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CONTINUOUS OCONE MARTINGALES AS WEAK LIMITS OF RESCALED MARTINGALES

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Abstract
Consider a martingale M with bounded jumps and two sequences $a_n, b_n \to \infty$. We show that if the rescaled martingales

$$M_t^n = \frac{1}{\sqrt{a_n}} M_{b_n t}$$

converge weakly, then the limit is necessarily a continous Ocone martingale. Necessary and sufficient conditions for the weak convergence of the rescaled martingales are also given.

1 Introduction

A continuous Ocone martingale is a continuous martingale that is conditionally Gaussian, given its quadratic variation process. This class of martingales was recently studied for instance in the papers Ocone (1993), Dubins et al. (1993) and Vostrikova and Yor (2000). In this note we show that continuous Ocone martingales arise as the only possible weak limits of rescaled sequences of martingales with bounded jumps. If M is a martingale with bounded jumps and a_n, b_n are sequences of positive numbers that both increase to infinity, we consider the sequence of rescaled martingales M^n defined by

$$M_t^n = \frac{1}{\sqrt{a_n}} M_{b_n t}. \tag{1}$$

We show that if we have weak convergence $M^n \Rightarrow N$ in the Skorohod space $D[0, \infty)$, then N is necessarily a continuous Ocone martingale.
We also prove that for rescaled martingales of this type, weak convergence of the martingales is equivalent to weak convergence of the brackets. Most martingale central limit theorems only provide sufficient conditions on the brackets. These results are often of the form 'if the brackets of a sequence of martingales converge in probability to the bracket of the limiting martingale, then the martingales themselves converge weakly' (if necessary, a Lindeberg-type condition

is added to control the jumps), see e.g. Jacod and Shiryaev (1987) or Liptser and Shiryayev (1989). However, the requirement that the brackets should converge in probability can be too strong in certain cases (see the examples in Section 5). We will prove that for sequences of rescaled martingales with bounded jumps, one only has to require that the brackets converge in distribution. It is well-known that this is also necessary (see Jacod and Shiryaev (1987), Corollary VI.6.6).

Necessary and sufficient conditions for the martingale central limit theorem in terms of convergence of the brackets are known in certain cases. See for instance Theorem VIII.3.8 of Jacod and Shiryaev (1987) for the case that the limit is Gaussian, or Theorem VIII.4.47 of the same book for the case that the law of the limit process is the unique solution of a certain martingale problem. To prove our functional central limit theorem for rescaled martingales, we do not need to work in the canonical setup of the latter result of Jacod and Shiryaev (1987) nor do we have to make an explicit reference to the martingale problem, since our proof relies on totally different arguments.

In the next section, we quickly recall some facts on continuous Ocone martingales. For the sake of easy reference, Section 3 recalls two theorems that play a crucial role in the proof of our main result: a general version of Skorohod's embedding theorem and a result on rescaled Brownian motions. The main theorem is stated and proved in Section 4. In Section 5 we give some simple examples to illustrate the result.

2 Continuous Ocone martingales

In the paper Ocone (1993), Daniel Ocone introduced and studied martingales M with the property that for every predictable process a with $|a| = 1$, the law of the martingale transform $\int a\,dM$ is equal to the law of M. Nowadays, martingales with this property are often called *Ocone martingales.* Several characterizations of this class of martingales were derived in Ocone (1993). In particular, it turned out that the continuous Ocone martingales are precisly the continuous martingales that are conditionally Gaussian, given their quadratic variation. So the continuous martingale M is Ocone if and only if, given $\langle M\rangle$, the process M is a Gaussian martingale with quadratic variation $\langle M\rangle$.

