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WIENER SOCCER AND ITS GENERALIZATION

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Abstract
The trajectory of the ball in a soccer game is modelled by the Brownian motion on a cylinder, subject to elastic reflections at the boundary points (as proposed in [KPY]). The score is then the number of windings of the trajectory around the cylinder. We consider a generalization of this model to higher genus, prove asymptotic normality of the score and derive the covariance matrix. Further, we investigate the inverse problem: to what extent the underlying geometry can be reconstructed from the asymptotic score.

0 Introduction

0.1 Let $D \subset \mathbb{R}^2$ be the soccer field, i.e. a rectangular area with two segments I_1, I_2 marked on two opposite sides of it, of equal length and centered at the middle of the sides. The movement of the ball is described by the trajectory of the plane Brownian motion subject to reflections at the boundary points (we neglect the event of hitting the corner as being of zero measure).
This model was proposed by Kozlov, Pitman and Yor [KPY] as a realistic description of the soccer game. They, however, investigated a far less realistic model with the field described by a half-plane $\Re z > 0$ with goalposts on the real axis.
The score, according to [3], is defined as follows. Gluing together two copies of D along the segments $I_{1,2}$ one obtains a (topological) cylinder $\bar{D}$. Choose a system of short paths on $\bar{D}$ (which associates to each point a path joining it to a base point p_0 on the cylinder so that the lengths of the paths are bounded). If $\gamma : [0, T] \to D$ is a trajectory of the ball, then one can lift it to $\bar{D}$ and close it by joining its endpoints to p_0. The score $\mathrm{Sc}(T)$ is the winding of the resulting closed curve around the cylinder. In other words, score represents an element of $H_1(\bar{D}, \mathbb{Z}) \cong \mathbb{Z}$. We are interested in the asymptotic behavior of the score. It is clear that the ambiguity of the short paths system contributes just a bounded term to the score.
0.2 The main result of [3] is that the score normed by $\sqrt{\log T}$ has asymptotically symmetric exponential distribution with the variance proportional to the ratio of certain elliptic integrals associated with the goalposts positions on the real line. Regarding the realistic model described

in the beginning, the authors noted that the score in this case is asymptotically Gaussian with variance growing as T (which follows quite easily from Kozlov's results [3]) but refer to difficulties of finding the expression for variance.
The log in this theorem is typical for wandering on the complex plane (see [6] for example) and reflects the growth of the local time at zero of the Bessel process of order 2. The score in this model is the normal one with the variance exponentially distributed.

0.3 In this paper we will address the following model generalising the Wiener soccer. Let $D \subset \mathbb{R}^2$ be a topological disk bounded by a Jordan curve ∂D, g a smooth Riemannian metric defined in an open neighborhood of D and $\{I_i\}_{i=0,\ldots,N}$ a collection of nonintersecting (closed) arcs in ∂D. Gluing two copies of D along the arcs I_i one obtains a topological 2-sphere $\bar{D}$ with $N+1$ holes. We represent the first homology group of $\bar{D}$ as the factorspace $H = \mathbb{Z}^{N+1}/(e)$ where $e = (1, \ldots, 1)$. This space is generated by the cycles corresponding to the coherently oriented circumferences of the $(N+1)$ holes of $\bar{D}$. Fix once and forever a system of short paths in $\bar{D}$. This given, one can associate to any continuous curve $\gamma : [0,T] \to \bar{D}$ an element of $H_1(\bar{D}, \mathbb{Z}) \cong \mathbb{Z}^N$ defined up to a bounded term depending on the short paths system.
Let $\gamma(t), t \in [0,T]$ be the Brownian motion on $\bar{D}$ associated to the metric g and subject to elastic reflections at $\partial\bar{D}$ (we address the question how to define precisely the Brownian motion in $\bar{D}$ in the next section).

