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SOME TWO-DIMENSIONAL FINITE ENERGY PERCOLATION PROCESSES

OLLE HÄGGSTRÖM[1]
Dept. of Mathematical Sciences, Chalmers University of Technology, 412 96 Göteborg, Sweden
email: `olleh@chalmers.se`

PÉTER MESTER[2]
Dept. of Mathematics, Indiana University, Bloomington, IN 47405-5701, United States
email: `pmester@indiana.edu`

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Abstract

Some examples of translation invariant site percolation processes on the $\mathbf{Z}^2$ lattice are constructed, the most far-reaching example being one that satisfies uniform finite energy (meaning that the probability that a site is open given the status of all others is bounded away from 0 and 1) and exhibits a.s. the coexistence of an infinite open cluster and an infinite closed cluster. Essentially the same example shows that coexistence is possible between an infinite open cluster and an infinite closed cluster that are both robust under i.i.d. thinning.

1 Introduction

By a **site percolation** on $\mathbf{Z}^2$, we mean an $\{0,1\}^{\mathbf{Z}^2}$-valued random object X. Focus in percolation theory is mainly on the connected components (clusters) of X. Two vertices $x, y \in \mathbf{Z}^2$ are said to communicate if there exists a path $\{z_1, z_2, \ldots, z_n\}$ from $z_1 = x$ to $z_n = y$ with $X(z_1) = X(z_2) = \cdots = X(z_n)$ (in the definition of path, we require z_i and z_j to be L_1-nearest neighbors for each i), and a connected component is a maximal set of vertices that all communicate with each other. A connected component is called an **open cluster** or a **closed cluster** depending on whether its vertices take value 1 or 0 in X.

Much of percolation theory deals with the i.i.d. case (see, e.g., Grimmett [7]), though various dependent settings have also received much attention. Here we will abandon the i.i.d. assumption in favor of the weaker but natural assumption of **translation invariance**, meaning that for any n and any $x_1, \ldots, x_n \in \mathbf{Z}^2$, the distribution of $(X(x_1 + y), \ldots, X(x_n + y))$ does not depend on $y \in \mathbf{Z}^2$.

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Intuitively, the planar structure of $\mathbf{Z}^2$ makes it difficult for an infinite open cluster and an infinite closed cluster to coexist. In order to prove a theorem to this extent, some further conditions beyond translation invariance are needed, as the following trivial example shows: assign the vertices on the x-axis independently value 0 or 1 with probability $\frac{1}{2}$ each, and let the values of any other vertex z be dictated by the value of the vertex on the x-axis sharing z's x-coordinate. This produces a translation invariant site percolation with both infinite open clusters and infinite closed clusters (in fact, infinitely many of each).

In a seminal paper, Gandolfi, Keane and Russo [5] showed that translation invariance and positive associations together with some auxiliary ergodicity conditions (later relaxed by Sheffield [11]) is enough to rule out such coexistence. In applications of this result, the hard part has typically been to establish positive associations; see for instance Chayes [4] and Häggström [8]. Partly for this reason, several researchers over the years have asked whether the positive associations condition can be replaced by the often easier-to-verify condition of **finite energy**, defined as follows.

Definition 1.1. *A site percolation X on $\mathbf{Z}^2$ is said to satisfy* **finite energy** *if it admits conditional probabilities such that for all $x \in \mathbf{Z}^2$ and all $\xi \in \{0,1\}^{\mathbf{Z}^2 \setminus \{x\}}$ we have*

$$0 < \mathbf{P}(X(x) = 1 \,|\, X(\mathbf{Z}^2 \setminus \{x\}) = \xi) < 1 .$$

It is said to satisfy **uniform finite energy** *if for some $\varepsilon > 0$ it admits conditional probabilities such that for all $x \in \mathbf{Z}^2$ and all $\xi \in \{0,1\}^{\mathbf{Z}^2 \setminus \{x\}}$ we have*

$$\varepsilon < \mathbf{P}(X(x) = 1 \,|\, X(\mathbf{Z}^2 \setminus \{x\}) = \xi) < 1 - \varepsilon .$$

In this paper, we show by means of concrete examples that translation invariance together with finite energy is *not* sufficient to rule out coexistence of an infinite open cluster, and an infinite closed cluster. What's more, it does not even help if we replace finite energy by uniform finite energy:

Theorem 1.2. *There exists a translation invariant site percolation on $\mathbf{Z}^2$ that satisfies uniform finite energy and that produces a.s. an infinite open cluster and an infinite closed cluster.*

It is a classical result of Burton and Keane [3] that translation invariance and finite energy together are enough to rule out the existence of more than one infinite open cluster (and, by symmetry, more than one infinite closed cluster), so any example witnessing Theorem 1.2 must have a.s. exactly one infinite cluster of each kind.

The rest of this paper is devoted to examples exhibiting such coexistence. The example witnessing Theorem 1.2 requires a somewhat elaborate construction, and is therefore postponed to Section 3. Along the way, we answer affirmatively (in Theorem 3.2) the question of whether, still assuming translation invariance, coexistence is possible between an infinite open cluster and an infinite closed cluster that are both robust under i.i.d. thinning. Before that, and in order to offer the reader some intuition for the problem, we first present a slightly less involved construction in Section 2, which satisfies finite energy but not uniform finite energy.

2 First construction

The purpose of this section is to give an example which proves the following weaker version of Theorem 1.2.

