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CENTRAL LIMIT THEOREMS FOR THE PRODUCTS OF RANDOM MATRICES SAMPLED BY A RANDOM WALK

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Abstract
The purpose of the present paper is to study the asymptotic behaviour of the products of random matrices indexed by a random walk following the results obtained by Furstenberg and Kesten [4] and by Ishitani [6].

1 Introduction and the main result

Let G be a countable group and p be a probability measure on G. The right random walk on G defined by p is the canonical Markov chain $(S_n)_{n\geq 0}$ with state space G and transition matrix

$$p^{(1)}(x,y) = p(x^{-1}y) \quad , x, y \in \ G.$$

For $x, y \in G$, we denote by $p^{(n)}(x,y)$ the probability to go from x to y in n steps. We denote by $(\Omega, \mathcal{F}, P)$ the probability space associated to the random walk $(S_n)_{n\geq 0}$ starting from the identity element e of the group G. Let $(A_x)_{x\in G}$ be a sequence of independent and identically distributed random $m \times m$-matrices with strictly positive elements defined on a probability space $(\tilde{\Omega}, \mathcal{A}, \mu)$. We are interested in the asymptotic behaviour of the product

$$M_N = A_{S_0} A_{S_1} \dots A_{S_N},$$

or more precisely in the terms

$$\alpha_{i,j}^{(N)} = \frac{1}{N} \log(M_N)_{i,j} \text{ for } \ i, j = 1, \dots, m.$$

($(M_N)_{i,j}$ means the entry in row i of column j of the matrix M_N).
This question is motivated by the study of random walks evolving in a disordered media and has real similarities with the model of random walks in random sceneries (see section 1.4). The results presented in this paper should have some applications in the study of very long molecules represented by the random walk $(S_k)_{k\geq 0}$ evolving in a disordered media which for instance randomly acts on each atom.
Let us fix some notation: the expectation with respect to the probability measure μ (resp. P) will be denoted by $\tilde{E}$ (resp. E). On the space $(\Omega \times \tilde{\Omega}, \mathcal{F} \otimes \mathcal{A})$, the probability $P \otimes \mu$ is denoted by $\mathbb{P}$ and expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$.
Firstly, the sequence $(A_{S_k})_{k\geq 0}$ is stationary and ergodic, then by a direct application of Liggett's version of Kingman's Theorem (see [2] p. 319), for every $i, j = 1, \ldots, m$, $\alpha_{i,j}^{(N)}$ converges $\mathbb{P}$-almost surely as $N \to \infty$ to the real $\gamma_{i,j} = \lim_{N\to\infty} \mathbb{E}(\log(M_N)_{i,j})/N$ if for all (k, l), $\tilde{E}(\log(A_e)_{k,l})$ is finite. We define for every $n \geq 1$, the sequence

$$\beta_n = \sum_{j=0}^{\infty} \sum_{l=j}^{\infty} p^{(n+l)}(e, e).$$

Let (H) be the hypothesis:
there exists some $\delta > 0$ such that

$$\sum_{n=1}^{\infty} \beta_n^{\frac{\delta}{2+\delta}} < \infty.$$

Theorem 1 *Let us assume that the random matrices $(A_x)_{x\in G}$ satisfy both conditions:*
i) there exists a positive constant C such that

$$1 \leq \frac{\max_{i,j}(A_e)_{i,j}}{\min_{i,j}(A_e)_{i,j}} \leq C \quad \mu - a.e.$$

ii) $\tilde{E}(|\log(A_e)_{1,1}|^{2+\delta}) < \infty$.
Then, under (H), $\gamma_{i,j} \equiv \gamma$ is independent of (i, j) and there exists a nonnegative constant σ^2 such that for all $1 \leq i, j \leq m$,

$$\frac{\log(M_N)_{i,j} - N\gamma}{\sqrt{N}}$$

converges in distribution to the Normal distribution $\mathcal{N}(0, \sigma^2)$ (where $\mathcal{N}(0, \sigma^2) = \delta_0$ when $\sigma^2 = 0$).