Definition 0.1 *We denote by $Sc(T)$ the random element of $H_1(\bar{D}, \mathbb{Z})$ obtained by joining the ends of the Bronian motion $\gamma : [0,T] \to \bar{D}$ to the basepoint using the short paths system .*

0.5 We are interested in the asymptotic behavior of the score. The variance of $\mathrm{Sc}(T)$ grows linearly in T and the expectation is bounded, so the correct scaling is $\mathrm{Sc}(T)/\sqrt{T}$. The contribution of the change of the basepoint and of the short paths system to $\mathrm{Sc}(T)$ is bounded and therefore negligible in the scaled score. The scaled score is by definition an element of $H_{\mathbb{R}} = H \otimes \mathbb{R} \cong \mathbb{R}^{N+1}/(e)$.
In Section 2 we investigate the asymptotics of the score. Using some general central limit theorems for martingales with bounded quadratic variations we prove the convergence of the scaled score $\mathrm{Sc}(T)/\sqrt{T}$ to a Gaussian vector in $\mathbb{R}^N$ and determine its covariance matrix. This covariant matrix is given by a certain quadratic form which can be given a 'physical meaning' as the energy loss for a conducting plate.

0.6 Further, in Section 3, we investigate the related *inverse* problem. It was, actually, first posed in [1] in the context of the conducting plates mentioned above and can be formulated as follows: assume that only the asymptotical behavior of the score is known; is it possible then to reconstruct the original data $(D, g, \{I_i\})$? There is, certainly, a large group acting on the data which preserves the only observable thing, the covariance matrix (or the energy loss form): the data $(hD, h^*g, \{hI_i\})$ will give the same matrix for any diffeomorphism h. One can therefore reduce the problem to the case with fixed D and g (say, the unit circle with the standard metrics). The group of diffeomorphisms preserving D and g is now three-dimensional, and the variable parameters are $2N+2$ endpoints of the intervals I.
We address this inverse problem in Section 3. It turns out that it can be reduced to the Shottky problem for hyperelliptic curves (solved by Mumford [5]) and thence settled completely.
I would like to mention in this connection the paper [7] where an inverse problem of similar spirit has been addressed.

0.7 In Section 4 we discuss some ramifications and generalizations.

1 Brownian Motion in $\bar{D}$

1.1 We start with the precise construction of the Brownian motion on the manifold with boundary $\bar{D}$. The boundary of D consists of $N+1$ arcs I_i interlaced with $N+1$ other arcs; we will denote them by I'_i and number them so that I'_i lies between I_i and I_{i+1}; addition in subscripts henceforth is always understood *modulo* $(N+1)$. Let D_1, D_2 be two copies of D. The $(N+1)$-holed sphere $\bar{D}$ is the factor space obtained upon identification of the 'same' points in I_i's. The lift of the metric g from $D_{1,2}$ defines a metric on an open dense subset of $\bar{D}$ which we will refer to as g again.

1.2 To define the Brownian motion with elastic reflections at the boundary points in the topological manifold $\bar{D}$ we use the standard trick describing Brownian motions under conformal maps [4].

Specifically, let $f : U \to D$ be a continuous one-to-one mapping of the upper half of the complex plane $U = \{\Re(\cdot) \geq 0\} \subset \bar{\mathbb{C}}$ to D which is smooth in the interior of U and takes the conformal class of the standard metric on $\mathbb{C}$ (given by $dx^2 + dy^2$) to that of g. Such a map exists by theorems of Riemann and Koebe.

Let the segments $J_i = [a_i, b_i] \in \partial U = \mathbb{R}\mathbf{P}^1$ be the preimages of the arcs $I_i \subset \partial D$ under f (one of these segments can contain the infinity). Define $\bar{S}$ to be the Riemann sphere $\bar{\mathbb{C}}$ with slits made along the intervals $J'_i = (b_i, a_{i+1}), i = 0, \ldots, N$. This surface is topologically a 2-sphere with $N+1$ holes. The map $f : U \to D$ defined on the upper half of this holed sphere can be extended to a map $\bar{f} : \bar{S} \to \bar{D}$ in an obvious way. The resulting map is a homeomorphism which is smooth outside the real line $\mathbb{R}\mathbf{P}^1$.

1.3 At this stage we could already define the Brownian motion on $\bar{D}$ with elastic reflections at boundary as the image under $\bar{f}$ of the Brownian motion on $\bar{S}$ with elastic reflections at the points of intervals J'_i. Changing the clock of this Brownian motion would result in a $\bar{D}$-valued Markov process with continuous trajectories such that the generator of the corresponding semigroup is Δ_g (the Laplacian in g metric) in the interiority of $\bar{D}$.