Proposition 2.1. *There exists a translation invariant site percolation on $\mathbf{Z}^2$ that satisfies finite energy and that produces a.s. an infinite open cluster and an infinite closed cluster.*

The construction will be based on the notion of a uniform spanning tree for the $\mathbf{Z}^2$ lattice, first studied by Pemantle [10] and later by Benjamini et al. [1] and others.
A **spanning tree** of a connected graph $G = (V, E)$ is a connected subgraph of G that contains all vertices $v \in V$ but no cycles. Any finite such G has a finite number of possible spanning trees, and a **uniform spanning tree** for G is therefore elementary to define in this finite setting: it is the random spanning tree for G obtained by choosing one of the possible spanning trees at random according to uniform distribution. This procedure may be identified with a probability measure μ on $\{0,1\}^E$ which we call the uniform spanning tree measure for G.
If we now move on to the case where $G = (V, E)$ is infinite but locally finite, the concept of a uniform spanning tree is less elementary, because there may be infinitely many (even uncountably many) possible spanning trees. Pemantle [10] showed that the following natural definition makes sense. By an **exhaustion** of G, we mean a sequence $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots$ of connected finite subgraphs that exhausts G in the sense that every $v \in V$ and every $e \in E$ is contained in all but at most finitely many G_i's. For each G_i we know how to pick a uniform spanning tree, so we may define a probability measure μ_i on $\{0,1\}^E$ whose projection on $\{0,1\}^{E_i}$ is the uniform spanning tree measure for G_i while the projection on $\{0,1\}^{E \setminus E_i}$ may be defined arbitrarily. It turns out (see [10]) that the μ_i's converge (in the product topology) to a limiting measure μ on $\{0,1\}^E$. Furthermore μ is concentrated on subgraphs of G consisting of a union of finitely or infinitely many infinite trees (i.e., not necessarily a single tree as might be tempting to believe). Pemantle considered the case where G is the $\mathbf{Z}^d$ lattice – having vertex set $V = \mathbf{Z}^d$ and edge set E consisting of edges connecting L_1-nearest neighbors – and showed that the number of trees is a μ-a.s. constant, equalling 1 for $d \leq 4$, and ∞ for $d \geq 5$. The case which concerns us is $d = 2$, where the resulting spanning tree, other than being unique, also has the following interesting properties:

One end. For every vertex $x \in \mathbf{Z}^2$, there exists μ-a.s. exactly one infinite self-avoiding path in the tree starting at x.

Self-duality. Consider the dual lattice $\tilde{\mathbf{Z}}^2$, with vertex set $\tilde{V} = \mathbf{Z}^2 = \mathbf{Z}^2 + (\frac{1}{2}, \frac{1}{2})$ and edge set $\tilde{E}$ consisting of edges connecting L_1-nearest neighbors. In the natural planar embeddings of (V, E) and $(\tilde{V}, \tilde{E})$, each edge $e \in E$ crosses exactly one edge $\tilde{e} \in \tilde{E}$. Suppose we pick $Y \in \{0,1\}^E$ according to μ, and then pick $\tilde{Y} \in \{0,1\}^{\tilde{E}}$ by declaring each $\tilde{e} \in \tilde{E}$ present in $\tilde{Y}$ if and only if the edge $e \in E$ that it crosses is absent in Y. Then, it turns out, the distribution of $\tilde{Y}$ is the same as that of Y (apart from the $(\frac{1}{2}, \frac{1}{2})$ shift). In particular, $\tilde{Y}$ consists of a single one-ended spanning tree for $\tilde{G}$.

Using Pemantle's spanning tree construction Y, we construct a site percolation $X \in \{0,1\}^{\mathbf{Z}^2}$ as follows; it should be viewed as a picture of Y and $\tilde{Y}$ scaled up by factor 2. Writing $x \in \mathbf{Z}^2$ in terms of its coordinates as $x = (x_1, x_2)$

$$X(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 \text{ and } x_2 \text{ are both even} \quad \text{(these sites represent the vertices of } Y\text{)} \\ 0 & \text{if } x_1 \text{ and } x_2 \text{ are both odd} \quad \text{(these sites represent the vertices of } \tilde{Y}\text{)}. \end{cases}$$

The remaining sites respresent crossing pairs of edges in E and $\tilde{E}$: a 1 indicates the presence of $e \in E$ and a 0 that of its dual edge $\tilde{e}$. More precisely, if x_1 is even and x_2 is odd, we set $X(x_1, x_2) = 1$ iff the edge $e \in E$ linking $(\frac{x_1}{2}, \frac{x_2-1}{2})$ to $(\frac{x_1}{2}, \frac{x_2+1}{2})$ is present in Y; while if x_1 is odd and x_2 is even, we set $X(x_1, x_2) = 1$ iff the edge $e \in E$ linking $(\frac{x_1-1}{2}, \frac{x_2}{2})$ to $(\frac{x_1+1}{2}, \frac{x_2}{2})$ is present in Y.

This defines $X \in \{0,1\}^{\mathbf{Z}^2}$. It is clear from the construction that X produces a.s. a single infinite open cluster, a single infinite closed cluster, and no finite clusters. To serve as a couterexample proving Proposition 2.1, it is however deficient in two ways, as (i) it fails to be translation invariant, and (ii) it fails to exhibit finite energy. Translation invariance is fixed by letting $\hat{X} \in \{0,1\}^{Z^2}$ equal X shifted by a random amount equalling $(0,0)$, $(0,1)$, $(1,0)$ or $(1,1)$, each with probability $\frac{1}{4}$; the resulting site percolation $\hat{X}$ is easily seen to be translation invariant.
To modify it again to give it finite energy, note first that for each $x \in \mathbf{Z}^2$ which is open in $\hat{X}$ there is a unique infinite self-avoiding open path in $\mathbf{Z}^2$ starting at x and using only open vertices, and analogously for each closed $x \in \mathbf{Z}^2$. Thus, each $x \in \mathbf{Z}^2$ has the following property: removing x would cut off a finite (possibly 0) number of vertices from either the infinite open or the infinite closed cluster of $\hat{X}$. Write $b(x)$ for this number, and construct $\bar{X} \in \{0,1\}^{\mathbf{Z}^2}$ by letting