2 Proof of Theorem 1

The proof of Theorem 1 is essentially based on the following result of Ishitani (see Theorem 2 page 571 and Remark 4 page 575 in [6]). In order to state this result, we shall introduce some notation. Let $(\tilde{A}_k)_{k\geq 0}$ be a stationary sequence of $m \times m$-random matrices with strictly positive elements defined on a probability space $(\Omega', \mathcal{F}', \mathbb{P})$. By the stationarity, there exists a measure preserving transformation T such that for all $k \geq 0$,

$$\tilde{A}_{k+1}(\omega) = \tilde{A}_k(T\omega).$$

Let $\{\mathcal{M}_a^b; a \leq b, a = 0,1,...; b = 0,1,...\}$ be a family of sub-σ-fields of $\mathcal{F}'$ satisfying the conditions:
P1) If $a \leq c \leq d \leq b$, then $\mathcal{M}_c^d \subset \mathcal{M}_a^b$.
P2) For all $a \leq b$, $T^{-1}\mathcal{M}_a^b = \mathcal{M}_{a+1}^{b+1}$.
We define for $n \geq 1$

$$\alpha(n) = \sup_{k\geq 0} \sup \left\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|; A \in \mathcal{M}_0^k, B \in \mathcal{M}_{k+n}^\infty\right\}.$$

Theorem 2 *Suppose that the sequence $(\tilde{A}_k)_{k\geq 0}$ satisfies both conditions:*
i) there exists a positive constant C such that

$$1 \leq \frac{\max_{i,j}(\tilde{A}_0)_{i,j}}{\min_{i,j}(\tilde{A}_0)_{i,j}} \leq C \quad a.e.$$

ii) there exists $\delta' > 0$ such that

$$\sum_{n=1}^{\infty} [\alpha(n)]^{\frac{\delta'}{2+\delta'}} < \infty$$

iii) $\mathbb{E}(|\log(\tilde{A}_0)_{1,1}|^{2+\delta'}) < \infty$.
Then, there exists a nonnegative constant σ^2 such that for all $1 \leq i,j \leq m$,

$$\frac{\log(\tilde{A}_N \tilde{A}_{N-1} \dots \tilde{A}_0)_{i,j} - N\gamma}{\sqrt{N}}$$

converges in distribution to the Normal distribution $\mathcal{N}(0,\sigma^2)$ (with $\mathcal{N}(0,\sigma^2) = \delta_0$ when $\sigma^2 = 0$).

Proof of Theorem 1
Let us define $\Omega' = \Omega \times \tilde{\Omega}$, $\mathcal{F}' = \mathcal{F} \otimes \mathcal{A}$, $\mathbb{P} = P \otimes \mu$ and $\mathcal{M}_a^b = \sigma(A_{S_a}, \dots, A_{S_b}), a, b \geq 0$. These σ-fields clearly satisfy the conditions P1 and P2. The random matrices $(A_{S_k})_{k\geq 0}$ form a stationary sequence of $m \times m$-random matrices with strictly positive elements defined on the probability space $(\Omega', \mathcal{F}', \mathbb{P})$, so proving Theorem 1 is equivalent to showing that the conditions i), ii) and iii) of Ishitani's theorem hold. Conditions i) and iii) are verified since by hypothesis, there exists a positive constant C such that

$$1 \leq \frac{\max_{i,j}(A_e)_{i,j}}{\min_{i,j}(A_e)_{i,j}} \leq C \quad \mu - a.e.$$

and

$$\tilde{E}(|\log(A_e)_{1,1}|^{2+\delta}) < \infty.$$

Let us establish that the condition ii) of Ishitani's Theorem is satisfied for the sequence $(A_{S_k})_{k\geq 0}$. Let $A \in \mathcal{M}_0^k$ and $B \in \mathcal{M}_{k+n}^\infty$. Denote by R_0^k the range of the random walk $(S_k)_{k\geq 0}$, that is to say

$$R_0^k = \{S_0, \dots, S_k\}$$

and

$$R_{k+n}^\infty = \{S_{k+n}, \dots\}.$$

We will use the notation $\tilde{R}_{k,n} = R_0^k \cap R_{k+n}^\infty$.