We make, however one step further in view of some applications to follow and describe the Brownian motion on $\bar{S}$ itself as the projection of a Brownian motion on an adequate twofold covering of $\bar{S}$, analogously to the description of the reflected Brownian motion on the halfline as $|w|$, for w Brownian motion on $\mathbb{R}$. To achieve this, consider the hyperelliptic curve $M \subset \mathbb{C}\mathbf{P}^2$ given by

$$w^2 = \prod_{i=0}^{N} (t - a_i)(t - b_i)$$

in inhomogeneous coordinates. It is clear that the curve M is of genus N and that the projection $p : M \to \bar{\mathbb{C}} : (t, w) \mapsto t$ is a 2-to-1 covering of $\bar{\mathbb{C}}$ ramified over $\{a_i, b_i\}_{i=0,\ldots,N}$. We denote the corresponding critical points on M by A_i, B_i. Cutting M along the preimage of the equator $\mathbb{R}\mathbf{P}^1 \subset \mathbb{C}\mathbf{P} = \bar{\mathbb{C}}$ one decomposes the surface into the union of two copies of the upper half-sphere and two copies of the lower one. We denote these as $N_{1,2}$ and $S_{1,2}$ respectively, so that N_i is glued to S_i along the intervals J_i. Now we define the *modified* projection $\bar{p} : M \to \bar{S}$ as follows: $\bar{p}$ coincides with p on $N_1 \cup S_1$ and with the composition of p with the complex conjugation on $N_2 \cup S_2$.

Lemma 1.1 *The map $\bar{p}$ is continuous and smooth outside of the preimage of the intervals $I'_i, i = 0, \ldots, N$. Moreover, it is conformal on the latter set. Near a point in the preimage of I'_i the map $\bar{p}$ is equivalent (that is can be reduced by a local change of coordinates in the image* *and preimage) to the folding*

$$x + iy \mapsto x + i|y|.$$

Proof:
Obvious. ∎

1.6 We assume that M is provided with a complete Riemannian metric, e.g. with the metric of constant curvature. The previous lemma shows that the $\bar{p}$ image of the corresponding Brownian motion on M is the Brownian motion on $\bar{S}$ with elastic reflections at the slits I'_i with changed time. The Brownian motion on $\bar{D}$ is therefore the image of the Brownian motion on M under the composition map $h = \bar{f} \circ \bar{p} : M \to \bar{D}$ with the clock

$$\tau(t) = \int_0^t |h'(w(s))|^2 ds.$$

Here $|h'|$ is the coefficient of dilatation: the image of the metric on M under h is the $|h'|$-multiple of g (recall that the conformal classes of these two metrics on $\bar{D}$ coincide by construction).

Proposition 1.1 *Almost surely,*

$$\frac{\tau(t)}{t} \to \frac{4S_D}{S_M} \quad as \quad t \to \infty,$$

where S_M is the area of M and S_D is the (g)-area of D.

Proof:
From the definition, it follows that $\int_M |h'(z)|^2 vol_M$, for vol_M the volume form on M is the area of $\bar{D} \times 2 =$ area of $D \times 4$ (as h is 2-to-1 and $|h'(z)|^2$ is the Jacobian of h). Therefore, by the ergodic theorem for Brownian motion on the compact manifold M,

$$\frac{1}{t} \int_0^t |h'(w(s))|^2 ds \to \frac{1}{S_M} \int_M |h'(z)|^2 vol_M$$

almost surely. ∎

1.8 Therefore, we can define the Bronian motion on $\bar{D}$ as $\gamma_{\tau(t)} = h(w(t))$, with w denoting the Brownian motion on M. The Proposition shows that the process γ is defined almost everywhere for all times τ.

2 Asymptotic Score and Conducting Plates

2.1 Let the data $(D, g, \{I_i\})$ be as above, that is g is a Riemannian metric on the disk D and $\{I_i\}$ is a system of nonintersecting arcs on ∂D. We introduce the $N+1$-dimensional real vector space V of locally constant functions on the disjoint union of the intervals $I = \amalg_i I_i$. Consider the following quadratic form Q on V:

$$Q(x) = \inf_{u \in C^\infty(D): u|_I = x} \int_D \|du\|_g vol_g.$$

In other words, the infimum is taken over all function assuming the boundary values x_i on the arcs I_i and is free elsewhere. The norm $\|\cdot\|_g$ and the volume form vol_g are induced by the metric g.
Another way to write the integrand is, of course, $du \wedge d^*u$, where $d^* = *d*$, and $*$ is the standard dualising operator on forms [2].