$$\bar{X} = \begin{cases} \hat{X}(x) & \text{with probability } 1-2^{-b(x)-1} \\ 1-\hat{X} & \text{with probability } 2^{-b(x)-1} \end{cases} \tag{1}$$

independently for each $x \in \mathbf{Z}^2$. This defines $\bar{X}$, which clearly satisfies finite energy. Furthermore, if x is a site which is open in $\hat{X}$, then the expected number of sites in the unique infinite self-avoiding path from x in $\hat{X}$ that flip in the mapping (1) is bounded by $\sum_{i=1}^{\infty} 2^{-i}$ and therefore finite. So we have a.s. that for some vertex in the path and onwards, no vertex is flipped. Hence $\bar{X}$ has an infinite open cluster. Similarly we get that it has an infinite closed cluster. Thus, it has all the properties needed to warrant the statement that **Proposition 2.1 is established.**

3 Second construction

Most of the work needed to prove Theorem 1.2 is contained in the proof of the following Theorem 3.2. We will need some additional standard terminology. For an infinite but locally finite graph G, define the site percolation critical value $p_{c,site}(G)$ to be the infimum over all $p \in [0,1]$ such that i.i.d. site percolation on G with retention parameter p produces a.s. at least one infinite open cluster. Also, let $p_{c,bond}(G)$ be the analogous critical value for i.i.d. bond percolation on G, i.e., for the percolation process where it is the edges (rather than the vertices) that are removed at random. The following result is well known; see, e.g., [9, Thm. 1.1].

Lemma 3.1. *For any graph G of bounded degree, we have $p_{c,site}(G) < 1$ if and only if $p_{c,bond}(G) < 1$.*

Given a site percolation $\hat{X}$ on $\mathbf{Z}^2$, we write $G_{open}(\hat{X})$ for the (random) graph whose vertex set consists of all $x \in \mathbf{Z}^2$ such that $\hat{X}(x) = 1$, and whose edge set consists of all pairs of such vertices at L_1-distance 1 from each other. Analogously, $G_{closed}(\hat{X})$ has vertex set consisting of all $x \in \mathbf{Z}^2$ such that $\hat{X}(x) = 0$, and edge set consisting of all pairs of such vertices at L_1-distance 1 from each other.

Theorem 3.2. *There exists a translation invariant site percolation $\hat{X}$ such that with probability 1 we have both $p_{c,bond}(G_{open}(\hat{X})) < 1$ and $p_{c,bond}(G_{closed}(\hat{X})) < 1$.*

Before proving this result, which is our main task, we show how it easily implies Theorem 1.2.

Proof of Theorem 1.2 from Theorem 3.2. Let $\hat{X}$ be as in Theorem 3.2. By Lemma 3.1, we then have a.s. that $p_{c,site}(G_{open}(\hat{X})) < 1$ and $p_{c,site}(G_{closed}(\hat{X})) < 1$. We can then find an $\varepsilon \in (0, \frac{1}{2})$ such that

$$\mathbf{P}(p_{c,site}(G_{open}(\hat{X})) < 1-\varepsilon\,,\, p_{c,site}(G_{closed}(\hat{X})) < 1-\varepsilon) > 0\,. \tag{2}$$

In fact, we may without loss of generality assume that the event in (2) has probability 1, because the event is translation invariant so that conditioning on it does not mess up translation invariance. Now obtain another site percolation $\bar{X}$ from $\hat{X}$ by letting, for each $x \in \mathbf{Z}^2$ independently,

$$\bar{X} = \begin{cases} \hat{X}(x) & \text{with probability } 1-\varepsilon \\ 1-\hat{X} & \text{with probability } \varepsilon \end{cases} \tag{3}$$

It is immediate that the translation invariance property of $\hat{X}$ is inherited by $\bar{X}$. Furthermore, the transformation (3) implies (regardless of the details of $\hat{X}$) that

$$\mathbf{P}(\bar{X}(x) = 1 \,|\, \bar{X}(\mathbf{Z}^2 \setminus \{x\}) \in [\varepsilon, 1-\varepsilon] \text{ a.s.},$$

so $\bar{X}$ satisfies uniform finite energy. Next, $p_{c,site}(G_{open}(\hat{X})) < 1-\varepsilon$ implies that the set of sites in $G_{open}(\hat{X})$ that remain unflipped through the transformation (3) contains an infinite cluster; and analogously for $G_{closed}(\hat{X})$. In summary, $\bar{X}$ has all the properties needed to warrant Theorem 1.2. □

It remains to prove Theorem 3.2. It is instructive to think about why the $\hat{X}$ from Section 2 will not do. In that example, for each open vertex $x \in \mathbf{Z}^2$, $G_{open}(\hat{X})$ contains only a single infinite self-avoiding path starting at x. Carrying out i.i.d. bond percolation with retention parameter $1-\varepsilon$ on $G_{open}(\hat{X})$ will, regardless of how small $\varepsilon > 0$ is, a.s. kill at least one edge on this path and thus cut off x from any infinite cluster. Thus, $p_{c,bond}(G_{open}(\hat{X})) = 1$ (and, analogously, $p_{c,bond}(G_{closed}(\hat{X})) = 1$), so this choice of $\hat{X}$ fails to be a witness to Theorem 3.2.