$$\begin{aligned}
\mathbb{P}(A\cap B) &= \int_{\Omega'} \mathbf{1}_A \mathbf{1}_B \; dP \; d\mu \\
&= \int_\Omega \mathbf{1}_{\{\tilde{R}_{k,n}=\emptyset\}} \left(\int_{\tilde{\Omega}} \mathbf{1}_A\mathbf{1}_B \; d\mu\right) dP + \int_{\Omega'} \mathbf{1}_{\{\tilde{R}_{k,n}\neq\emptyset\}} \mathbf{1}_A\mathbf{1}_B \; dP \; d\mu \\
&= \int_\Omega \mathbf{1}_{\{\tilde{R}_{k,n}=\emptyset\}} \tilde{E}(\mathbf{1}_A)\tilde{E}(\mathbf{1}_B) \; dP + \int_{\Omega'} \mathbf{1}_{\{\tilde{R}_{k,n}\neq\emptyset\}} \mathbf{1}_A\mathbf{1}_B \; dP \; d\mu \\
&= \int_\Omega \tilde{E}(\mathbf{1}_A) E(\tilde{E}(\mathbf{1}_B)\mathbf{1}_{\{\tilde{R}_{k,n}=\emptyset\}}|S_0,\ldots,S_k) \; dP + \int_{\Omega'} \mathbf{1}_{\{\tilde{R}_{k,n}\neq\emptyset\}} \mathbf{1}_A\mathbf{1}_B \; dP \; d\mu \\
&= \int_\Omega \tilde{E}(\mathbf{1}_A) E(\tilde{E}(\mathbf{1}_B)|S_0,\ldots,S_k) \; dP \\
&\quad - \int_\Omega \tilde{E}(\mathbf{1}_A) E(\tilde{E}(\mathbf{1}_B)\mathbf{1}_{\{\tilde{R}_{k,n}\neq\emptyset\}}|S_0,\ldots,S_k) \; dP + \int_{\Omega'} \mathbf{1}_{\{\tilde{R}_{k,n}\neq\emptyset\}} \mathbf{1}_A\mathbf{1}_B \; dP \; d\mu
\end{aligned}$$

Now, let us prove that for every $B \in \mathcal{M}_{k+n}^\infty$,

$$E(\tilde{E}(\mathbf{1}_B)|S_0,\ldots,S_k) = \mathbb{P}(B).$$

It is enough to prove this equality for the particular sets $B = \{A_{S_{k+p}} \in C_1, A_{S_{k+p+1}} \in \; C_2, \ldots\}$ where $p \geq n$ and $C_i, i \geq 1$ are Borel sets of matrices. Using the Markov property for $(S_n)_n$, we can write

$$E(\tilde{E}(\mathbf{1}_B)|S_0,\ldots,S_k) = E_{S_k}(\tilde{E}(\mathbf{1}_{\{A_{S_p}\in C_1, A_{S_{p+1}}\in C_2,\ldots\}})).$$

The sequence of random matrices $(A_x)_{x\in G}$ being stationary, we have that, for any $x \in G$,

$$E_x(\tilde{E}(\mathbf{1}_{\{A_{S_p}\in C_1, A_{S_{p+1}}\in C_2,\ldots\}})) = E_e(\tilde{E}(\mathbf{1}_{\{A_{S_p}\in C_1, A_{S_{p+1}}\in C_2,\ldots\}}))$$

$$= \mathbb{P}(A_{S_p} \in C_1, A_{S_{p+1}} \in C_2, \ldots) = \mathbb{P}(A_{S_{p+k}} \in C_1, A_{S_{p+k+1}} \in C_2, \ldots) = \mathbb{P}(B)$$

and consequently,

$$\int_\Omega \tilde{E}(\mathbf{1}_A) E(\tilde{E}(\mathbf{1}_B)|S_0,\ldots,S_k) \; dP = \mathbb{P}(A)\mathbb{P}(B).$$

Then,

$$|\mathbb{P}(A\cap B) - \mathbb{P}(A)\mathbb{P}(B)| \leq 2P(\tilde{R}_{k,n} \neq \emptyset)$$

Let us estimate $P(\tilde{R}_{k,n} \neq \emptyset)$. This probability is bounded above by

$$\begin{aligned}
\sum_{j=0}^{k} \sum_{l=n+k}^{\infty} P(S_j = S_l) &= \sum_{x\in G}\sum_{j=0}^{k}\sum_{l=n+k}^{\infty} P(S_l = x|S_j = x)P(S_j = x) \\
&= \sum_{j=0}^{k}\sum_{l=n+k}^{\infty} p^{(l-j)}(e,e) \\
&= \sum_{j=0}^{k}\sum_{l=n+j}^{\infty} p^{(l)}(e,e)
\end{aligned}$$

Then, for every $n \geq 1$,

$$\alpha(n) \leq 2 \sum_{j=0}^{\infty} \sum_{l=n+j}^{\infty} p^{(l)}(e,e)$$

Under the hypothesis (H): there exists $\delta > 0$ such that

$$\sum_{n=1}^{\infty} [\alpha(n)]^{\frac{\delta}{2+\delta}} < \infty,$$

so condition ii) follows.