Lemma 2.1 *The quadratic form Q has one-dimensional kernel generated by $e = (1, \ldots, 1) \in V$. The restriction of Q to any N-hyperplane complementary to e is postive definite.*

Proof:
This is obvious: if all boundary values are equal, the infimum is attained at a constant function; if some of the values are different, an easy estimate using Cauchy inequality proves positivity. ∎

2.3. Remark. The quadratic form Q has a physical meaning: if (D, g) describes a conducting plate with (generally anisotropic) resistance g, then $Q(x)$ is the energy loss resulting from applying a constant electric potential x_i to the i-th arc on the boundary of D. In the simplest case, when D is a rectangle with sides a, b and $I_{1,2}$ are just its a sides, then $Q(x_0, x_1) = (\frac{a}{b})(x_0 - x_1)^2$. In particular, identifying the quadratic form Q on V with the linear map $Q : V \to V^*$, the image of the vector $(x_0, \ldots, x_N)$ is the vector of total currents through the corresponding segments of the boundary.
2.4 Let $(\tilde{D}, \tilde{g})$ be a disk with Riemannian metric and $f : \tilde{D} \to D$ be homeomorphic and smooth in the interior of $\tilde{D}$. Assume that f takes the conformal class of the Riemannian metric $\tilde{g}$ to that of g.

Lemma 2.2 *For any function u on D smooth in its interiority, the energies of u and f^*u coincide:*

$$\int_D |du|^2 vol_g = \int_{\tilde{D}} |df^*u|^2_{\tilde{g}} vol_{\tilde{g}}.$$

Proof:
Well known (the integrand is divided, and the volume is multiplied by the squared dilatation coefficient). ∎
2.6 Hence, the functional Q is invariant under mappings of $(D, g, \{I_i\})$ which respect the conformal classes and take boundary arcs to boundary arcs.
The factor space of V by the line generated by the vector $e = (1, \ldots, 1)$ is isomorphic to the first cohomology group of $\bar{D}$ with real coefficients. An element a of the latter is defined by its value $a_i = (a, A_i)$ on the cycle A_i encircling the i-th hole (the preimage of I'_i). Clearly, the numbers a_i satisfy $\sum a_i = 0$ as the cycles A_i do. We fix an isomorphism $s : V/(e) \to H^1(\bar{D}, \mathbb{R})$ which sends $(x_0, \ldots, x_N)$ to $2(x_1 - x_0, x_2 - x_1, \ldots, x_0 - x_N)$ (the 2 will be explained below). The lift of Q via s^{-1} to $H^1(\bar{D}, \mathbb{R}) \cong \mathbb{R}^N$ we will denote as Q_H.
The relevance of the quadratic form Q_H to the score Brownian motion on the surface $\bar{D}$ is given by the following Theorem.

Theorem 2.1 *The random variable $Sc(T)/\sqrt{T} \in H_{\mathbb{R}} \cong \mathbb{R}^N$ converges almost surely to a Gaussian random value $\xi \in H_{\mathbb{R}}$. For any vector $a \in H^1$ the variance of (a, ξ) is $Q_H(a)/S_D$, where S_D is the area of D (measured with respect to the g-area form).*

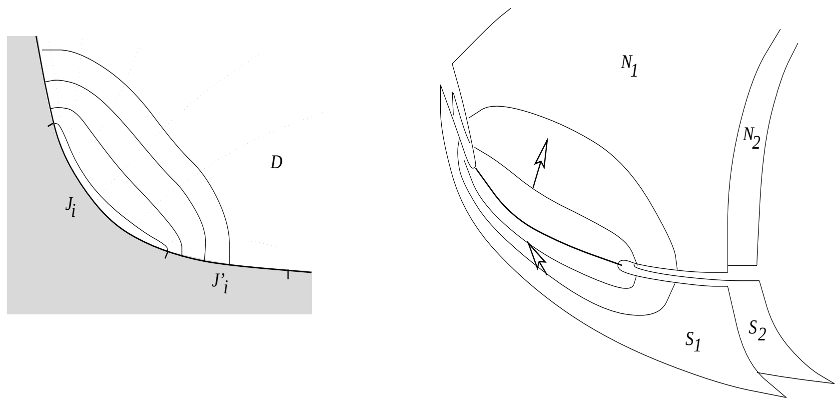

Figure 1. The lift of the form du . The levels of the form are shown as solid lines in D ; the dotted lines show the levels of its orthogonal form.