What made the infinite clusters of $\bar{X}$ of Section 2 survive was the inhomogeneity of the retention probabilities, sufficiently rapidly approaching 1 as we moved from $x \in \mathbf{Z}^2$ off along its single self-avoiding path to infinity. When the retention parameter is set fixed at $1-\varepsilon$, we could try another approach: to replace the single path from x to infinity by a road that becomes progressively broader (and therefore more robust to random thinning) as we move along it. Some intuitive evidence that this should be doable comes from the work of Grimmett [6] and others concerning i.i.d. bond percolation on graphs G_f arising by restricting the $\mathbf{Z}^2$ lattice to vertices $x = (x_1, x_2) \in \mathbf{Z}^2$ with $x_1 \geq 0$ and $0 \leq x_2 \leq f(x_1)$ (and the usual nearest-neighbor edges connecting them), where $f : \mathbf{Z}_+ \to \mathbf{R}_+$ is a function that grows towards infinity as its argument goes to infinity. It turns out that a relatively slow growth of f suffices to ensure that $p_{c,bond}(G_f) < 1$; in particular, Grimmett showed that the critical value $p_{c,bond}(G_f)$ equals that of the full $\mathbf{Z}^2$ lattice (i.e., $p_{c,bond}(G_f) = 1/2$) if and only if $\lim_{n\to\infty} f(n)/\log(n) = \infty$.

The fact that such slow growth of f is enough suggests that it should be possible to modify the tree-structure of the $\hat{X}$ of Section 2 in such a way as to obtain a witness to Theorem 3.2. This is what we set out to do in the following. For technical reasons, we opt for a tree-like structure with a lot more regularity than the example in Section 2. Our construction will be built up from rectangular sets on a sequence of larger and larger scales. The percolation theory developed in the last few decades offers an abundance of results concerning crossing probabilities in i.i.d. percolation on such rectangles. We will settle for one which is due to Bollobás and Riordan – see Lemma 3.3 below – although other choices would certainly have been possible.

Due to the amount of work needed to prove Theorem 3.2, we divide it into a number of smaller portions. First, in Section 3.1, we introduce the terminology needed for a precise discussion of crossing probabilities for i.i.d. percolation and the Bollobás–Riordan result. Then, in Section 3.2, we define the basic building blocks of our construction. In Sections 3.3 and 3.4, we go on to some preliminary considerations that will be crucial for showing $p_{c,bond}(G_{open}) < 1$ and $p_{c,bond}(G_{closed}) < 1$, respectively. The construction is completed in Section 3.5, and in Section 3.6 we finally establish $p_{c,bond}(G_{open}) < 1$ and $p_{c,bond}(G_{closed}) < 1$, thus completing the proof.

3.1 Rectangles and crossing probabilities

If S_1, S_2 are subsets of $\mathbf{Z}^d$ then we will call them **congruent** if there exists a $v \in \mathbf{Z}^d$ so that $S_1 = S_2 + v$. Note that for us d will be 1 or 2. If $B \subset \mathbf{Z}$ and $B = (a, b) \cap \mathbf{Z}$, then we say that B is a **block**. A subset R of $\mathbf{Z}^2$ will be called a **rectangle** if it can be written as $R = B_1 \times B_2$, where the B_i's are blocks; if the blocks are congruent, we call it a **square**. If B_1 has l elements and B_2 has k, we say that R is an $l \times k$ rectangle. The sets $(\min B_1) \times B_2$ and $(\max B_1) \times B_2$ are called the (**left** and **right**, respectively) **vertical sides** of R. The sets $B_1 \times (\min B_2)$ and $B_1 \times (\max B_2)$ will be the (**bottom** and **top**, respectively) **horizontal sides**.
We shall need the notion of **crossing** in a rectangle when preforming i.i.d. bond percolation with retention probability p – indicated by writing $\mathbf{P}_p$ for the probability measure – on it. For such a percolation process on a rectangle R, the event $H(R)$ defined as the set of those subgraphs of R containing a path between the two different vertical sides will be called a **horizontal crossing** in R. The event $V(R)$ which we define by interchanging the words vertical and horizontal above called a **vertical crossing** in R.
Furthermore, we say that a rectangle $Q = A_1 \times A_2$ is **well-joined** to the rectangle $R = B_1 \times B_2$ if *either*

> $A_1 \subset B_1$ and $B_2 \subset A_2$ (in which case we say that their being "well-joined" is of type **vertical to horizontal** or $V \to H$)

or

> $B_1 \subset A_1$ and $A_2 \subset B_2$ (in which case we say that their being "well-joined" is of type **horizontal to vertical** or $H \to V$).

If $R_1, R_2, \ldots, R_m, \ldots$ is a sequence of rectangles, we say that it is well-joined if every pair of consecutive rectangles from the sequence is well-joined and the sequence of their types is alternating (i.e: $\ldots V \to H, H \to V, V \to H, H \to V, \ldots$). The importance of this concept will be the following: if we have a sequence of well-joined rectangles $R_1, R_2, \ldots, R_m, \ldots$ and the first type is (say) $V \to H$ and if all the events

$$V(R_1), H(R_2), V(R_3), H(R_4), \ldots, V(R_{2k-1}), H(R_{2k}), \ldots$$

hold then we can easily extract an infinite path from the individual crossings (given the appropriate vertical or horizontal crossings for the rectangles in the sequence). Moreover, if we know a lower bound for the individual probabilities of the above events, then by the well-known Harris–FKG inequality which states that for i.i.d. percolation any two increasing events are positively correlated (see, e.g., [7, Thm. 2.4]) we can get a lower bound for the probability of an infinite path simply by multiplication. We shall make use of the following result of Bollobás and Riordan [2].

