Elect. Comm. in Probab. **8** (2003)170–178

ELECTRONIC COMMUNICATIONS in PROBABILITY

MICROSCOPIC STRUCTURE OF A DECREASING SHOCK FOR THE ASYMMETRIC K-STEP EXCLUSION PROCESS

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Submitted 10 September 2003, *accepted in final form* 28 November 2003

AMS 2000 Subject classification: Primary 60K35; Secondary 82C22
Keywords: asymmetric k-step exclusion; Non-convex or non-concave flux; microcopic shock ; rightmost particle.

Abstract

The asymmetric k-step exclusion processes are the simplest interacting particle systems whose hydrodynamic equation may exhibit both increasing and decreasing entropic shocks under Euler scaling. We prove that, under Riemann initial condition with right density zero and adequate left density, the rightmost particle identifies microscopically the decreasing shock.

1 Introduction

The asymmetric k-step exclusion process is a conservative attractive process on $\mathbf{X} = \{0,1\}^{\mathbb{Z}}$ that generalizes simple exclusion (general class of processes of this type was introduced in [G]). The hydrodynamic behavior of these processes were studied in [BGRS] (for a specific review see equally [FGRS]). One of the interesting features of these processes is that their macroscopic flux function is neither convex nor concave leading to both increasing and decreasing entropic shock solutions of the hydrodynamic equation. In this note we investigate the microscopic counterpart of a decreasing shock solution in the asymmetric nearest neighbor case, under Riemann initial condition with right density zero.

Indeed, remember that the nearest neighbor simple exclusion process with an asymmetry to the right has a concave flux function, and its hydrodynamic equation can exhibit (only) an increasing shock (see for instance [KL], chapters VIII and IX). The microscopic structure of this shock was analyzed in a series of papers by P. Ferrari *et al.* (the first ones were [FKS]

and [F]; see [L99] for a unified presentation and a complete reference list), and in more general settings in [R] and [S]. These authors proved that the shock was characterized by the evolution of a second class particle, which moved at the shock speed, and followed the characteristic lines and shocks of the hydrodynamic equation; moreover, under Riemann initial condition with densities λ (resp. ρ) to the left (resp. right) of the origin, the process seen by this second class particle possessed an invariant measure with asymptotic densities λ (resp. ρ) to the left (resp. right) of the origin. Unfortunately, we cannot adapt the techniques developed in those papers to k-step exclusion, because on the one hand jumps are not restricted to *stricto sensu* nearest-neighbor sites, and on the other hand both [R] and [S] rely on the concavity of the flux function. We point out that following along the same lines as we do here for k-step exclusion one can obtain the microscopic structure of (the increasing) shock for finite range non-nearest neighbor asymmetric exclusion process with $(0,\rho)$ initial profile.
We consider an asymmetric (probability p - resp. q - to jump to the right - resp. left -) starting with an initial measure $\mu_{\lambda,0}$: *i.e.* a product measure with density λ to the left of (and at) the origin and 0 to its right. Our candidate for a microscopic object which identifies the shock is the rightmost particle (cf. [DKPS] where the asymmetric simple exclusion process was studied in the case of an increasing shock, with left density 0; there, the leftmost particle identified the shock). We prove that the rightmost particle evolves at speed v_{shock}, and that the process seen by this particle has an invariant measure with asymptotic density λ to the left of the origin. We illustrate our method for the totally asymmetric 2-step exclusion process. When $\lambda \in (0, 1/4)$ this corresponds to an initial shock profile for the hydrodynamic equation. The shock (discontinuity) at zero propagates at a speed $v_{shock} = (p-q)(1+\lambda-2\lambda^2)$ (see e.g. [FGRS]). Comments will be made to show how to extend the result to the asymmetric nearest neighbor case.
We present our results in Section 2, and prove them in Section 3.

Remark: For the k-step asymmetric case with $k > 2$, the flux function starts out being convex, the shock speed and the allowed range of densities for a decreasing entropic shock are determined by the convex envelope of the initial part of the flux function. However our argument remains valid within suitable changes.