A nice characterization of continuous Ocone martingales can be formulated in terms of their so-called DDS Brownian motion. By the Dambis-Dubins-Schwarz time-change theorem, each divergent local martingale M (i.e. a martingale M for which $\langle M\rangle_\infty = \infty$ a.s.) can be written as $M_t = W_{\langle M\rangle_t}$, where W is a Brownian motion that is defined on the same probability space as M (see for instance Revuz and Yor (2001), p. 181). The Brownian motion W is called the DDS Brownian motion of M. It has been proved in Dubins et al. (1993) that a divergent continuous martingale M is Ocone if and only if the DDS Brownian motion of M is independent of $\langle M\rangle$. The 'converse' of this result is also true. Indeed, if on some probability space we have a Brownian motion W and a continuous, increasing process C that starts in 0, then $M = W_C$ is a continuous local martingale with respect to its natural filtration and its bracket is given by $\langle M\rangle = C$ (see Exercise IV.1.44 of Revuz and Yor (2001)). Conditionally on its bracket C, the local martingale M is clearly Gaussian, hence M is Ocone.

For more characterizations of Ocone martingales, we refer to Vostrikova and Yor (2000). See also the related paper of Beghdadi-Sakrani (2002).

3 Preliminaries

Throughout the paper we use the standard notations of general martingale theory, see for instance Jacod and Shiryaev (1987) of Liptser and Shiryayev (1989). All martingales M are assumed to start in 0, i.e. $M_0 = 0$, and the optional quadratic variation of a martingale M is denoted by $[M]$. The jumps of a càdlàg process X are denoted by $\Delta X_t = X_t - X_{t-}$. $D[0,\infty)$ is the Skorohod space of càdlàg functions on $[0,\infty)$, $C[0,\infty)$ is the space of continuous functions on $[0,\infty)$ and $C_0^+[0,\infty)$ is the space of continuous, increasing functions on $[0,\infty)$ that start in 0. By $\mathcal{L}(X \mid P)$ we denote the law of a random element X under the probability measure P. The symbol $\Rightarrow$ denotes weak convergence.
We now recall two known theorems, which are the basic ingredients in the proof of the main result of this paper. The first one is a general version of Skorohod's embedding theorem, developped in the papers Monroe (1972), Kubilius (1985) and Coquet et al. (1994). See the latter paper for a proof.

Theorem 3.1. *Let M be a square integrable martingale, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$. Then there exists a filtered probability space $(\Omega', \mathcal{F}', \mathbb{F}', P')$ supporting a square integrable martingale M' and a càdlàg, adapted, increasing process $\tau = (\tau_t)_{t\geq 0}$ with the following properties:*

(i) $\mathcal{L}((M,[M]) \mid P) = \mathcal{L}((M',[M']) \mid P')$,

(ii) $M'_t = W_{\tau_t}$, *where W is a process on $(\Omega', \mathcal{F}', P')$ that is a Brownian motion with respect to its natural filtration,*

(iii) $[M'] - \tau$ *is a martingale,*

(iv) if $E \sup_{s\leq t} |\Delta M_s|^4 < \infty$ for all $t \geq 0$, then the process

$$\left(\sum_{s\leq t} \left(4(\Delta M'_s)^4 - (\Delta \tau_s)^2\right) \right)_{t\geq 0}$$

is a submartingale.

The second theorem that we will use states that rescaled Brownian motions are asymptotically independent of any other random element. It is a special case of more general result presented in Van Zanten (2000). See also Exercise (1.17) in Chapter XIII of Revuz and Yor (2001) for related results.

Theorem 3.2. *Let W be a Brownian motion and let a_n be a sequence of positive numbers, increasing to infinity. For every n, define the Brownian motion W^n by $W^n_t = W_{a_n t}/\sqrt{a_n}$. Then for every random element X on the same probability space, with values in a Polish space $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$, we have weak convergence $(W^n, X) \Rightarrow (B, X)$ in $C[0,\infty) \times \mathcal{X}$, where B is a Brownian motion that is independent of X.*

4 Main result

The following theorem is the main result of the paper.

Theorem 4.1. *Let M be a martingale with bounded jumps and let a_n, b_n be sequences of positive numbers, both increasing to infinity. For each n, define the rescaled martingale M^n by*

$$M_t^n = \frac{1}{\sqrt{a_n}} M_{b_n t}.$$

Then the following statements hold:

(i) If we have weak convergence $M^n \Rightarrow N$ in $D[0,\infty)$, then the process N is necessarily a continuous Ocone martingale.