Lemma 3.3. *Fix an integer $\lambda > 1$ and a $p \in (\frac{1}{2}, 1)$. We can then find constants $\gamma = \gamma(\lambda, p) > 0$ and $n_0 = n_0(\lambda, p)$ such that if $n > n_0$, then for each $\lambda n \times n$ rectangle R we have $\mathbf{P}_p(H(R)) > 1 - n^{-\gamma}$.*

From this result, we obtain the following.

Corollary 3.4. *For any fixed $p \in (\frac{1}{2}, 1)$, we can find a $c > 0$, a positive integer n_0 and a $\gamma > 0$ such that for any positive integer L the following holds. If R is an $Ln \times n$ rectangle where $n > n_0$, then $\mathbf{P}_p(H(R)) \geq c^{L/n^{\gamma}}$.*

Proof. Take $\lambda = 3$ in Lemma 3.3 and let γ be the corresponding $\gamma(3, p)$ and n_0 be the corresponding $n_0(3, p)$. Let R be an $Ln \times n$ rectangle. R can be covered by overlapping "little" $3n \times n$

rectangles in such a way that the intersection of a consecutive pair of them is an $n \times n$ square and we can do it in such a way that altogether the number of the $3n \times n$ rectangles and $n \times n$ squares is not greater than L. Notice that if we have horizontal crossings for all the $3n \times n$ rectangles and vertical crossings for all the $n \times n$ squares, then we have a horizontal crossing for the whole $Ln \times n$ rectangle. Then, by the Harris–FKG inequality, we can estimate $\mathbf{P}_p(H(R))$ from below as $(1-n^{-\gamma})^L$. But this quantity equals $((1-n^{-\gamma})^{n^\gamma})^{L/n^\gamma}$, so the corollary follows from the fact that $(1-n^{-\gamma})^{n^\gamma}$ is bounded away from zero. $\square$

3.2 Building blocks

For a finite set $K \subset \mathbf{Z}$, we let $\mathbf{conv}(K)$ denote the smallest block containing K. If C has the form

$$C = \bigcup_{k \in \mathbf{Z}} (B + (l+d)k)$$

where $d > 1$ is some integer, and B is a block with $|B| = l$, then we say that C is a **block progression**, and we refer to l as the **block length** and d as the **block distance** in C. We say that (l, d) is the **parameter** of C. We will refer to the sets $B_k = B + (l+d)k$ as the **blocks** of C. We call a block D a **gap** of C if it is in the complement of C and maximal with that property. Note that in that case $|D| = d$. If L is a positive integer, $T \subset \mathbf{Z}$ and C is a block progression as above, then we say that T is a **block progression over** C **with factor** L if

$$T = \bigcup_{k \in \mathbf{Z}} (D + (l+d)Lk)$$

where D is a gap of C. Let C_1 and C_2 be two congruent block progressions. The blocks of C_i will be denoted as B^i_j where $j \in \mathbf{Z}$. Let

$$V_j = B^1_j \times \mathbf{Z} \text{ and } H_j = \mathbf{Z} \times B^2_j.$$

Then the set $G \subset \mathbf{Z}^2$ defined as

$$G = \left(\bigcup_{k \in \mathbf{Z}} V_k \right) \cup \left(\bigcup_{j \in \mathbf{Z}} H_j \right)$$

will be called the **grid determined by** C_1 and C_2. The **parameter of the grid** above will be the parameter of C_i. If G and H are grids we say that H is a **grid over** G with factor L if, whenever G is determined by C_i and H is determined by T_i for $i \in \{1,2\}$, T_i is a block progression over C_i with factor L.

We now go on to define finite analogues of the above concepts. If B is a block and $C = \bigcup_{k=0}^{q-1}(B + (l+d)k)$, then we say that C is a **block complex**. Next we define the notion of a window. Let C_1 and C_2 be two congruent block complexes. The blocks of C_i will be denoted as B^i_j, where $j \in \{0, 1, \ldots, q-1\}$. Let

$$V_j = B^1_j \times \mathbf{conv}(C_2) \text{ and } H_j = \mathbf{conv}(C_1) \times B^2_j.$$

Then the set $W \subset \mathbf{Z}^2$ defined as

$$W = \left(\bigcup_{k=0}^{q-1} V_k \right) \cup \left(\bigcup_{j=0}^{q-1} H_j \right)$$

will be called the **window determined by** C_1 and C_2.
We shall call the V_i's and H_j's the **frames** of the given window. The convex hull of a window in $\mathbf{R}^2$ is a square whose intersection with $\mathbf{Z}^2$ is the **shade** of the window. If we take the set theoretic complement of the frames in the shade, then the resulting set splits into squares in $\mathbf{Z}^2$. We refer to those squares as the **panes** of the window. For a window W as above let us refer to the corresponding block length (independent of i and j) $|B^i_j|$ as the "frame width" of W, denoted as $\mathbf{w}(W)$. Also $|\mathbf{conv}(C_i)|$ will be called the "side length" of W and we denote it as $\mathbf{s}(W)$.
For a window W as above we define its **fork** as follows: It will be the union of $q-1$ vertical parts and one horizontal part. The vertical parts (we shall call them the **cut-frames** of the fork) are the sets of the form

$$\mathrm{cut}(V_i) := V_i \setminus H_{q-1}$$

where $i \in \{1, \ldots, q-1\}$. That is, we cut off each vertical strip at the top and we throw away the leftmost vertical strip. The horizontal part (which we shall call the **bottom** of the fork) will be

$$H_0 \setminus V_0 .$$

Thus altogether the fork of W is defined as

$$F(W) := \left(\left(\bigcup_{i=1}^{q-1} V_i \right) \cup H_0 \right) \setminus \left(V_0 \cup H_{q-1} \right) .$$