2 Notation and results

We denote by $S(t)$ the evolution semi-group of the asymmetric two-step exclusion process $(\eta_t)_{t\geq 0}$ on $\mathbf{X} = \{0,1\}^{\mathbb{Z}}$ with generator L given by

$$\begin{aligned} Lf(\eta) &= p\sum_{x\in\mathbb{Z}} \eta(x)[1-\eta(x+1)]\left(f(\eta^{x,x+1}) - f(\eta)\right) \\ &\quad + q\sum_{x\in\mathbb{Z}} \eta(x)[1-\eta(x-1)]\left(f(\eta^{x,x-1}) - f(\eta)\right) \\ &\quad + p^2\sum_{x\in\mathbb{Z}} \eta(x)\eta(x+1)[1-\eta(x+2)]\left(f(\eta^{x,x+1,x+2}) - f(\eta)\right) \\ &\quad + q^2\sum_{x\in\mathbb{Z}} \eta(x)\eta(x-1)[1-\eta(x-2)]\left(f(\eta^{x,x-1,x-2}) - f(\eta)\right) \end{aligned}$$

on all bounded cylinder functions f on $\mathbf{X}$, where $p = 1-q \in [0,1]\setminus\{1/2\}$, $\eta^{x,y}$ is the configuration η where the states of sites x and y have been interchanged and $\eta^{x,y,z}$ is the configuration η where the states of sites x, y and z have been shifted; *i.e.* $\eta^{x,y}(z) = \eta(z)$ when $z \neq x, y$; $\eta^{x,y}(x) = \eta(y)$; $\eta^{x,y}(y) = \eta(x)$, and $\eta^{x,y,z}(w) = \eta(w)$ when $w \neq x, y, z$; $\eta^{x,y,z}(y) = \eta(x)$; $\eta^{x,y,z}(z) = \eta(y)$; $\eta^{x,y,z}(x) = \eta(z)$. Notice that we chose a 'pushing interpretation' (a particle may jump to its neighboring site pushing eventually a particle that could stand there provided the next neighboring site has a vacancy) of the evolution, so that particles always keep the same respective order.
Like the simple exclusion process, the k-step exclusion process is attractive (with respect to the usual order on configurations, *i.e.* for $\eta_1, \eta_2 \in \mathbf{X}$, $\eta_1 \leq \eta_2$ means that $\eta_1(x) \leq \eta_2(x)$ for all $x \in \mathbb{Z}$), and has a one parameter family $\{\nu_\alpha, \alpha \in [0,1]\}$ of extremal invariant and translation invariant measures, where ν_α is the Bernoulli product measure on $\mathbf{X}$ with density $\alpha \in [0,1]$, *i.e.* with marginal $\nu_\alpha(\eta(x) = 1) = \alpha$ for all $x \in \mathbb{Z}$.

In the sequel we set $p = 1$ (total asymmetry); appropriate comments will be made for the $1/2 < p < 1$ case (the case $0 \leq p < 1/2$ being symmetric).

Let $\widehat{\mathbf{X}} = \{\eta \in \mathbf{X} : \eta(0) = 1\}$. The two-step exclusion process as seen from a pushing tagged particle starting at zero evolves on $\widehat{\mathbf{X}}$ according to the semi group $\widehat{S}(t)$ with generator $\widehat{L} = \widehat{L}_0 + \widehat{L}_1$ which acts on all bounded cylinder functions f on $\{0,1\}^{\mathbb{Z}}$ as follows:

$$
\begin{aligned}
\widehat{L}_0 f(\eta) &= \sum_{x\neq 0,-1} \eta(x)[1-\eta(x+1)]\left(f(\eta^{x,x+1}) - f(\eta)\right) \\
&\quad + \sum_{x\neq 0,-1,-2} \eta(x)\eta(x+1)[1-\eta(x+2)]\left(f(\eta^{x,x+1,x+2}) - f(\eta)\right) \\
\widehat{L}_1 f(\eta) &= [1-\eta(1)]\left(f(\tau_1\eta^{0,1}) - f(\eta)\right) \\
&\quad + \eta(-1)[1-\eta(1)]\left(f(\tau_1\eta^{-1,0,1}) - f(\eta)\right) \\
&\quad + \eta(1)[1-\eta(2)]\left(f(\tau_1\eta^{0,1,2}) - f(\eta)\right)
\end{aligned}
$$

where τ denotes the shift operator, *i.e.* for $n \in \mathbb{Z}$, $\tau_n(\eta)(x) = \eta(x+n)$, for all $\eta \in \mathbf{X}, x \in \mathbb{Z}$. By [BGRS] Theorem 5.1 the Palm measure $\nu_\alpha(\cdot|\eta(0) = 1) = \widehat{\nu}_\alpha$ is invariant for this process.

In this note we consider the totally asymmetric 2-step exclusion process with initial measure $\mu_{\lambda,0}$. Due to the pushing interpretation of the dynamics, it has a rightmost particle, of initial position $Z_0 = Z(\eta)$, whose distribution G is geometric of mean $(1/\lambda) - 1$, and of position $S(t)Z = Z_t$ at time t. The 2-step exclusion induces a process seen by the rightmost particle, $(\widehat{\eta}_t)_{t\geq 0} = (\tau_{Z_t}\eta_t)_{t\geq 0}$, supported on $\widehat{\mathbf{X}}' = \{\eta \in \widehat{\mathbf{X}} : \eta(x) = 0 \text{ if } x > 0\}$, with initial measure $\widetilde{\mu}_{\lambda,0}$. A configuration $\widehat{\eta}$ in $\widehat{\mathbf{X}}'$ is obtained from a configuration η on $\mathbf{X}$ distributed according to $\mu_{\lambda,0}$ by shifting it so that the rightmost particle is at the origin: $\widehat{\eta} = \tau_{Z(\eta)}\eta$. Under $\widetilde{\mu}_{\lambda,0}$, $\{\eta(x) : x < 0\}$ is distributed according to a product measure with density λ. Note that the process $(\widehat{\eta}_t)_{t\geq 0}$ has semi-group $\widehat{S}(t)$ and generator $\widehat{L}$, where the last term in the definition of $\widehat{L}$ is equal to zero since the process is supported on configurations with $\widehat{\eta}(1) = 0$, and for the same reason in the two previous terms $1 - \widehat{\eta}(1) = 1$. We also observe that $\mu_{\lambda,0}\tau_{Z_t}S(t) = \widetilde{\mu}_{\lambda,0}\widehat{S}(t)$.

For $s \geq 0$, define $\widetilde{\jmath}_{0,-1} : \widehat{\mathbf{X}'} \to \mathbb{N}$ as $\widetilde{\jmath}_{0,-1}(\widehat{\eta}_s) = 1 + \widehat{\eta}_s(-1)$. It is the flux of holes crossing the bond between 0 and -1 at time s. This is also the rate at which the rightmost particle jumps right at time s. Indeed, Z_s is the sum of Z_0, and net change in the position of the rightmost particle in time s. This net change can then be obtained as a functional of the process $(\widehat{\mathbf{X}}, \widetilde{\mu}_{\lambda,0}, \widehat{S}(t))$:

$$E_{\mu_{\lambda,0}}(Z_t) = E_G(Z_0) + \int_0^t E_{\widetilde{\mu}_{\lambda,0}}\left(\widetilde{\jmath}_{0,-1}(\widehat{\eta}_u)\right) du. \tag{1}$$

In the next section we will prove, using the previous notation,

Theorem 2.1

$$\lim_{t\to\infty} \frac{Z_t}{t} = v_{shock}$$

in L^1 with respect to $P_{\mu_{\lambda,0}}$.