(ii) Let N be a continuous Ocone martingale. Then $M^n \Rightarrow N$ in $D[0,\infty)$ if and only if $[M^n] \Rightarrow [N]$ in $D[0,\infty)$.

Proof. By Theorem 3.1 we can 'copy' the martingale M to another filtered probability space and embed it in a Brownian motion. Slightly abusing notation we simply write $M = W_{\tau_t}$, omitting the primes. Let K be a bound on the absolute value of the jumps of M, i.e. $|\Delta M| \le K$.
(i). Recall that the weak limit of a sequence of martingales with bounded jumps is necessarily a martingale with respect to its natural filtration (see Jacod and Shiryaev (1987), Corollary IX.1.19). Since weak convergence of martingales with bounded jumps implies weak convergence of their brackets (see Jacod and Shiryaev (1987), Corollary VI.6.6), we have $[M^n] \Rightarrow [N]$. Note that by the boundedness of the jumps of M and Proposition VI.3.26 of Jacod and Shiryaev (1987), the sequence $[M^n]$ is C-tight, i.e. every weak limit of a subsequence of $[M^n]$ is necessarily continuous. It follows that the limit bracket $[N]$ is a continuous process.
Let us first prove that $[M^n] \Rightarrow [N]$ implies weak convergence of the rescaled time-change

$$\tau^n := \left(\frac{\tau_{b_n t}}{a_n}\right)_{t \ge 0} \Rightarrow [N] \tag{2}$$

in $D[0,\infty)$. Consider the purely discontinuous martingale $Z = [M] - \tau$ (cf. part (iii) of Theorem 3.1). Noting that $[M^n]_t = [M]_{b_n t}/a_n$ we see that it suffices to show that for all $t \ge 0$

$$\sup_{s \le t} \left(\frac{1}{a_n} Z_{b_n s}\right)^2 \xrightarrow{P} 0. \tag{3}$$

Now observe that $\Delta Z_t = \Delta[M]_t - \Delta\tau_t = (\Delta M_t)^2 - \Delta\tau_t$. So for the quadratic variation of Z we have

$$[Z]_t = \sum_{s \le t} (\Delta Z_s)^2 = \sum_{s \le t} ((\Delta M_s)^2 - \Delta\tau_s)^2 \le 2 \sum_{s \le t} ((\Delta M_s)^4 + (\Delta\tau_s)^2).$$

Using the fact that $Z^2 - [Z]$ is a martingale we thus find that Z^2 is Lenglart dominated by the process $2\sum_{s\le \cdot}((\Delta M_s)^4 + (\Delta\tau_s)^2)$. From part (iv) of theorem 3.1 we see that the process $\sum_{s \le \cdot} (\Delta\tau_s)^2$ is Lenglart dominated by $4\sum_{s\le\cdot}(\Delta M_s)^4$. It follows that Z^2 is Lenglart dominated by $10\sum_{s\leq\cdot}(\Delta M_s)^4$. So by Lenglart's inequality (see e.g. Jacod and Shiryaev (1987), Lemma I.3.30) we have for all $\varepsilon,\eta,T>0$

$$\begin{aligned} P\left(\sup_{s\leq T} Z_s^2 \geq \varepsilon\right) &\leq \frac{1}{\varepsilon}\left(\eta + 10\,E\sup_{s\leq T}(\Delta M_s)^4\right) + P\left(10\sum_{s\leq T}(\Delta M_s)^4 \geq \eta\right) \\ &\leq \frac{\eta}{\varepsilon} + \frac{10K^4}{\varepsilon} + P(10K^2[M]_T \geq \eta). \end{aligned} \tag{4}$$

For given n and $\lambda,\mu,t>0$, take $\varepsilon=\lambda a_n^2$, $\eta=\mu a_n^2$ and $T=b_n t$ in (4) to find that

$$P\left(\sup_{s\leq t}\left(\frac{1}{a_n}Z_{b_n s}\right)^2 \geq \lambda\right) \leq \frac{\mu}{\lambda} + \frac{10K^4}{\lambda a_n^2} + P\left(\frac{10K^2}{a_n}[M^n]_t \geq \mu\right).$$

