Elect. Comm. in Probab. **11** (2006), 242–251

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON FRACTIONAL STABLE FIELDS INDEXED BY METRIC SPACES

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Submitted 2 February 2006, *accepted in final form* 26 September 2006

AMS 2000 Subject classification: 60G15
Keywords: Fractional Stable Fields, Metric spaces

Abstract
We define and build H-fractional α-stable fields indexed by a metric space (E,d). We mainly apply these results to spheres, hyperbolic spaces and real trees.

1 Introduction

The H-Fractional Brownian Motion B_H [8, 18], indexed by the Euclidean space $(\mathbb{R}^n, ||.||)$, is a centered Gaussian field such that the variance of its increments is equal to a fractional power of the norm:

$$\mathbb{E}(B_H(M) - B_H(N))^2 \quad = \quad ||MN||^{2H} \quad \forall M,N \in \mathbb{R}^n .$$

In other words, the normalized increments of the H-Fractional Brownian are constant in distribution:

$$\frac{B_H(M) - B_H(N)}{||MN||^H} \quad \stackrel{\mathcal{D}}{=} \quad Z \quad \forall M,N \in \mathbb{R}^n ,$$

where Z is a centered Gaussian random variable with variance 1. It is well-known that the H-Fractional Brownian Motion B_H exists iff $0 < H \leq 1$. [5] proposes to build Fractional Brownian Motions indexed by a metric space (E,d) as centered Gaussian fields which normalized increments are constant in distribution:

$$\frac{B_H(M) - B_H(N)}{d^H(M,N)} \quad \stackrel{\mathcal{D}}{=} \quad Z \quad \forall M,N \in E ,$$

where Z is still a centered Gaussian random variable with variance 1. When (E,d) is the sphere or the hyperbolic space endowed with their geodesic distances, [5] proves that the Fractional Brownian Motion exists iff $0 < H \leq 1/2$. When (E,d) is a real tree with its natural distance, [5] proves that the Fractional Brownian Motion exists at least for $0 < H \leq 1/2$.
The following question then arises: what happens when we move from the Gaussian case to the stable case? One knows that there exists several H-self-similar α-stable fields with stationary

increments indexed by $(\mathbb{R}^n, ||.||)$, with various conditions on the fractional index H, providing $0 < H \leq 1/\alpha$ if $0 < \alpha \leq 1$ and $0 < H < 1$ if $1 < \alpha < 2$ [10]. Let us mention some of them (cf. [13]):

- Linear Fractional Stable Motions: $0 < H < 1$ and $H \neq 1/\alpha$,
- α-Stable Lévy Motions: $H = 1/\alpha$,
- Log-fractional Stable Motions: $H = 1/\alpha$,
- Real Harmonizable Fractional Stable Motions: $0 < H < 1$,
- Lévy-Chentsov fields: $H = 1/\alpha$,
- β-Takenaka fields: $0 < \beta < 1$ and $H = \beta/\alpha$.

All these fields have normalized increments that are constant in distribution:

$$\frac{X(M) - X(N)}{||MN||^H} \quad \stackrel{\mathcal{D}}{=} \quad S_\alpha \quad \forall M, N \in \mathbb{R}^n \ ,$$

where S_α is a standard symmetric α-stable random variable, i.e. a random variable which characteristic function is given by:

$$\mathbb{E}(e^{i\lambda S_\alpha}) \quad = \quad e^{-|\lambda|^\alpha} \ .$$

We propose to call H-fractional α-stable field an α-stable field $X(M), M \in E$ which normalized increments are constant in distribution:

$$\frac{X(M) - X(N)}{d^H(M, N)} \quad \stackrel{\mathcal{D}}{=} \quad S_\alpha \quad \forall M, N \in E \ ,$$

where S_α is a standard symmetric α-stable random variable.
Let us summarize our main results.

- Non-existence.

  Let $\beta_E = \sup\{\beta > 0$ such that d^β is of negative type$\}$. For instance, β_E is equal to 1 for spheres and hyperbolic spaces. We prove that there is no H-fractional α-stable field when $\alpha H > \beta_E$.

- Existence.

  We mainly prove the following. Assume that E contains a dense countable subset and that d is a measure definite kernel:

  - if $0 < \alpha \leq 1$, we construct H-fractional α-stable fields for any $0 < H \leq 1/\alpha$.
  - if $1 < \alpha < 2$, we construct H-fractional α-stable fields for any $H \in (0, 1/(2\alpha)] \cup [1/2, 1/\alpha]$.