3.3 Preliminaries for $p_{c,bond}(G_{open}) < 1$

If we have two windows W and W^+, we write $W \prec W^+$ to indicate that the shade of W is a pane of W^+ (note that this relation is not transitive). If we have a sequence

$$\mathscr{S} = W_1, \ldots, W_k, \ldots$$

of windows then we write $W_1 \prec W_2 \prec \cdots \prec W_k \prec W_{k+1} \prec \ldots$ to indicate $W_k \prec W_{k+1}$ for each k.
If we have a sequence as above we define the set $\mathtt{ERB}_k(\mathscr{S})$, and when the sequence $\mathscr{S}$ is understood we write simply $\mathtt{ERB}_k$; $\mathtt{ERB}$ stands for "Escape route to the Right and to the Bottom". Note that for $k > 1$, $\mathtt{ERB}_k$ will be the union of two rectangles. First observe that if $W \prec W^+$ holds, then there is a unique vertical frame V^+ of W^+ which is attached to W from the right in the sense that $(W + (1,0)) \cap V^+$ is nonempty. Now consider $W_{k-1} \prec W_k \prec W_{k+1}$. Let V_k^+ be the unique vertical frame attached to W_{k-1} from the right. Then $\mathrm{cut}(V_k^+)$ will be one of the rectangles whose union is $\mathtt{ERB}_k$. To define the other rectangle we take the bottom B_k of $F(W_k)$ and we extend it to the right to get the "extended bottom"

$$E_k := \bigcup_{j=0}^{\mathbf{w}(W_{k+1})} (B_k + (j,0)) .$$

Now let us define

$$\mathtt{ERB}_k := \mathrm{cut}(V_k^+) \cup E_k .$$

Note that

$$\mathrm{cut}(V_k^+) \text{ is a } \mathbf{w}(W_k) \times (\mathbf{s}(W_k) - \mathbf{w}(W_k)) \text{ rectangle} \tag{4}$$

while

$$E_k \text{ is a } (\mathbf{s}(W_k) - \mathbf{w}(W_k) + \mathbf{w}(W_{k+1})) \times \mathbf{w}(W_k) \text{ rectangle.} \tag{5}$$

Now we want to extend this definition to $k = 1$ as well. Note that the definition for E_k can be adapted to the case $k = 1$ with no difficulty. The only thing that we do not have a natural choice for is a cut-frame. We simply define $\mathtt{ERB}_{W_1}$ as $F(W_1) \cup E_1$. Finally we define the **road** $\mathbf{r}(\mathscr{S})$ of the sequence $\mathscr{S}$ above as

$$\mathbf{r}(\mathscr{S}) := \bigcup_{k=1}^{\infty} \mathtt{ERB}_k .$$

The importance of the road is the following. If in the i.i.d. percolation each edge inside $\mathtt{ERB}_1$ remains open and for each $\mathtt{ERB}_k$ for $k > 1$ we have a vertical crossing for the corresponding cut-frame and a horizontal crossing for the corresponding bottom, then for each point of $F(W_1)$ there is an open path to infinity. Note that, besides the exceptional $k = 1$ case, the remaining parts of the road can be considered as a sequence of well-joined rectangles.

3.4 Preliminaries for $p_{c,bond}(G_{closed}) < 1$

If we are in the shade of a window but not in its fork, then we can move to the left top corner of the window by moving always outside of the fork. More specifically, if we have

$$\mathscr{S} = W_1 \prec W_2 \prec \cdots \prec W_k \prec W_{k+1} \prec \ldots$$

then we define the corresponding $\mathtt{ELT}_k(\mathscr{S})$ as follows, $\mathtt{ELT}$ being short for "Escape route to the Left and to the Top". Consider the pair $W_{k-1} \prec W_k$. Take the leftmost vertical frame V^l_{k-1} of W_{k-1} note that this is contained in the complement of $F(W_{k-1})$. Let $\mathrm{ext}(V^l_{k-1})$ be the rectangle maximal for the following properties. It is contained in the shade of W_k, while its "horizontal component" is the same as that of V^l_{k-1} in the sense that if $V^l_{k-1} = A \times B$ and $\mathrm{ext}(V^l_{k-1}) = \hat{A} \times \hat{B}$ then $A = \hat{A}$, and we also have

$$V^l_{k-1} \subseteq \mathrm{ext}(V^l_{k-1})$$

and

$$\mathrm{ext}(V^l_{k-1}) \subseteq (F(W_k) \cup F(W_{k-1}))^C .$$

Also let H^t_k be the topmost horizontal frame of W_k and let

$$\mathtt{ELT}_k := \mathrm{ext}(V^l_{k-1}) \cup H^t_k .$$

For the record, note the size of these two rectangles: $\mathrm{ext}(V^l_{k-1})$ is a $\mathbf{w}(W_{k-1}) \times (\mathbf{s}(W_k) - \mathbf{w}(W_k))$ one, while H^t_k is a $\mathbf{s}(W_k) \times \mathbf{w}(W_k)$ one. If we take a similar union for the $\mathtt{ELT}$'s as we had for the $\mathtt{ERB}$'s then we will have an infinite "road" to infinity moving strictly outside of the forks of the windows in the sequence (but still in the windows).

3.5 The actual construction

The site percolation $\hat{X} \in \{0,1\}^{\mathbf{Z}^2}$ that we are about to define will depend on two initial parameters d_0 and l_0 and a sequence of positive integers $L_1, \ldots, L_k, \ldots$ where the latter sequence "grows fast" in a later-specified way. We choose $d_0 \geq l_0 > n_0$ where n_0 is from Corollary 3.4.