Figure 1: Figure 1. The lift of du. The solid lines denote the levels of u on M; the dotted lines — the levels of the orthogonal form.

Proof:
To start the proof we will construct a linear mapping from the space V of locally constant functions on $I = \amalg_i I_i$ to the space of harmonic 1-forms on M. For $x = (x_0, \ldots, x_N) \in V$, let u_x be the continuous function on the upper halfplane $U \subset \bar{\mathbb{C}}$ which has the following properties:

- The function is harmonic in the interiority of U;
- The restriction of u_x to J_i is identically x_i;
- The normal derivative of u at the points of $\partial U \cong \mathbb{R}\mathbf{P}^1 - \cup_i J_i$ vanishes.

The existence of such a function is standard; one could define u_x as the expected value of the x_I, where I is the number of the first of the intervals J_i hitted by the reflected Brownian motion in U.
Let $f : \bar{\mathbb{C}} \to U$ be the folding of the complex sphere to the upper halfplane, $x + iy \mapsto x + i|y|$. The composite mapping of the hyperelliptic curve M to U given by $k = f \circ p$ is smooth (and even conformal) outside the preimage of the real projective line $\mathbb{R}\mathbf{P} \subset \bar{\mathbb{C}}$. On this set $M - k^{-1}(\mathbb{R}\mathbf{P}) = N_1 \cup S_1 \cup N_2 \cup S_2$ the mapping k preserves orientation on N's and reverses it on S's. Define the 1-form on this set as

$$\omega_x = k^*(du_x) \text{ on } N_1 \cup S_2 \text{ and } \omega_x = -k^*(du_x) \text{ on } N_2 \cup S_1.$$

Lemma 2.3 *The form ω_x extends to a smooth harmonic form on M (also denoted by ω_x).*

Proof:
Indeed, the continuity of ω_x outside of the ramification points of p (that is, preimages of a_i, b_i's) is clear from the fact that the annulator of du_x is tangent to the real line at points of J_i and orthogonal to it at points of J'_i (see Figure 1).

Further, the fact that ω_x is harmonic at the interior points of the preimages of intervals J_i or J'_i (that is, outside the ramification locus $\{p^{-1}(a_i), p^{-1}(b_i)\}_{i=0,\ldots,N}\}$ of the projection p) follows

from the symmetry principle. The fact that the ramification points are removable singularities follows from the boundedness of the integral of ω_x, (which is clearly exact in the neighborhood of each of the ramification points). ∎
The p-preimage of the segment J_i is a smooth closed curve in M. We fix the orientation of this curve by demanding that it coincides with the lift of the natural orientation of $\partial U = \mathbb{R}\mathbf{P}$ to $N_1 \cup S_1$. This defines an element of $H_1(M, \mathbb{Z})$. We will denote this element as B_i. Analogously, we will denote as A_i the cycle corresponding to the preimage of J_i' oriented in accordance with the lift from ∂U to $N_1 \cup S_2$.
Clearly, the elements $\{A_i, B_i\}$ generate $H_1(M, \mathbb{Z})$ and satisfy

$$\sum_{i=0}^{N} A_i = \sum_{i=0}^{N} B_i = 0.$$

Lemma 2.4 $\int_{A_i} \omega_x = 2(x_{i+1} - x_i)$.

Proof:
The integration contour splits into two parts, one along the preimage of J_i' in $N_1 \cup S_2$, another along the preimage in $N_2 \cup S_1$. The form along the first part is $k^*(du)$ and its integral is $x_{i+1} - x_i$, along the second part of the contour, $\omega_x = -k^* du$ and the integral is $x_{i+1} - x_i$ again. ∎
Proof of Theorem 2.1
Pick an element $a \in H^1(\bar{D}, \mathbb{R})$. The evaluation of a on the score $Sc(T) \in H_1(\bar{D}, \mathbb{Z})$ differs, by definition, just by a bounded term from the evaluation of $\bar{p}^*(a)$ on the cycle in $H_1(M, \mathbb{Z})$ obtained by closing the trajectory of the Brownian motion $w \in M$ run to the time t such that $\tau(t) = T$. Let $x = s^{-1}(a)$, with s defined in 2.6. Then the (closed) 1-form ω_x represents $\bar{p}^*(a)$ under the de Rham isomorphism. Therefore, $(Sc(T), a)$ differs just by a bounded term from $I(t) = \int_{\{w(s)\}_{0 \le s \le t}} \omega_x$. The process $I(t)$ is a martingale as ω_x is harmonic, and its characteristic is $\int_0^t |\omega_x(w(s))|^2 ds$. Therefore, $I(t)/t$ converges a.e. to a normal random value with variance $\frac{1}{S_M} \int_M |\omega_x|^2 vol$. Applying Proposition and the fact that