3 Examples

- **The centered random walk on $\mathbb{Z}^d$.**
In the case of the abelian group $G = (\mathbb{Z}^d, +)$, $(S_n)_{n\geq 1}$ can be written as a sum of independent and identically distributed random vectors $(X_i)_{i\geq 1}$ with values in $\mathbb{Z}^d$ and $S_0 = 0$. Under the hypothesis that the random vectors $(X_i)_{i\geq 1}$ are centered, with finite covariance matrix and that the random walk is strongly aperiodic, there exists a constant $C_d > 0$ such that
$$p^{(n)}(0,0) \sim C_d n^{-d/2}$$
as $n \to \infty$ (see Spitzer [9]). The hypothesis (H) is clearly satisfied as soon as $d \geq 7$. Notice however that the method used for proving Theorem 1 is not applicable in the case when the dimension d of the space is between 3 and 6. The probability of self-intersection of a $\mathbb{Z}^d$-random walk ($3 \leq d \leq 6$) between the first k steps and after the step $k+n$ is too large (see Lemma 7 in [3]) to conclude.
Consider the particular case $d = 1$, when the random variables X_i are centered, independent, identically distributed, strongly aperiodic and in the attraction domain of a stable distribution of index $\alpha \in]0,2[$, a local limit theorem (see Stone ([10])) can be obtained: there exists a constant $C'_d > 0$ such that
$$p^{(n)}(0,0) \sim C'_d n^{-1/\alpha}$$
as $n \to \infty$. The hypothesis (H_δ) for $\delta > 2\alpha/(1-3\alpha)$ and thus Theorem 1 holds as soon as $\alpha < \frac{1}{3}$.

- **The non-centered random walk on $\mathbb{Z}^d$.**

Let $(S_n)_{n\geq 1}$ be a sum of independent and identically distributed random vectors $(X_i)_{i\geq 1}$ with values in $\mathbb{Z}^d$ and $S_0 = 0$. Assume that the mean vector of X_1 exists and is not equal to the null vector. Let us solve the case $d = 1$. Without losing generality, we assume that $m = E(X_1) > 0$. We denote by ϕ the Laplace transform of the random variable X_1. For $\lambda \geq 0$,
$$\phi(\lambda) = E(e^{-\lambda X_1}).$$
As $E(|X_1|) < \infty$, we have
$$\phi(\lambda) = 1 - m\lambda + o(\lambda),$$

and, for any $\lambda \geq 0$,
$$p^{(n)}(0,0) = P(S_n = 0) \leq \phi(\lambda)^n.$$
The condition (H) is satisfied when we choose $\lambda > 0$ small enough and then Theorem 1 holds. This reasoning can obviously be adapted to dimension larger than 1.

- **The random walk on the homogeneous tree.**
We consider the case where G is the free product of $q \geq 3$ copies of $\mathbb{Z}_2$ i.e. G has generators $\{a_1, \ldots, a_q\}$ and $a_i^2 = e,\ \forall i = 1, \ldots, q$. The random walk on this group G corresponds to the nearest-neighbours random walk on the homogeneous tree with degree q. It is well-known that this random walk is transient and a local limit theorem can even be obtained (see [5]): there exists a strictly positive constant C such that
$$p^{(n)}(e,e) \sim CR^{-n}n^{-3/2}$$
as $n \to \infty$ where R is the spectral radius of the random walk defined by $R = (\limsup_n p^{(n)}(e,e)^{1/n})^{-1}$ strictly greater than one. The condition (H) is clearly satisfied for every q and Theorem 1 follows.

Remark: The hypothesis (H) is satisfied if $\limsup_n p^n(e,e)^{1/n} < 1$. This condition applies when the group G is non amenable by an old result of Kesten [7].