Since the set of all probability measures on the compact set $\widehat{\mathbf{X}}$ is compact, there exists an increasing sequence of times $(t_n)_{n\geq 0}, t_n \to \infty$, such that,

$$\lim_{n\to\infty} \frac{1}{t_n} \int_0^{t_n} \widetilde{\mu}_{\lambda,0}\widehat{S}(t)\ dt = \widetilde{\mu}, \tag{2}$$

a stationary measure for the $(\widehat{\eta}_t)_{t\geq 0}$ process (see [L85], Proposition I.1.8 (e)). As a consequence of Theorem 2.1, we obtain that $\widetilde{\mu}$ (which has density 0 to the right of the origin) is asymptotically equal to (in the Cesáro sense) ν_λ far to the left from the origin: Let $\widetilde{\mu}'$ be any invariant measure for a Markov process with semigroup $\widehat{S}(t)$ starting from an initial measure $\widetilde{\mu}_{\lambda,0}$. Then

Corollary 2.1 *If $\{n_k\}_{k\in\mathbb{N}}$ is a subsequence of $\mathbb{N}$ with $n_k \to \infty$ when $k \to \infty$ such that*

$$\lim_{k\to\infty} \frac{1}{n_k} \sum_{i=1}^{n_k} \widetilde{\mu}'\tau_{-i} = \gamma.$$

then $\gamma = \nu_\lambda$.

3 Proofs

The k-step exclusion process is attractive, that is coordinatewise partial order between configurations is preserved by the k-step evolution. The process seen by the rightmost particle does not have this property. On the other hand it preserves a partial order between configurations which compares the number of holes between successive particles appropriately. We now introduce this partial order on configurations in $\mathbf{X}$ which will play a crucial role in the proof of Theorem 2.1.

We consider $\eta \in \widehat{\mathbf{X}}'' \subset \widehat{\mathbf{X}}$, which either have infinitely many particles to the right and left of the origin, or infinitely many particles to the left of the origin and no particles to the right of the origin. We label particles as follows: If there are infinitely many particles to the right

as well as to the left of the origin, particles are labelled by their natural ordering on the line with $X_0(\eta) = 0$. Let $\gamma_i(\eta)$ be the number of holes between $i+1$-st and i-th particle, *i.e.* $\gamma_i(\eta) = X_{i+1}(\eta) - X_i(\eta) - 1$. If there are no particles to the right of the origin then we let $\gamma_0(\eta) = +\infty$ and $X_n(\eta) = \gamma_n(\eta) = \infty$ for all $n \geq 1$. It is easy to show that γ_i is a continuous function of η at all η such that $\gamma_i(\eta) < \infty$.
Given $\eta_1, \eta_2 \in \widehat{\mathbf{X}}''$,
(a) if η_1 and η_2 have infinitely many particles to the right and left of origin,

$$\eta_1 \preceq \eta_2 \quad \text{if and only if} \quad \gamma_i(\eta_1) \leq \gamma_i(\eta_2), \qquad \text{for all } i \in \mathbb{Z};$$

(b) if $X_j(\eta_2) = \infty$ for all $j \geq 1$, and η_1 has infinitely many particles to the right and left of origin,

$$\eta_1 \preceq \eta_2 \quad \text{if and only if} \quad \gamma_i(\eta_1) \leq \gamma_i(\eta_2), \qquad \text{for all } i \leq -1;$$

(c) if $X_j(\eta_1) = X_j(\eta_2) = \infty$ for all $j \geq 1$,

$$\eta_1 \preceq \eta_2 \quad \text{if and only if} \quad \gamma_i(\eta_1) \leq \gamma_i(\eta_2), \qquad \text{for all } i \leq -1.$$