2 Non-existence

Let us first recall the definitions of functions of positive or negative type. Let X be a set.

- A symmetric function $(x, y) \mapsto \phi(x, y),\ X \times X \to \mathbb{R}^+$ is of positive type if, $\forall x_1, \ldots, x_n \in X,\ \forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \phi(x_i, x_j) \quad \geq \quad 0 \, .$$

- A symmetric function $(x, y) \mapsto \psi(x, y),\ X \times X \to \mathbb{R}^+$ is of negative type if
  - $\forall x \in X,\ \psi(x, x) = 0$
  - $\forall x_1, \ldots, x_n \in X,\ \forall \lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 0$

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi(x_i, x_j) \quad \leq \quad 0 \, .$$

Schoenberg's Theorem [14] implies the equivalence between

- Function ψ is of negative type.
- $\forall x \in X,\ \psi(x, x) = 0$ and $\forall t \geq 0$, function $\exp(-t\psi)$ is of positive type.

Lemma 2.1
Let ψ be a function of negative type and let $0 < \beta \leq 1$. Then ψ^β is of negative type.

Proof.
For $x \geq 0$, and $0 < \beta < 1$, by performing the change of variable $y = \lambda x$, one has:

$$x^\beta \quad = \quad C_\beta \int_0^{+\infty} \frac{e^{-\lambda x} - 1}{\lambda^{1+\beta}} d\lambda \, ,$$

with

$$C_\beta \quad = \quad \left(\int_0^{+\infty} \frac{e^{-\lambda} - 1}{\lambda^{1+\beta}} d\lambda \right)^{-1} \quad .$$

Let $\lambda_1, \ldots, \lambda_n$ such that $\sum_{i=1}^{n} \lambda_i = 0$:

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi^\beta(x_i, x_j) \quad = \quad C_\beta \int_0^{+\infty} \frac{\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda \psi(x_i, x_j)}}{\lambda^{1+\beta}} d\lambda \, .$$

By Schoenberg's Theorem, $\sum_{i,j=1}^{n} \lambda_i \lambda_j e^{-\lambda \psi(x_i, x_j)} \geq 0$. Since $C_\beta \leq 0$:

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \psi^\beta(x_i, x_j) \quad \leq \quad 0 \, ,$$

and Lemma 2.1 is proved. □

For the metric space (E, d), let us define:

$$\beta_E \quad = \quad \sup\{\beta > 0 \text{ such that } d^\beta \text{ is of negative type}\} \;, \tag{1}$$

with the convention $\beta_E = 0$ if d^β is never of negative type. Let us note that if E contains three points M_1, M_2, M_3 such that:

$$\begin{aligned} d(M_1, M_2) &> d(M_1, M_3) \;, \\ d(M_1, M_2) &> d(M_2, M_3) \;, \end{aligned}$$

then $\beta_E < \infty$. Indeed, with $\lambda_1 = -1/2,\ \lambda_2 = -1/2\ , \lambda_3 = 1$, one has
$\sum_{i,j=1}^{3} \lambda_i\lambda_j d^\beta(M_i, M_j) \sim 1/2\ d^\beta(M_1, M_2) > 0$ as $\beta \to +\infty$.

Corollary 2.1
If $\beta_E \neq 0$, then $\{\beta > 0$ such that d^β of negative type$\} = (0, \beta_E]$.

Proof.
It follows from Lemma 2.1 that d^β is of negative type for $\beta < \beta_E$, and is never of negative type for $\beta > \beta_E$. Let $(\beta_p)_{p\geq 0}$ be an increasing sequence converging to β_E (when $\beta_E < \infty$). For all $M_1, \ldots, M_n \in E$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ such that $\sum_{i=1}^{n} \lambda_i = 0$:

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j d^{\beta_p}(M_i, M_j) \quad \leq \quad 0 \;. \tag{2}$$

Let now perform $\beta_p \to \beta_E$ in (2):

$$\sum_{i,j=1}^{n} \lambda_i\lambda_j d^{\beta_E}(M_i, M_j) \quad \leq \quad 0 \;.$$

It follows that d^{β_E} is of negative type. □

Let us now give some values of β_E.