Hence it follows from the fact that $[M^n]\Rightarrow[N]$ that we indeed have (3) for every $t\geq 0$, which completes the proof of (2).
From the fact that $\tau^n \Rightarrow [N]$, it follows that the pair (W,τ^n) is tight in $C[0,\infty)\times D[0,\infty)$. Hence, there exists a subsequence $n'\to\infty$ such that

$$(W,\tau^{n'}) \Rightarrow (B,[N]),$$

where B is of course a Brownian motion. By the almost sure representation theorem there exist, on an auxiliary probability space, random elements $(\overline{W},\overline{\tau}^{n'})$ and $(\overline{B},\overline{[N]})$ such that

$$(\overline{W},\overline{\tau}^{n'}) \overset{\mathrm{d}}{=} (W,\tau^{n'}), \quad (\overline{B},\overline{[N]}) \overset{\mathrm{d}}{=} (B,[N]) \tag{5}$$

and

$$(\overline{W},\overline{\tau}^{n'}) \overset{\mathrm{as}}{\to} (\overline{B},\overline{[N]})$$

(here $\overset{\mathrm{d}}{=}$ denotes equality in distribution). Now consider the rescaled Brownian motions $\overline{W}^{n'}$, defined by

$$\overline{W}_t^{n'} = \frac{1}{\sqrt{a_{n'}}}\overline{W}_{a_{n'}t}.$$

By Theorem 3.2, it follows from the convergence $\overline{\tau}^{n'} \overset{\mathrm{as}}{\to} \overline{[N]}$ that

$$(\overline{W}^{n'},\overline{\tau}^{n'}) \Rightarrow (\beta,\overline{[N]}), \tag{6}$$

where β is a Brownian motion that is independent of $\overline{[N]}$.
Since β and $[N]$ are continuous processes, it follows from (5) that the pair $(\beta,\overline{[N]})$ is a random element of $C[0,\infty)\times C_0^+[0,\infty)$. The relative topology induced on this space by the Skorohod topology is simply the local uniform topology. It is easily verified that composition of functions, viewed as a map $C[0,\infty)\times C_0^+[0,\infty)\to C[0,\infty)$, is continuous with respect to the local uniform topology. Hence, the continuous mapping theorem and (6) imply that

$$\left(\overline{W}^{n'}_{\overline{\tau}_t^{n'}}\right)_{t\geq 0} \Rightarrow \beta_{\overline{[N]}}$$

in $D[0,\infty)$. By construction, the process on the left-hand side is in distribution equal to $M^{n'}$, so we have shown that

$$M^{n'} \Rightarrow \beta_{\overline{[N]}}$$

in $D[0,\infty)$. It follows that $N \stackrel{\mathrm{d}}{=} \beta_{\overline{[N]}}$, so N is indeed a continuous Ocone martingale with respect to its natural filtration.
(ii). The proof of the second statement of the theorem uses exactly the same arguments as the proof of part (i). Indeed, if $[M^n] \Rightarrow [N]$, then the previous arguments show that every subsequence $n' \to \infty$ has a further subsequence $n'' \to \infty$ such that $M^n \Rightarrow \beta_{\overline{[N]}}$, where $\overline{[N]} \stackrel{\mathrm{d}}{=} [N]$ and β is a Brownian motion that is independent of $\overline{[N]}$. By the characterization of Ocone martingales in terms of their DDS Brownian motions, the process $\beta_{\overline{[N]}}$ has the same law as N. □

5 Examples

5.1 An example related to Donsker's theorem

Theorem 4.1 includes the following simple version of the classical Donsker theorem.
Let $X_1, X_2, \ldots$ be bounded and centered i.i.d. random variables with unit variance. Consider the rescaled martingales

$$M^n_t = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} X_i.$$

By the law of large numbers,

$$[M^n]_t = \frac{1}{n} \sum_{i=1}^{[nt]} X_i^2 \stackrel{\mathrm{as}}{\to} t$$

for every $t \geq 0$. This implies that $[M^n]$ converges almost surely to the function $t \mapsto t$ in $C[0,\infty)$. Hence, by Theorem 4.1, $M^n \Rightarrow W$, where W is a Brownian motion.