This order extends to probability measures: We denote by $\mathcal{M}$ the set of bounded monotone (w.r.t. $\preceq$) functions on $\widehat{\mathbf{X}}''$. Then, since the distribution of $\{\eta(x), x < 0\}$ under $\widetilde{\mu}_{\lambda,0}$ is product with density λ, we have

$$\widetilde{\mu}_{\lambda,0} \succeq \widehat{\nu}_\lambda \tag{3}$$

which means that, for any increasing $f \in \mathcal{M}$, $\int f \, d\widetilde{\mu}_{\lambda,0} \geq \int f \, d\widehat{\nu}_\lambda$. Moreover, if $\eta_1, \eta_2 \in \widehat{\mathbf{X}}''$ and $\eta_1 \preceq \eta_2$ then $\widehat{L}(\gamma_i(\eta_1)) \leq \widehat{L}(\gamma_i(\eta_2))$, for all relevant i. It follows that if $f \in \mathcal{M}$ is increasing on $\widehat{\mathbf{X}''}$ then so is $\widehat{S}(t)f$ for all $t > 0$ since

1) $\widehat{S}(t)$ is defined on $\widehat{\mathbf{X}}$ so that all configurations have a particle at the origin which remains at the origin because of the tagged particle evolution of $\widehat{S}(t)$.
2) When one compares two configurations (from cases (b) and (c) in the definition of $\preceq$) the fact that there is always a particle at the origin implies that the labelling of the γ's are unchanged by the evolution for both configurations.

In other words, $\widehat{S}(t)$ is an attractive semi-group with respect to the partial order $\preceq$ we have introduced, and using (3), $\widetilde{\mu}_{\lambda,0}\widehat{S}(t) \succeq \widehat{\nu}_\lambda$ for all $t \geq 0$, so that by (2),

$$\widetilde{\mu} \succeq \widehat{\nu}_\lambda.$$

Remark:
The attractivity of $\widehat{S}(t)$ can also be seen by using a particle to particle coupling described as follows: Let us denote by η_t and ξ_t the processes starting with initial measures $\widetilde{\mu}_{\lambda,0}$ and $\widehat{\nu}_\lambda$ respectively. We couple the two processes by requiring that the particles in η_t and ξ_t with the same labels $i \in \mathbb{Z}$ use the same clock for jumps if $X_i(\eta_t) < \infty$ and $X_i(\xi_t) < \infty$.
Even though we have sketched the attractivity argument for the totally asymmetric case we point out that it can be extended to the asymmetric case straightforwardly.

Proof of Theorem 2.1.
Step 1. We first prove that

$$\lim_{t\to\infty} E_{\mu_{\lambda,0}} \left(\frac{Z_t}{t} \right) \geq v_{shock} \tag{4}$$

by contradiction. Suppose that $\lim_{t\to\infty} E_{\mu_{\lambda,0}}(Z_t/t) < v_{shock}$. Then there exists an $\epsilon > 0$ and an increasing subsequence $(t'_n)_{n\geq 0}$ of $(t_n)_{n\geq 0}$, $t'_n \to \infty$, (cf. (2)) such that

$$\lim_{n\to\infty} \mu_{\lambda,0}\left(\frac{Z_{t'_n}}{t'_n} - v_{shock} < -\epsilon\right) > 0.$$

Let $H(u)$ be a positive real valued continuous function on $\mathbb{R}$ with support in $(v_{shock} - 2\epsilon/3, v_{shock} - \epsilon/3)$, with $\int H(u)du > \delta > 0$. Then

$$\liminf_{n\to\infty} \mu_{\lambda,0} S(t'_n)\left(\left|\frac{1}{t'_n}\sum_{x\in\mathbb{Z}} H\left(\frac{x}{t'_n}\right)\eta(x) - \int \lambda H(u)du\right| > \lambda\delta\right) > 0. \tag{5}$$