- Euclidean space $(\mathbb{R}^n, ||.||)$.

  One easily checks that function $(x, y) \mapsto ||x - y||^2$ is of negative type. Indeed, take $\lambda_1, \ldots, \lambda_p$ with $\sum_{i=1}^{p} \lambda_i = 0$ and $x_1, \ldots, x_p \in \mathbb{R}^n$:

$$\sum_{i,j=1}^{p} \lambda_i\lambda_j ||x_i - x_j||^2 \quad = \quad -2||\sum_{i=1}^{p} \lambda_i x_i||^2 \leq 0 \;.$$

Then, by Lemma 2.1, function $(x,y) \mapsto ||x-y||^\beta$ is of negative type when $0 < \beta \leq 2$. Consider now four vertices M_1, M_2, M_3, M_4 of a square with side length 1 and take $\lambda_1 = \lambda_3 = 1$ and $\lambda_2 = \lambda_4 = -1$. Then $\sum_{i,j=1}^{4} \lambda_i \lambda_j ||M_i M_j||^\beta = -8 + 4\sqrt{2}^\beta$ and is strictly positive when $\beta > 2$. It follows that $\beta_{\mathbb{R}^n} = 2$.

- Space $(\mathbb{R}^n, ||.||_{\ell^q})$ where $||x||_{\ell^q}^q = \sum_{i=1}^{n} |x_i|^q$. When $n \geq 3$, $q > 2$, [6, 7] imply $\beta_{\mathbb{R}^n} = 0$.
- Spheres $S_n = \{x \in \mathbb{R}^{n+1},\ ||x|| = 1\}$ with its geodesic distance. It follows from [5] that $\beta_{S_n} = 1$.
- Hyperbolic spaces $H_n = \{x \in \mathbb{R}^{n+1},\ \sum_{i=1}^{n} x_i^2 - x_{n+1}^2 = -1,\ x_{n+1} \geq 1\}$ with its geodesic distance d. It has been proved by [4] that d is of negative type. [4, Prop. 7.6] implies that $\beta_{H_n} = 1$.
- Real trees. A metric space (T,d) is a real tree (e.g. [3]) if the following two properties hold for every $x, y \in T$.
  - There is a unique isometric map $f_{x,y}$ from $[0, d(x,y)]$ into T such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x,y)) = y$.
  - If ϕ is a continuous injective map from $[0,1]$ into T, such that $\phi(0) = x$ and $\phi(1) = y$, we have
  $$\phi([0,1]) = f_{x,y}([0, d(x,y)]).$$

  It has been proved by [17] that the distance on real trees is of negative type: $\beta_T \geq 1$. One can build trees with $\beta_T > 1$. Nevertheless, we give a family of simple trees $(T_p)_{p\geq 1}$ such that $\lim_{p\to+\infty} \beta_{T_p} = 1$. A_0 is the root of the tree. A_0 has p sons $A_1, \dots A_p$, with:
  $$\begin{aligned} d(A_0, A_i) &= 1 \quad i \neq 0\ , \\ d(A_i, A_j) &= 2 \quad i \neq j,\ i,j \neq 0\ . \end{aligned}$$
  Choose $\lambda_0 = 1$ and $\lambda_i = -1/p$ for $i = 1, \dots, p$. Then
  $$\sum_{i,j=0}^{p} \lambda_i \lambda_j d^\beta(A_i, A_j) = -2 + 2^\beta \frac{p-1}{p}\ .$$
  $-2 + 2^\beta \frac{p-1}{p}$ is positive for $\beta \geq 1 + \log_2\left(\frac{p}{p-1}\right)$. It follows that $\beta_{T_p} \leq 1 + \log_2\left(\frac{p}{p-1}\right)$.

Proposition 2.1
There is no H-fractional α-stable fields when $\alpha H > \beta_E$.