Note that there are only finitely many different translates of a given grid so we can choose uniformly a grid G_0 with parameter (l_0, d_0) among the finitely many congruent copies. If G_k has been defined for a positive integer k, then let G_{k+1} be a uniformly chosen grid over G_k with factor L_{k+1}. If the L_i grow fast enough, then a.s. any $x \in \mathbf{Z}^2$ will be in G_k for only finitely many k.

If the grid G_i has parameter (l_i, d_i), then G_{i+1} will have parameter $(l_{i+1}, d_{i+1}) = (d_i, L_{i+1}l_i + (L_{i+1} - 1)d_i)$. Then for $l_i + d_i$ we have the simple recursion

$$l_{i+1} + d_{i+1} = L_{i+1}(l_i + d_i)$$

which clearly implies

$$l_{i+1} + d_{i+1} = (\prod_{j=1}^{i+1} L_j)(l_0 + d_0)\,. \tag{6}$$

Now color the points x of $\mathbf{Z}^2$ with colors $-1, 0, 1, 2, \ldots, n, \ldots$ as follows: if x is not in any of the G_k, then x gets the color -1, otherwise it gets the largest k for which $x \in G_k$.
It is crucial to make sure that any vertex be in only finitely many of the G's, for which a Borel–Cantelli argument is enough if the L_k's grows fast enough. We now give a sufficient condition for that. Let us estimate the probability that the origin is in G_k (by invariance the same estimate works for any given vertex). If we have a grid H with parameter (l, d) then instead of looking at this as a union of certain vertical and horizontal "infinite rectangles" we can visualize $\mathbf{Z}^2$ as partitioned into a disjoint union of $(l + d) \times (l + d)$ squares and consider the portion H has within each of the squares. These portions will give us the probabilities that a particular point is contained in H. To compute these portions we choose the squares so that their intersection with H is especially simple, namely for each square K from the partition the following holds: $H \cap K$ is the union of two rectangles R_v and R_h (here h, v refers to "horizontal" and "vertical" respectively) so that R_v has type $l \times (l + d)$ and R_h has type $(l + d) \times l$ and R_v is the "leftmost" rectangle of that type contained in K while R_h is the "topmost" one, meaning that neither $(-1, 0) + R_v$ nor $R_h + (0, 1)$ is contained in K. This gives us that $|H \cap K| = ld + l(l + d)$ while obviously $|K| = (l + d)^2$. Then the probability of the origin being in H (if H is uniformly selected as was the case with the G_k's) equals

$$(ld + l(l + d))/(l + d)^2\,.$$

Now let us check what condition on $L_1, L_2, \ldots$ needed to make the Borel–Cantelli argument work. In order to do that consider $H = G_{i+1}$ so the probability of the origin being in H is

$$(l_{i+1}d_{i+1} + l_{i+1}(l_{i+1} + d_{i+1}))/(l_{i+1} + d_{i+1})^2\,,$$

which, with a little bit of arithmetic, becomes

$$2l_{i+1}/(l_{i+1} + d_{i+1}) - l_{i+1}^2/(l_{i+1} + d_{i+1})^2\,.$$

For Borel–Cantelli to work we need that summing these positive numbers over i yields a finite value, and for that it is clearly enough that the sum of $l_{i+1}/(l_{i+1} + d_{i+1})$ converges. To see how it relates to $L_1, L_2, \ldots$ we spell out our recursions again:

$$l_{i+1}/(l_{i+1} + d_{i+1}) = d_i/(L_{i+1}(l_i + d_i)) < 1/L_{i+1}\,.$$

So it is enough to have

$$\sum_{i=1}^{\infty} 1/L_i < \infty\,.$$

After this Borel–Cantelli interlude, we now turn back to the construction. Observe that each color class splits into a disjoint union of windows. Actually a more precise **"structural observation"** is true: A point x of color class k is always contained in a window $W(x)$ each of whose points has the same color with $\mathbf{w}(W(x)) = l_k$ and $\mathbf{s}(W(x)) = d_{k+1}$. Also for this $W(x)$ there exists a $W^+(x)$ each of whose points has color $k+1$ so that $W(x) \prec W^+(x)$. Altogether we find that for an x of color class k we have a sequence $W_1(x) \prec W_2(x) \prec \dots \prec W_j(x) \prec W_{j+1}(x) \prec \dots$ of windows where each point of $W_j(x)$ is of color class $k+j-1$.

The construction is simply to take the forks of all of the windows: our translation invariant site percolation $\hat{X} \in \{0,1\}^{\mathbf{Z}^2}$ arises by assigning value 1 to x precisely for those $x \in \mathbf{Z}^2$ that belong to such a fork. Let us write $\mathtt{RF}$ (short for "Random Forks") for $G_{open}(\hat{X})$ and $\mathtt{RF}^*$ for $G_{closed}(\hat{X})$.

3.6 Nontriviality of the critical values

It remains to show that $p_{c,bond}(\mathtt{RF}) < 1$ and $p_{c,bond}(\mathtt{RF}^*) < 1$; we begin with the former. If $x \in \mathbf{Z}^2$ is in $\mathtt{RF}$ consider $W_1(x) \prec W_2(x) \prec \dots \prec W_j(x) \prec W_{j+1}(x) \prec \dots$ as above. With positive probability x is of color class 1. Moreover, still with positive probability, each edge in $\mathtt{ERB}_1$ remains open. We will condition on this event.