$$\int_M |\omega_x|^2 vol = 4 \int_U |\omega_x|_g^2 vol_g,$$

(M covers D fourfold) we obtain the desired result. ∎
2.10. Example. For the rectangular field of size $a \times b$ and $I_{1,2}$ being the b-sides, (and standard metric) one finds the variance of the asymptotical score ξ is a^{-2}. This is, of course, of little surprise, as in this case the score $\mathrm{Sc}(T)$ is just some rounding of $w(T)/b$, where w the 1-dimensional Brownian motion.
We notice here that *any* disk D with just two marked arcs on its boundary can be mapped conformally onto one of the rectangular areas with the intervals being the opposite sides. Moreover, one can adjust the sizes to preserve the area under this mapping.

3 Inverse Problem

3.0 Theorem 2.1 describes the asymptotic behavior of the score in terms of the g-area of the plane domain D and of a certain quadratic form Q. It makes sense to pose an *inverse* problem: to what extent do these data characterize the original ones, that is $(D, g, \{I\}_i)$?

3.1 Firstly, the Lemma 2.2 implies that the energy form Q is an invariant of the conformal class of the metric g only. Therefore, it cannot distinguish between 'soccer fields' differing by a conformal diffeomorphism and the inverse problem can be meaningfully posed only if one fixes one within each equivalence class. As was mentioned, one can choose the upper half plane U on the Riemannian sphere $\bar{S} = \mathbb{C}\mathbf{P}^1$ as D provided with its standard (Fubini-Study) metric. The unspecified data then are the positions of the 'goalposts' $\{a_i, b_i\}_{i=0,\ldots,N}, a_i, b_i \in \mathbb{R}\mathbf{P}^1$. These positions form, apparently, a $2(N+1)$-dimensional family $\mathcal{C}$ (diffeomorphic to S^1 times a $2N+1$-dimensional open simplex). The group $PSL(2,\mathbb{R})$ of conformal diffeomorphisms of U is three-dimensional, and the associated Teichmüller space is a cell of dimension $2N-1$, $\mathcal{T} = \mathcal{C}/PSL(2,\mathbb{R}) \cong \mathbb{R}^{2N-1}$. The best way to introduce coordinates on $\mathcal{T}$ is to employ the cross ratios of the points a_i, b_i. Indeed, for any $0 \leq i, j \leq N$ we set

$$c_{ij} = \Big(\frac{a_i - a_j}{b_i - a_j}\Big) / \Big(\frac{a_i - b_j}{b_i - b_j}\Big).$$

The parameters c_{ij} are real, invariant with respect to automorphisms of U and take values in $\mathbb{R}_{>1}$. Clearly, they determine an embedding of $\mathcal{T}$ into the $\mathbb{R}^n_{>1}$ for some large n.

3.2 The energy form Q is a function on $\mathcal{T}$ with values in the space of quadratic forms Q on the $(N+1)$-dimensional real vector space V, which have a one-dimensional kernel and are nonnegative definite (we will denote this space as $\mathrm{Sym}_0(V)$). The set of such quadratic forms is a cell of dimension $N(N-1)/2$. The dimension of the image is not less than that of the source and one can ask whether the energy form defines an embedding of the Teichmueller space. If this is the case, one can solve the inverse problem unambiguously.

3.3 Remark. The question about the properites of the energy mapping $E : \mathcal{T} \to \mathrm{Sym}_0(V)$ was raised by Colin de Verdière [1] in connection with his studies of the energy forms of plane electric circuits.

3.4 The main result of this section is that the energy functional is indeed an embedding. This follows quite easily from the next Proposition.