5.2 An example related to the Papanicolaou-Stroock-Varadhan theorem

The following example concerns the weak convergence of a sequence of rescaled martingales whose brackets do not converge is probability, but only in law.
Let W be a standard Brownian motion and $f \in L^2(\mathbb{R})$. Consider for every n the continuous martingale M^n defined by

$$M^n = \frac{1}{n^{1/4}} \int_0^{nt} f(W_s)\, dW_s.$$

These are clearly of the form (1), with $a_n = \sqrt{n}$, $b_n = n$ and $M = \int f(W)\, dW$. Their brackets are given by

$$[M^n]_t = \langle M^n \rangle_t = \frac{1}{\sqrt{n}} \int_0^{nt} f^2(W_s)\, ds.$$

The asymptotic behaviour of Brownian additive functionals of the latter type is well-known. Indeed, using the scaling property of Brownian motion it is easy to see that

$$\langle M^n \rangle \Rightarrow \|f\|_2^2 L$$

in $C[0,\infty)$, where L is standard Brownian local time at 0 (see Revuz and Yor (2001), Chapter XIII, Proposition (2.2)). So by Theorem 4.1, it follows that $M^n \Rightarrow \beta_{\|f\|_2^2 L} \stackrel{\mathrm{d}}{=} \|f\|_2 \beta_L$, where β is a Brownian motion that is independent of L. Note that this also follows from the Papanicolaou-Stroock-Varadhan theorem (Revuz and Yor (2001), p. 526).

5.3 Two Ocone martingales related to Brownian motion

Let W be a Brownian motion. Then we can write $W = W^+ + W^-$, with

$$W_t^+ = \int_0^t 1_{(0,\infty)}(W_s)\, dW_s, \quad W_t^- = \int_0^t 1_{(-\infty,0]}(W_s)\, dW_s.$$

In this subsection we show that W^+ and W^- are Ocone martingales. Note that the martingales W^+ and W^- are examples of non-Gaussian Ocone martingales, with respect to a filtration that is generated by a single Brownian motion W. For this to be possible, it is essential that the brackets $\langle W^+\rangle$ and $\langle W^-\rangle$ are not strictly increasing, see Vostrikova and Yor (2000), Theorem 3.
By the scaling property of Brownian motion, the law of the process

$$\left(\frac{1}{n}\int_0^{nt} 1_{(0,\infty)}(W_s)\, ds\right)_{t\geq 0}$$

is independent of n. By Theorem 4.1, it follows that

$$\frac{1}{\sqrt{n}}\int_0^{n\bullet} 1_{(0,\infty)}(W_s)\, dW_s \Rightarrow \beta_{\int_0^\bullet 1_{(0,\infty)}(W_s)\, ds},$$

where β is a Brownian motion that is independent of W. But on the other hand, the scaling property of Brownian motion shows that the distribution of the martingales on the left-hand side of the preceding display is independent of n. It follows that

$$W^+ \stackrel{\mathrm{d}}{=} \beta_{\int_0^\bullet 1_{(0,\infty)}(W_s)\, ds}. \tag{7}$$

In particular, we see that W^+ is Ocone. Using similar reasoning, we see that

$$W^- \stackrel{\mathrm{d}}{=} \gamma_{\int_0^\bullet 1_{(-\infty,0]}(W_s)\, ds}, \tag{8}$$

where γ is a Brownian motion that is independent of W, hence W^- is also Ocone.
Note that by the Arcsine Law (see Karatzas and Shreve (1991), p. 273), relations (7) and (8) imply that for every fixed $t \geq 0$,

$$W_t^+ \stackrel{\mathrm{d}}{=} W_t^- \stackrel{\mathrm{d}}{=} \beta_{tA},$$

where A has the arcsine distribution on $[0,1]$ and β is a Brownian motion that is independent of A.

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