Proof.
We prove Proposition 2.1 by contradiction. Let $\lambda, \lambda_1, \dots, \lambda_n \in \mathbb{R}$ and $M_1, \dots, M_n \in E$. On one hand:

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \mathbb{E}\left[exp(i\lambda(X(M_i) - X(M_j)))\right] = \mathbb{E}\left|\sum_{i=1}^{n} \lambda_i exp(i\lambda X(M_i))\right|^2 \geq 0\ .$$

On the other hand:

$$\sum_{i,j=1}^{n} \lambda_i \lambda_j \mathbb{E}\left[exp(i\lambda(X(M_i) - X(M_j)))\right] \quad = \quad \sum_{i,j=1}^{n} \lambda_i \lambda_j \exp(-|\lambda|^\alpha d^{\alpha H}(M_i, M_j)) \; .$$

If $\alpha H > \beta_E$, Schoenberg's Theorem implies that there exists λ such that $\exp(-|\lambda|^\alpha d^{\alpha H}(M, N))$ is not of positive type and Proposition 2.1 is proved. $\square$

3 Construction of H-fractional α-stable fields

3.1 Main result

Let us recall the definition of a measure definite kernel (cf. [12]).

Definition 3.1 *Measure definite kernel.*
A function $(M, N) \mapsto \psi(M, N)$, from $E \times E$ onto $\mathbb{R}^+$, is a measure definite kernel if there exists a measure space $(\mathbf{H}, \sigma(\mathbf{H}), \mu)$ and a map $M \mapsto H_M$ from E onto $\sigma(\mathbf{H})$ such that:

$$\psi(M, N) \quad = \quad \mu(H_M \Delta H_N) \; ,$$

where Δ denotes the symmetric difference of sets.

For $\beta > 0$, $f \in L^\beta(\mathbf{H}, \mu)$, define the pseudo-norm:

$$||f||_\beta \quad = \quad \left(\int_{\mathbf{H}} |f|^\beta d\mu \right)^{1/\beta} \; .$$

It follows that:

$$\begin{aligned} \psi(M, N) \quad &= \quad \int_{\mathbf{H}} |\mathbf{1}_{H_M} - \mathbf{1}_{H_N}| d\mu \\ &= \quad ||\mathbf{1}_{H_M} - \mathbf{1}_{H_N}||_\beta^\beta \; . \end{aligned}$$

Theorem 3.1
Let $1/2 \le H \le 1/\alpha$. The following formula, with $n \ge 1, \lambda_1, \ldots, \lambda_n \in \mathbb{R},\ M_1, \ldots, M_n \in E$,

$$\mathbb{E}\left(\exp\left(i \sum_{j=1}^{n} \lambda_j X(M_j) \right) \right) \quad = \quad \exp\left(- \left\| \sum_{j=1}^{n} \lambda_j \mathbf{1}_{H_{M_j}} \right\|_{1/H}^{\alpha} \right) \tag{3}$$

defines the distribution of an α-stable field $X(M), M \in E$ satisfying:

$$\frac{X(M) - X(N)}{\psi^H(M, N)} \quad \stackrel{\mathcal{D}}{=} \quad S_\alpha \quad \forall M, N \in E \; , \tag{4}$$

where S_α is a standard symmetric α-stable random variable.

Proof.
We follow Theorem 1 and Lemma 4 of [1]. We have seen that function $(x, y) \mapsto |x - y|^\gamma,\ x, y \in \mathbb{R}$ is of negative type if $0 < \gamma \le 2$. It follows that function $(f, g) \mapsto ||f - g||_{1/H}^{1/H},\ f, g \in L^{1/H}(\mathbf{H}, \mu)$

is of negative type when $H \geq 1/2$. Since $\alpha H \leq 1$, one can apply Lemma 2.1: function $(f,g) \mapsto ||f-g||_{1/H}^{\alpha}$, $f,g \in L^{1/H}(\mathbf{H},\mu)$ is of negative type. Schoenberg's Theorem implies that, for all $\lambda \in \mathbb{R}$, function $(f,g) \mapsto \exp(-|\lambda|^{\alpha}||f-g||_{1/H}^{\alpha})$, $f,g \in L^{1/H}(\mathbf{H},\mu)$ is of positive type. (3) is therefore a characteristic function.
Fix now $1 \leq j_0 \leq n$ in (3). We clearly have:

$$\begin{aligned}\lim_{\lambda_{j_0}\to 0} \mathbb{E}\left(\exp\left(i\sum_{j=1}^{n}\lambda_j X(M_j)\right)\right) &= \exp\left(-\left|\left|\sum_{j=1, j\neq j_0}^{n}\lambda_j \mathbf{1}_{H_{M_j}}\right|\right|_{1/H}^{\alpha}\right) \\ &= \mathbb{E}\left(\exp\left(i\sum_{j=1, j\neq j_0}^{n}\lambda_j X(M_j)\right)\right) .\end{aligned}$$

The Kolmogorov consistency theorem then proves that (3) defines the distribution of an α-stable stochastic fields.
Choosing $n=2$ and $\lambda_1 = -\lambda_2$ in (3) leads to (4).