Let M be the hyperelliptic curve constructed in the previous section and $A_i(B_i)$, $i = 0, \ldots, N$ be the cycles defined in 2.8. The cycles $A_i, B'_i = \sum_{j=1}^{i} B_j, i = 1, \ldots, N$ form a basis in the integer homologies $H_1(M, \mathbb{Z})$ with respect to which the intersection form is standard: $\langle A_i, B'_j \rangle = \delta_{ij}; \langle A_i, A_j \rangle = \langle B'_i, B'_j \rangle = 0$.

Let $\{\omega_i\}$ be the basis of the space of holomorphic 1-forms on M dual to $\{A_i\}$, that is such that $\int_{A_i} \omega_j = \delta_{ij}$. The matrix $Z = (z_{ij}), z_{ij} = \int_{B'_i} \omega_j$ is called the period matrix (of the curve M) and determines the Jacobian variety of the curve M: $J(M) = \mathbb{C}^N / L, L = \{\mathbf{n} + Z\mathbf{m}, \mathbf{n}, \mathbf{m} \in \mathbb{Z}^N\}$.

Proposition 3.1 *The period matrix Z is $i/2$ times $T^T Q T$, where T is the $(N+1) \times N$ matrix with entries $t_{ij} = 1 \quad if \quad i \leq j$ and 0 otherwise .*

Proof:

We recall from (2.8) that for any $x \in V$, we have a 1-form du_x which is harmonic on U. Let $\imath$ be the operator defining the complex structure on U. The 1-form $du_x \circ \imath$ is again harmonic. The annulator of this form is tangent to ∂U at points of intervals J'_i and orthogonal to it at points of ∂U (as follows from the opposite behavior of du_x). The standard calculation shows that

$$Q(x) = \sum_i x_i \int_{J_i} \omega_x \imath$$

whence

$$\int_{J_i} du_x \imath = \sum_j Q_{ij}.$$

The properties of the form $du_x\imath$ stated above show that its lift from U to M, with properly adjusted signs on different branches of k (we set it to be the lifting on $N_1 \cup S_1$ and opposite to it on $N_2 \cup S_2$), we obtain, analogously to the case of ω_x, a smooth harmonic form on M. One can check immediately that this form coincide with $\omega_x \imath$ (where this time $\imath$ is the complex structure on M). Therefore the form $\omega_x^{\mathbb{C}} = \omega_x - i\omega_x\imath$ is a holomorphic 1-form on M. Its integrals over a cycle A_i are, obviosly, the integrals of ω_x over it, as the $\omega_x\imath$ part vanishes on the lifts of the vectors tangent to intervals J.
It follows that for $x_i = (1/2)(e_0 + \sum_{j=i+1}^N e_j)$, e_j the standard basis vectors in V, the corresponding holomorphic differential $\omega_i^{\mathbb{C}} = \omega_{x_i}^{\mathbb{C}}$ (as well as its 'real part' $\omega_i = \omega_{x_i}$) has the property that $\int_{A_i} \omega_j = \delta_{ij}, i, j = 1, \ldots, N$.
By the definition, the entries of the period matrix Z are

$$z_{ij} = \int_{B'_j} \omega_i^{\mathbb{C}} = -i \sum_{l=1}^{j} \int_{B_l} \omega_i \imath.$$

As was established above,

$$int_{B_l} \omega_j \imath = (1/2)(Q_{l0} + \sum_{k=j+1}^{N} Q_{lk}) = -(1/2) \sum_{k=1}^{j} Q_{lk},$$

where we used the fact $\sum_{k=0}^N Q_{lk} = 0$.
Combining all together we obtain the claimed formula. ∎
As a corollary we immediately obtain the solutiuon of our inverse problem:

Theorem 3.1 *The energy map associating the form Q to the data $(U, g, \{I\}_i)$ $(g = dx^2 + dy^2)$ is a proper imbedding of the Teichmüller space $\mathcal{T}$ into the space $Sym_0(V)$.*