On the other hand, from local equilibrium (see [BGRS], Theorem 2.1), there is weak convergence

$$\mu_{\lambda,0}\tau_{(ut'_n)} S(t'_n) \Rightarrow \nu_{\rho(u,1)}$$

where $\rho(x,t) = \rho(x/t, 1)$ is the self similar weak entropic solution of

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho + \rho^2 - 2\rho^3) = 0\,;$$

with initial condition $\rho(x,0) = \lambda$ if $x \leq 0$; 0 otherwise. This implies (see [KL], Proposition III.0.4) that $\mu_{\lambda,0}S(t'_n)$ is a weak equilibrium profile of density $\rho(u,1)$. That is

$$\limsup_{n\to\infty} \mu_{\lambda,0} S(t'_n)\left(\left|\frac{1}{t'_n}\sum_{x\in\mathbb{Z}} G\left(\frac{x}{t'_n}\right)\eta(x) - \int \rho(u,1)G(u)du\right| > \delta'\right) = 0.$$

for any bounded continuous function G on $\mathbb{R}$ and any $\delta' > 0$.
Applying this to H, and using the fact that $\rho(u,1) = \lambda$ for all $u < v_{shock}$ we get a contradiction with equation (5). This yields (4).
Moreover, combining (1), (2) and (4) we obtain

$$\begin{aligned} v_{shock} &\leq \lim_{n\to\infty} E_{\mu_{\lambda,0}}\left(\frac{Z_{t_n}}{t_n}\right) \\ &= \lim_{t\to\infty} E_{\mu_{\lambda,0}}\left(\frac{Z_t}{t}\right) \\ &= E_{\widetilde{\mu}}(\widetilde{\jmath}_{0,-1}) \end{aligned} \tag{6}$$

Step 2. In this step we are interested in understanding the behavior of $\widetilde{\mu}$ on the left tail sigma algebra of $\mathbf{X}$. The sequence of measures $(N^{-1}\sum_{n=1}^{N}\widetilde{\mu}\tau_{-n})_{N>0} := (\widetilde{\mu}_N)_{N>0}$ has a convergent subsequence $(\widetilde{\mu}_{N_k})_{k>0}$ by compactness. Let $\widetilde{\mu}_\infty = \lim_{k\to\infty}\widetilde{\mu}_{N_k}$. Notice that $\widetilde{\mu}_\infty$ is a translation invariant measure by definition.
Let f be any cylinder function on $\mathbf{X}$, then there exists an $m(f) = m \in \mathbb{Z}^+$ such that the cylinder function $\tau_{-m}f$ depends only on $\{\eta(x) : x < 0\}$.
For all $\eta \in \widehat{\mathbf{X}}$ and $n \geq 0$, define $f^n(\eta) := \tau_{-n}f(\eta) = f(\tau_{-n}\eta)$.

Since $\widetilde{\mu}$ is an invariant measure for $\widehat{S}(t)$ we have

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=1}^{N_k} \widehat{L}(f^n)\; d\widetilde{\mu} \\ &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=m+2}^{N_k} \widehat{L}(f^n)\; d\widetilde{\mu} \\ &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=m+2}^{N_k} (\widehat{L}_0 + \widehat{L}_1)(f^n)\; d\widetilde{\mu}\,. \end{aligned}$$

Because for all $n > m+2$, $f^n(\tau_1\eta^{0,1}) = f^n(\tau_1\eta^{-1,0,1}) = f^n(\tau_1\eta^{0,1,2}) = f^{n-1}(\eta)$, we have

$$\begin{aligned} & \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=m+2}^{N_k} \widehat{L}_1(f^n)\; d\widetilde{\mu} \\ = & \lim_{k\to\infty} \int \frac{1}{N_k} \sum_{n=m+2}^{N_k} [1-\eta(1)] \left(f^n(\tau_1\eta^{0,1}) - f^n(\eta)\right)\; d\widetilde{\mu} \\ & + \lim_{k\to\infty} \int \frac{1}{N_k} \sum_{n=m+2}^{N_k} \eta(-1)[1-\eta(1)] \left(f^n(\tau_1\eta^{-1,0,1}) - f^n(\eta)\right)\; d\widetilde{\mu} \\ & + \lim_{k\to\infty} \int \frac{1}{N_k} \sum_{n=m+2}^{N_k} \eta(1)[1-\eta(2)] \left(f^n(\tau_1\eta^{0,1,2}) - f^n(\eta)\right)\; d\widetilde{\mu} \\ = & \; 0 \end{aligned}$$