□

Remark 3.1
One should wonder if function $(f,g) \mapsto ||f-g||_{1/H}^{\alpha}$ *is of negative type for* $H < 1/2$. *Assume that we can choose three disjoints sets* A_1, A_2 *and* A_3 *such that* $\mu(A_1) = \mu(A_2) = \mu(A_3) = c > 0$. *Put* $f = \sum_1^3 \lambda_i \mathbf{1}_{A_i}$. *Then:*

$$||f||_{1/H}^{\alpha} = c^{\alpha H}\left(\sum_1^3 |\lambda_i|^{1/H}\right)^{\alpha H} .$$

But one knows [6, 7] that function $(x,y) \mapsto ||x-y||_{\ell^q}^{p}$, $x,y \in \mathbb{R}^n$ *is never of negative type when* $n \geq 3$, $0 < p \leq 2$ *and* $q > 2$. *Function* $(f,g) \mapsto ||f-g||_{1/H}^{\alpha}$ *is not of negative type for* $H < 1/2$.

3.2 Direct applications of Theorem 3.1

3.2.1 Euclidean spaces $(\mathbb{R}^n, ||.||)$, $n \geq 1$

Although it is not our goal, we have a look to the Euclidean spaces. One knows that, for $0 < \beta \leq 1$, functions $(x,y) \mapsto ||x-y||^{\beta}$ $x,y \in \mathbb{R}^n$ are measure definite kernels. This is known as Chentsov's construction ($\beta = 1$) [2] and Takenaka's construction [15] ($0 < \beta < 1$), see [13, p. 400-402] for a general presentation. Let us briefly describe these two constructions.

- Chentsov's construction ($\beta = 1$).

  For any hyperplane h of $\mathbb{R}^n$, let r be the distance of h to the origin of $\mathbb{R}^n$ and let $s \in S_{n-1}$ be the unit vector orthogonal to h. The hyperplane h is parametrized by the pair (s,r). Let $\mathbf{H}$ be the set of all hyperplanes that do not contain the origin. Let $\sigma(\mathbf{H})$ be the Borel σ-field. Let $\mu(ds,dr) = dsdr$, where ds denotes the uniform measure on S_{n-1} and dr the Lebesgue measure on $\mathbb{R}$. Let H_M be the set of all hyperplanes separating the origin and the point M. Then, there exists a constant $c > 0$ such that

  $$||MN|| = c\mu(H_M \Delta H_N) .$$

- Takenaka's construction ($0 < \beta < 1$).

  A hypersphere in $\mathbb{R}^n$ is parametrized by a pair (x, λ), where $x \in \mathbb{R}^n$ is its center and $\lambda \in \mathbb{R}^+$ its radius. Let $\mathbf{H}$ be the set of all hyperspheres in $\mathbb{R}^n$. Let $\sigma(\mathbf{H})$ be the Borel σ-field. μ_β is the measure $\mu_\beta(dx, d\lambda) = \lambda^{\beta-n-1} dx d\lambda$. Let H_M be the set of hyperspheres separating the origin and the point M. Then, there exists a constant $c_\beta > 0$ such that

$$||MN||^\beta = c_\beta \mu_\beta(H_M \Delta H_N) .$$

Theorem 3.1 can therefore be applied with $\psi = ||.||^\beta$, $0 < \beta \leq 1$. This leads βH-fractional α-stable fields for any $0 < \beta \leq 1$ and any H providing $1/2 < H \leq 1/\alpha$. The range of feasible parameters is therefore $0 < H \leq 1/\alpha$.

3.2.2 Spheres S_n

When [9] introduces the Spherical Brownian Motion, he proves that the geodesic distance d is a measure definite kernel. Indeed, for any point M on the unit sphere, define a half-sphere by:

$$H_M = \{N \in S_n,\ d(M, N) \leq \pi/2\} .$$

Let ds be the uniform measure on S_n, let ω_n be the surface of the sphere, and define the measure μ by:

$$\mu(ds) = \frac{\pi}{\omega_n} ds .$$

Then:

$$d(M, N) = \mu(H_M \Delta H_N) .$$

Theorem 3.1 can be applied with $\psi = d$. This leads to H-fractional α-stable fields for any H providing $1/2 \leq H \leq 1/\alpha$.