Then we can use the notion of road $\mathbf{r}(x)$ introduced in Section 3.3. Because of the conditioning we just declared, we can focus on the parts in the road which corresponded to indices $k > 1$. Let us apply the strategy we described in Section 3.1 in connection with the notion of being well-joined. We need the side lengths of the rectangles constituting the road, and to substitute the frame widths and side lengths of $\mathtt{RF}$ into the formulas (4) and (5) in Section 3.3. The vertical rectangle at the ith step of the road for x is an $l_i \times (d_{i+1} - l_i)$ one while the next horizontal one is a $(d_{i+1} + l_{i+1} - l_i) \times l_i$ one. Note that from the point of view of crossing an $2d_{i+1} \times l_i$ (horizontal) rectangle and an $l_i \times 2d_{i+1}$ (vertical) is just worse than any of the above so if we find a lower bound for their having the appropriate crossings then that bound works for the original rectangles as well.

Now by using the recursion (6) we obtain estimates for the side lengths:

$$2d_{i+1} > d_{i+1} + l_{i+1} = \left(\prod_{j=1}^{i+1} L_j \right) (l_0 + d_0) > d_{i+1}\,.$$

Note that (simply because $l_{i+2} = d_{i+1}$) we also have

$$2l_{i+2} > \left(\prod_{j=1}^{i+1} L_j \right) (l_0 + d_0)\,,$$

and furthermore

$$2L_{i+1}L_i l_i = 2L_{i+1}L_i d_{i-1} > L_{i+1}L_i(d_{i-1} + l_{i-1}) = d_{i+1} + l_{i+1} > d_{i+1}\,.$$

In other words we have $2L_{i+1}L_i > d_{i+1}/l_i$. Now apply Corollary 3.4 to the above $l_i \times 2d_{i+1}$ rectangle R. Then $4L_{i+1}L_i$ may play the role of L in the corollary, which then tells us that

$$\mathbf{P}_p(V(R)) > c^{4L_{i+1}L_i/l_i^{\gamma}}\,. \tag{7}$$

We next use the fact that the sequence of rectangles defined above (i.e. the "road" we get when we take a vertical strip from the fork and go down to the bottom horizontal one and the move to the vertical strip in the next level and so on...) is well-joined. The estimate (7) together with the Harris–FKG inequality implies that the probability of the sequence containing an infinite path is greater than

$$\prod_{i=2}^{\infty} c^{2(4L_{i+1}L_i/l_i^{\gamma})} \tag{8}$$

where the factor 2 in the power corresponds to taking both the horizontal and vertical rectangles into account at a given step, and the index i going from 2 corresponds to the conditioning declared at the beginning of Section 3.6. The product (8) is positive exactly when

$$\sum_{i=2}^{\infty} L_{i+1}L_i/l_i^{\gamma} < \infty\,. \tag{9}$$

Recall the balance we need to establish: on one hand, the L's need to grow fast enough so that Borel–Cantelli applies to show only finitely many of the events $x \in G_k$ hold, while on the other hand they need to grow slowly enough to make sure that the sum (9) converges. But of course with the given conditions there is *plenty* of room for that because as we saw the l_i is essentially the product of all L_k's up to index i. We can even allow the L's to grow exponentially. Indeed, let $L_i = 2^i$. Then we see that the term corresponding to index $i+2$ of the above sum is $2^{2i+5}/l_{i+2}^{\gamma}$. Now note that

$$2l_{i+2} > (\prod_{j=1}^{i+1} L_j)(l_0+d_0) = 2^{((i+1)(i+2)/2)}(l_0+d_0)\,.$$

We note that $2^{2i+5}/l_{i+2}^{\gamma}$ can be bounded from above as some constant multiplied by $2^{-\alpha i^2+\beta i+\delta}$ (where $\alpha > 0, \beta, \delta \in \mathbf{R}$) whose sum (over i) is clearly convergent. (In fact, we could consider even faster growing L's as long as we make sure that the product of the first some terms should be much bigger than the next two terms.)

This justifies our claim that $p_{c,bond}(\mathtt{RF}) < 1$ for $\mathtt{RF}$, and it remains only to establish the analogous claim $p_{c,bond}(\mathtt{RF}^*) < 1$. For that purpose we do a computation very similar to the above one but now applied to the road defined by the $\mathtt{ELT}$'s. Note that the sizes of the vertical and horizontal rectangles in $\mathtt{ELT}_k$ are $d_{k+1}\times l_k$ for the horizontal one and $l_{k-1}\times(d_{k+1}-l_k)$ for the vertical one, and furthermore that in this case both crossing probabilities for the above considered two rectangles is not less then the horizontal crossing probability for a $d_{k+1}\times l_{k-1}$ one.

First we need an estimate for the ratio d_{i+1}/l_{i-1}. We use again the basic recursion for the $(l+d)$'s we had at the "structural observation":

$$2L_{i+1}L_iL_{i-1}l_{i-1} = 2L_{i+1}L_iL_{i-1}d_{i-2} > L_{i+1}L_iL_{i-1}(d_{i-2}+l_{i-2}) = d_{i+1}+l_{i+1} > d_{i+1}\,.$$

So now the quantity $2L_{i+1}L_iL_{i-1}$ can play the role of L from Corollary 3.4. So we need

$$\sum_{i=1}^{\infty} L_{i+2}L_{i+1}L_i/l_i^{\gamma} < \infty\,. \tag{10}$$

Now if we make the same kinds of estimates as for $\mathtt{RF}$, we see that the i'th term in this case will be $2^{3i+3}/l_i^{\gamma}$. So in the numerator we still have an exponent linear in i, while in the denominator we have an exponent of second order, so the sum in (10) is indeed finite, and the proof of Theorem 3.2 is complete.

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