and as $\widehat{L}_0(f^n) = L(f^n)$, for $n \geq m+2$

$$\begin{aligned} 0 &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=m+2}^{N_k} \widehat{L}(f^n)\; d\widetilde{\mu} \\ &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=m+2}^{N_k} L(f^n)\; d\widetilde{\mu} \\ &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=1}^{N_k} L(\tau_{-n}f)\; d\widetilde{\mu} \\ &= \lim_{k\to\infty} \frac{1}{N_k} \int \sum_{n=1}^{N_k} \tau_{-n}Lf\; d\widetilde{\mu} \\ &= \int Lf\; d\widetilde{\mu}_\infty \end{aligned}$$

where we have used the commutativity of L and τ in next to the last step. This proves that $\widetilde{\mu}_\infty$ is an invariant measure for the semi-group $S(t)$. Since $\widetilde{\mu}_\infty$ is a translation invariant measure by definition we have that $\widetilde{\mu}_\infty$ is a convex combination of product measures (see [G] Theorem 1.3 which is a slight adaptation of the corresponding result for simple exclusion, see [L85], Theorem VIII.3.9 (a)). That is $\widetilde{\mu}_\infty = \int_0^1 \nu_\alpha\; d\pi(\alpha)$ where π is a measure on $[0,1]$. Now we want to show that $\pi((\lambda, 1]) = 0$. Let $\eta \in \widehat{\mathbf{X}}''$. Recall that for $i > 0, X_{-i}(\eta)$ denotes the location of the i-th particle in η to the left of the origin. For all $n < 0$ define $l_n(\eta) = \max\{i \geq 0 : X_{-i}(\eta) \geq n\}$. The random variable $l_n(\eta)$ counts the number of particles in η which are in $[n, 0] \cap \mathbb{Z}$. Now since $\widetilde{\mu} \succeq \widehat{\nu}_\lambda$, there exists a coupling measure $\overline{\mu}$ on $\{(\eta, \xi) \in \widehat{\mathbf{X}}'' \times \widehat{\mathbf{X}}''\}$ of marginals $\widetilde{\mu}$ and $\widehat{\nu}_\lambda$, with $\overline{\mu}(\{\gamma_i(\eta) \geq \gamma_i(\xi) : i < 0\}) = 1$. From this it follows that $l_n(\eta) \leq l_n(\xi)$ for all $n < 0$, $\overline{\mu}$ almost surely. Define $A = \{\eta \in \mathbf{X} : \liminf_{k\to\infty} N_k^{-1} \sum_{j=1}^{N_k} \eta(-j) > \lambda\}$. Let $f(\eta) = \eta(-1)$. Then A is measurable with respect to the left tail sigma algebra of $\{\eta(i) : i \in \mathbb{Z}\}$ and $\tau_{-j}A = A$ for all $j \in \mathbb{N}$. We have for all $k \geq 1$

$$\frac{1}{N_k + 1} \sum_{n=0}^{N_k} \tau_{-n} f(\eta) \leq \frac{1}{N_k + 1} \sum_{n=0}^{N_k} \tau_{-n} f(\xi)$$

$\overline{\mu}$ almost surely. Therefore

$$\frac{1}{N_k + 1} \sum_{n=0}^{N_k} \mathbf{1}_{\tau_{-n}A}(\eta) = \mathbf{1}_A(\eta) \leq \mathbf{1}_A(\xi)$$

$\overline{\mu}$ almost surely. Taking expectations and limit in k we get

$$\widetilde{\mu}_\infty(A) \leq \widehat{\nu}_\lambda(A).$$

Since $\widehat{\nu}_\lambda(A) = 0$ we have that $\pi((\lambda, 1]) = 0$ and we conclude that $\widetilde{\mu}_\infty$ is a convex combination of product measures with density at most λ.