4 Further results and open problems

It is quite natural to extend Theorem 1 to recurrent random walks. Let $(A_x)_{x\in G}$ be a sequence of matrices with strictly positive elements, independent and identically distributed. When they commute, the product M_N can be rewritten using properties of formal series as
$$\log M_N = \sum_{k=0}^{N} \log(A_{S_k}).$$
So, for every $i, j = 1, \ldots, m$, $(\log(M_N))_{i,j}$ is a random walk in a random scenery, the random scenery being given here by $(\log\ A_x)_{i,j}, x \in G$. Functional limit theorems for the random walks in random sceneries were well studied when $G = \mathbb{Z}^d, d \geq 1$ (see [1],[8]). So we can deduce functional limit theorems for the sequence of random variables $(\log M_N)_{i,j}$ where $i, j = 1, \ldots, m$, it gives us the asymptotic behaviour in distribution of the element (i, j) of the matrix $\log M_N$, not the one of $(M_N)_{i,j}$. Let $(A_x)_{x\in\mathbb{Z}^d}$ be a sequence of $m \times m$-random matrices with strictly positive elements; the matrices are assumed independent and identically distributed. We assume that they commute.
In the case $d = 1$, using Kesten and Spitzer's Theorem (see [8]), we obtain the following

Proposition 3 *Let $S_n = X_1 + \ldots + X_n$ be a random walk on $\mathbb{Z}$ such that the increments $X_i, i \geq 1$ be independent and identically distributed random variables belonging to the domain of attraction of a stable law of index $\alpha \in]0, 2[$ then, for every $1 \leq i, j \leq m$,*
$$\left(\frac{(\log M_{[Nt]})_{i,j} - [Nt]\tilde{E}\left((\log A_0)_{i,j}\right)}{N^\beta}\right)_{t\geq 0}$$
converges weakly in $D[0,\infty)$ (the set of right continuous real-valued functions with left limits) to a self-similar process with index $\beta = 1 - \frac{1}{2\alpha}$, with stationary increments.

When $d = 2$, Bolthausen's Theorem (see [1]) allows us to deduce

Proposition 4 *Let $S_n = X_1 + \ldots + X_n$ be a strongly aperiodic random walk on $\mathbb{Z}^2$ such that the increments $X_i, i \geq 1$ be independent and identically distributed random variables, centered, with finite covariance matrix Σ, then for every $1 \leq i, j \leq m$,*

$$\sqrt{2\pi}(\det \Sigma)^{1/4} \left(\frac{(\log M_{[Nt]})_{i,j} - [Nt]\tilde{E}\left((\log A_0)_{i,j}\right)}{\sqrt{N \log N}} \right)_{t \geq 0}$$

converges weakly in $D[0, \infty)$ to a standard Brownian motion.

The previous link established between the products of random matrices sampled by a random walk and the random walks in random sceneries suggest us the following conjectures. Both these results are trivial in the case when all matrices are diagonal with positive elements, but for general matrices these are not at all trivial problems.

Conjecture 1

Let $(A_x)_{x \in \mathbb{Z}}$ be a sequence of $m \times m$-random matrices with strictly positive elements, assumed independent and identically distributed. Let $S_n = X_1 + \ldots + X_n$ be a $\mathbb{Z}$-random walk such that the increments $X_i, i \geq 1$, are random variables belonging to the domain of attraction of a stable law of index $\alpha \in]0, 2[$. Then, under the conditions i) and ii) of Theorem 1, for every $1 \leq i, j \leq m$,

$$\frac{\log(M_N)_{i,j} - \mathbb{E}\log(M_N)_{i,j}}{N^{1-\frac{1}{2\alpha}}}$$

converges in distribution to a non-degenerate distribution.

Conjecture 2

Let $(A_x)_{x \in \mathbb{Z}^2}$ be a sequence of $m \times m$-random matrices with strictly positive elements, assumed to be independent and identically distributed. Let $S_n = X_1 + \ldots + X_n$ be a strongly aperiodic random walk on $\mathbb{Z}^2$ such that the increments $X_i, i \geq 1$, are independent and identically distributed random variables, centered, with finite covariance matrix. Then, under the conditions i) and ii) of Theorem 1, for every $1 \leq i, j \leq m$,

$$\frac{\log(M_N)_{i,j} - \mathbb{E}\log(M_N)_{i,j}}{\sqrt{N \log N}}$$

converges in distribution to a Normal distribution.

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