Proof:
The main claim, that the energy map is one-to-one, follows immediately from the Torelli theorem. Indeed, by the Proposition , the form Q defines the period matrix and thence the Jacobian. By Torelli theorem, the curve M can be unambiguously reconstructed form $J(M)$. In our case, the curve is hyperelliptic, and admits the unique (up to an automorphism in the image) 2-to-1 holomorphic map to $\mathbb{C}\mathbf{P}^1$. The critical values of the map are just the numbers $a_i, b_i, i = 0, \ldots, N$, up to an automorphism of $\mathbb{C}\mathbf{P}^1$, and, apparently, define the interval system $\{I\}_i$.
In fact, one can give a pretty detailed description of the image of the Teichmueller space using the theory in [5]. Indeed, given Z one can form 2^N theta constants

$$\theta[{\eta' \atop \eta''}](Z) = \sum_{\mathbf{n} \in \mathbb{Z}^N} exp(\pi i(\mathbf{n} + \eta', Z(\mathbf{n} + \eta')) + 2\pi i(\mathbf{n}, \eta'')),$$

where the vectors $\eta', \eta'' \in (1/2)\mathbb{Z}^N$ have coodinate entries 0 or 1/2. According to [5], the Abelian variety $\mathbb{C}^N/L, L = \{\mathbf{n} + Z\mathbf{m}, \mathbf{n}, \mathbf{m} \in \mathbb{Z}^N\}$ is the Jacobian of a hyperelliptic curve M if and only if all but $\binom{2N+2}{N+1}$ of these theta constants vanish. The remaining theta constants can be used to find the cross ratios parametrizing $\mathcal{T}$: each squared cross ratio c_{ij}^2 can be expressed (via Frobenius formulae) as some ratio of fourth degrees of certain theta-constants [5] p. 3.136. In our situation we are bounded by the requirement that all the cross ratios are real. One sees immediately that it follows from the reality of the theta constants; inversely, the reality of cross-ratios results in the reality of iZ and therefore in that of the theta constants (as each term in the series defining them is real).

The properness of the energy mapping is also immediate from the Frobenius formula. ∎

4 Concluding Remarks

The results of the previous section yield quite of lot of different corollaries and generalizations.

4.1 Returning to the initial motivation of this study, the soccer field, the Theorem implies, that given the size of the soccer field and assuming that the goalposts are placed symmetrically on the opposite sides of it, the variance of the asymptotic score determines unambigously their width. This is obvious, as the (only) cross ratio involved is that of the images of the goalposts under the conformal mapping taking the field to the upper half plane, and it depends monotonously on the width of the goalposts.

4.2 The results of this paper can be easily extended to the case when the metric has singularities, provided that the total area of the domain remains finite. If the singularities are strong enough, so that the area becomes infinite, we arrive at various generalizations of the results of [3]. The original result of Kozlov, Pitman and Yor describes the case when M has genus 1 and the metric g is the smooth one times a positive function which has at two points (preimages of infinity on M) poles of order 2. Nearly all the time, the trajectory of the motion wanders near these singular points and the characteristic of the integral of any harmonic on M form is, asymptotically, a local time process L_t^0 at zero for the Bessel process (of order 2). An immediate generalization can be obtained if one considers an algebraic curve M, a rational function $f : M \to \mathbb{C}$ on it and the Riemannian metric on M given by the lift of $dx^2 + dy^2$ via f. One can see quite easily that the resulting distribution of the asymptotical score (scaled by $\sqrt{\log t}$) has the Laplace transform $(1 + (\lambda, Q_H\lambda))^{-1}$ for the quadratic form Q_H defined in 2.6. Similarly, one can address the question about the metrics with singularities of other orders. Once again, this can be solved by finding the asymptotical distribution of the local time at zero for the Bessel process of relevant order. Notice, however, that the order of the pole cannot be larger than two: in this case the process is transient, the local time remains finite and one cannot apply the ergodic theorems. Geometrically, this corresponds to Riemannian surfaces with nonempty killing boundary.

4.3 The real Teichmueller space $\mathcal{T}$, like its complex counterpart, admits a compactification by divisors on which some points come together pairwise. It is well known that near such a divisor some entries of the period matrix tend to infinity as the logarithm of the function defining the divisor. One very interesting site in this compactified space is the vicinity of the maximal ($(N+1)$-fold) selfintersection of its boundary. For example, one can consider the situation where all J'-intervals are very small (or, equivalently, all J ones are very large. The entries of the period matrix tend to infinity and the theta constant becomes dominated by a finite number of terms. The condition of vanishing of a theta constant reduces then to coincidence of the values of the energy form on certain points of the integer lattice. One can show that near the infinity, the image of the energy map is (asymptotically) piecewise-linear and has the combinatorial type of the fan dual to associahedron.

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