3.2.3 Hyperbolic spaces H_n

The geodesic distance is a measure definite kernel [16, 11]. The proof is more technical and we give only a rough outline. H_n is considered as a subset of the real projective space $P^n(\mathbb{R})$. Let H_M be the set of hyperplanes that separates M and the origin 0 of H_n in $P^n(\mathbb{R}) - l_\infty$. Let μ be a measure on H_n invariant under the action of the Lorentz group. Then, up to a normalizing constant, the geodesic distance d can be written as:

$$d(M, N) = \mu(H_M \Delta H_N) .$$

Theorem 3.1 can be applied with $\psi = d$. This leads to H-fractional α-stable fields for any H providing $1/2 \leq H \leq 1/\alpha$.

3.2.4 Real trees

Let us shortly explain the construction given in [17]. Fix O in the tree T. Set H_M be the geodesic path between O and M. Then $d(M, N) = \mu(H_M \Delta H_N)$ where μ is the Valette's measure of the tree: distance d is a measure definite kernel.
Theorem 3.1 can be applied with $\psi = d$. This leads to H-fractional α-stable fields for any H providing $1/2 \leq H \leq 1/\alpha$.

4 Space with countable dense subspace

We will now extend the result of Theorem 3.1.
Assume that E contains a countable dense subset Γ and that distance d is a measure definite kernel. A measure definite kernel is always of negative type [12, Prop. 1.1]. It follows from Lemma 2.1 that $(M,N) \mapsto d^{\beta}(M,N),\ M,N \in E,\ 0<\beta\leq 1$, is of negative type. [12, Prop. 1.4] proves that the square root of a function of negative type defined on a countable space is a measure definite kernel. It follows that $(M,N) \mapsto d^{\beta/2}(M,N),\ M,N \in \Gamma,\ 0<\beta\leq 1$, is a measure definite kernel since Γ is countable. Theorem 3.1 can be applied with $\psi(M,N) = d^{\beta/2}(M,N),\ M,N \in \Gamma,\ 0<\beta\leq 1$: we have build a field $X(M), M \in \Gamma$. Since this field is α-stable, it has finite moments of order $0<\alpha'<\alpha$ and there exists a constant $c>0$ (cf. [13, Prop. 1.2.17]) such that, for all $N, N' \in \Gamma$:

$$\mathbb{E}|X(N)-X(N')|^{\alpha'} \quad = \quad c d^{\alpha'\beta H/2}(N,N') \ . \tag{5}$$

Let $M \in E-\Gamma$ and let $N \to M,\ N \in \Gamma$. From (5), one can define $X(M)$ as:

$$X(M) \quad \stackrel{\mathbb{P}}{=} \quad \lim_{N \to M, N \in \Gamma} X(N) \ .$$

We have therefore build an α-stable field $X(M), M \in E$ satisfying:

$$\frac{X(M)-X(N)}{d^{\beta H/2}(M,N)} \quad \stackrel{\mathcal{D}}{=} \quad S_\alpha \quad \forall M,N \in E \ ,$$

where S_α is a standard symmetric α-stable random variable.

Let us now apply this construction to the spheres and hyperbolic spaces with their geodesic distances. We build $\beta H/2$-fractional α-stable fields with any $0<\beta\leq 1$, $1/2 \leq H \leq 1/\alpha$.
Let us summarize this construction with the previous construction of sections 3.2.2 and 3.2.3. We are able to build H-fractional α-stable fields in the following cases:

- when $\alpha \leq 1$, with any $0 < H \leq 1/\alpha$; and one knows from Proposition 2.1 that $H > 1/\alpha$ is forbidden;
- when $1<\alpha<2$, with any $0<H\leq 1/(2\alpha)$ and $1/2 \leq H \leq 1/\alpha$; and one knows from Proposition 2.1 that $H>1/\alpha$ is still forbidden; the interval $(1/(2\alpha), 1/2)$ is “missing”.

Remark 4.1
One doesn't know if d^γ is a measure definite kernel for $1/2<\gamma<1$. This is the reason of the “missing” interval $(1/(2\alpha), 1/2)$.

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