theorem (c.f., Theorem 2.7 of Billingsley (1999)) it follows that

$$Y_{n,\epsilon}(t) = H_\epsilon(W_n)(t) + o_P(1) \xrightarrow{\mathscr{D}} H_\epsilon(W)(t) \text{ in } D[0,1], \quad \text{as } n \to \infty. \tag{13}$$

Combining (11)–(13) yields (10) by Theorem 3.2 of Billingsley (1999). □

Theorem 2 *Let $\{S_k; k \geq 1\}$ be a sequence of positive random variables. Suppose there exists a standard Wiener process $\{W(t); t \geq 0\}$ and two positive constants μ and σ such that*

$$S_n - n\mu - \sigma W(n) = o\big(\sqrt{n \log\log n}\big) \ a.s. \tag{14}$$

Let

$$\mathcal{F} = \left\{ f(t) = \int_0^t f'(u)du : f(0) = 0, \int_0^1 (f'(u))^2 du \leq 1, 0 \leq u \leq 1 \right\}.$$

Then with probability one

$$\left\{ \Big(\prod_{k=1}^{[nt]} \frac{S_k}{k\mu} \Big)^{\gamma/\sqrt{2n\log\log n}}; 0 \leq t \leq 1 \right\}_{n=3}^{\infty} \text{ is relatively compact} \tag{15}$$

and the limit set is

$$\left\{ \exp\Big\{ \int_0^x \frac{f(u)}{u} du \Big\} : f \in \mathcal{F}, 0 \leq x \leq 1 \right\}.$$

In particular,

$$\limsup_{n\to\infty} \left(\prod_{k=1}^{n} \frac{S_k}{k\mu} \right)^{\gamma/\sqrt{2n\log\log n}} = e^{\sqrt{2}} \ a.s. \tag{16}$$

Proof of Theorem 2. Similar to (3), we have

$$\log \prod_{k=1}^{n} \frac{S_k}{k\mu} = \frac{\sigma}{\mu} \int_0^n \frac{W(x)}{x} dx + o(\sqrt{n\log\log n}) \ a.s.$$

Notice

$$\frac{1}{\sqrt{2n\log\log n}} \int_0^{nt} \frac{W(x)}{x} dx = \int_0^t \frac{1}{u} \frac{W(nu)}{\sqrt{2n\log\log n}} du$$

and with probability one

$$\left\{ \frac{W(nt)}{\sqrt{2n\log\log n}} : 0 \leq t \leq 1 \right\}_{n=3}^{\infty} \text{ is relatively compact}$$

with $\mathcal{F}$ being the limit set (c.f., Theorem 1.3.2 of Csőrgö and Révész (1981) or Strassen (1964)). The first part of the conclusion follows immediately. For (16), it suffices to show that

$$\sup_{f\in\mathcal{F}} \sup_{0\leq t\leq 1} \int_0^t \frac{f(u)}{u} du \leq \sqrt{2} \tag{17}$$

and

$$\sup_{f\in\mathcal{F}} \int_0^1 \frac{f(u)}{u} du \geq \sqrt{2}. \tag{18}$$

For any $f \in \mathcal{F}$, using the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\int_0^t \frac{f(u)}{u} du &= \int_0^t \frac{1}{u} \int_0^u f'(v) dv du = \int_0^t \int_v^t f'(v) \frac{1}{u} du dv \\ &= \int_0^t f'(v) \log \frac{t}{v} dv \leq \left(\int_0^t \left(\log \frac{t}{v} \right)^2 dv \right)^{1/2} \left(\int_0^t \left(f'(v) \right)^2 dv \right)^{1/2} \\ &\leq \left(\int_0^t \left(\log \frac{t}{v} \right)^2 dv \right)^{1/2} = \sqrt{2t} \leq \sqrt{2},\end{aligned}$$

where $0 \leq t \leq 1$. Then (17) is proved. Now, let $f(t) = (t - t \log t)/\sqrt{2}$, $f(0) = 0$. Then $f \in \mathcal{F}$ and

$$\int_0^1 \frac{f(u)}{u} du = \frac{1}{\sqrt{2}} \int_0^1 (1 - \log u) du = \sqrt{2}.$$

Hence (18) is proved. □

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References

[1] B.C. Arnold and J. A. Villaseñor. The Asymptotic Distributions of Sums of Records. *Extremes*, **1**, No.3 (1998), 351-363. MR1814709 (2002a:60025) MR1814709

[2] P. Billingsley. *Convergence of Probability Measures*, Joh Wiley & Sons, INC, New York (1999). MR1700749

[3] M. Csörgő and P. Révész. *Strong Approximations in Probability and Statistics*, Akadémiai Kiadó, Budapest (1981). MR0666546

[4] K. Gonchigdanzan and G. A. Rempala. A note on the almost sure limit theorem for the product of partial sums. *Applied Math. Lett.*, **19**, No. 2 (2006), 191-196. MR2198407 MR2198407

[5] G. Rempala and J. Wesolowski. Asymptotics for products of sums and U-statistics. *Elect. Comm. in Prob.*, **7** (2002), 47-54. MR1887173 (2002k:60070) MR1887173

[6] V. Strassen. An invariance principle for the law of the iterated logarithm. *Z. Wahrsch. Keitsth. verw. Gebiete*, **3** (1964), 211-226. MR0175194 (30 #5379) MR0175194