Now define as $\widetilde{\jmath}_{i,i-1}(\eta)$, $i < 0$, the flux of holes jumping across the $-i$-th particle to the left of the origin for the $(\widehat{\eta}_t)_{t\geq 0}$ process

$$\widetilde{\jmath}_{i,i-1}(\eta) = \mathbf{1}_{\{\gamma_i(\eta)>0\}}(\mathbf{1}_{\{\gamma_{i-1}(\eta)>0\}} + 2\mathbf{1}_{\{\gamma_{i-1}(\eta)=0\}}) + \mathbf{1}_{\{\gamma_i(\eta)=0\}}\mathbf{1}_{\{\gamma_{i+1}(\eta)>0\}}$$

(we point out that the 2 in the second term in parenthesis comes from the fact that a hole in front of the $-i$-th particle can jump in between $-i$-th and $-i-1$-st particle or behind $-i-1$-st particle at the same rate). By an elementary computation, $E_{\nu_\lambda}(\widetilde{\jmath}_{i,i-1}(\eta)) = 1 + \lambda - 2\lambda^2 = v_{shock}$, for all $i < 0$.

Since $\widetilde{\mu}$ is an invariant measure for $\widehat{S}(t)$ and $\widetilde{\jmath}_{i,i-1} - \widetilde{\jmath}_{i-1,i-2} = \widehat{L}(\gamma_{i-1})$, we have $E_{\widetilde{\mu}}(\widetilde{\jmath}_{0,-1}) = E_{\widetilde{\mu}}(\widetilde{\jmath}_{n,n-1})$ for all $n < 0$. This implies that

$$\begin{aligned}
\int \widetilde{\jmath}_{0,-1}\, d\widetilde{\mu} &= \lim_{k\to\infty} \frac{1}{N_k} \int \Big(\sum_{n=1}^{N_k} (\tau_{-n})\widetilde{\jmath}_{-1,-2}\Big)\, d\widetilde{\mu} \\
&= E_{\widetilde{\mu}_\infty}(\widetilde{\jmath}_{-1,-2}) \\
&= \int_0^\lambda E_{\nu_\alpha}(\widetilde{\jmath}_{-1,-2})\, d\pi(\alpha) \\
&\leq 1 + \lambda - 2\lambda^2 = v_{shock}. \qquad (7)
\end{aligned}$$

We have used the fact that the shock speed is a monotone increasing function of the particle density α if $\alpha \in (0, 1/4)$ in the last line. Combining (7) and (6) we conclude that $E_{\widetilde{\mu}}(\widetilde{\jmath}_{0,-1}) = v_{shock} = \lim_{t\to\infty} E_{\mu_{\lambda,0}}(Z_t/t)$ thus proving Theorem 2.1. Notice that if $\pi([0, \lambda)) > 0$ then the inequality in (7) would be strict, contradicting (6). Therefore $\pi(.)$ is the Dirac measure concentrated on λ:

$$\widetilde{\mu}_\infty = \nu_\lambda\,. \qquad \square$$

Proof of Corollary 2.1.
Since we obtained the result of the corollary for $\widetilde{\mu}$ in the proof of the Theorem, and the assumptions on $\widetilde{\mu}$ that we needed are satisfied for any $\widetilde{\mu}'$ considered in the corollary, the result follows from the previous proof. □

Acknowledgements. We thank S.R.S. Varadhan for illuminating discussions; Pablo Ferrari for interesting discussions and pointing out the extension to non-nearest neighbor exclusion process; Tom Mountford for valuable discussions. We thank the referee for a careful reading of a previous version of the paper and for helpful suggestions. H.G. and K.R. were partially supported by Université de Rouen. H.G. thanks IME-USP, K.R. and E.S. thank Institut Henri Poincaré, where this work was begun during the semester *Limites hydrodynamiques*, and E.S. thanks IMA-EPFL for hospitality.

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