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A LOG-SCALE LIMIT THEOREM FOR ONE-DIMENSIONAL RANDOM WALKS IN RANDOM ENVIRONMENTS

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Abstract
We consider a transient one-dimensional random walk X_n in random environment having zero asymptotic speed. For a class of non-i.i.d. environments we show that $\log X_n / \log n$ converges in probability to a positive constant.

1 Introduction

In this note we consider a one-dimensional random walk X_n in ergodic environments and prove a log-scale result which is consistent with limit theorems known for i.i.d. and Markovian environments, namely that $\log X_n / \log n$ converges in probability to a positive constant.
The environment is a stationary and ergodic sequence indexed by the sites of $\mathbb{Z}$ which represents the probabilities that a nearest-neighbour random walk X_n moves to the right. The (annealed) law of X_n is obtained by averaging its quenched law (i.e. given a fixed environment) over the set of environments.
More formally, let $\Omega = (0,1)^{\mathbb{Z}}$, $\mathcal{F}$ be its Borel $\sigma-$algebra, and P be a stationary and ergodic probability measure on $(\Omega, \mathcal{F})$. The set Ω serves as the set of *environments* for a random walk X_n on $\mathbb{Z}$. The *random walk in the environment* $\omega = (\omega_i)_{i \in \mathbb{Z}} \in \Omega$ is the time-homogeneous Markov chain $X = (X_n)_{n \in \mathbb{N}}$ taking values on $\mathbb{Z}$ and governed by the *quenched* law

$$P_\omega(X_0 = 0) = 1 \quad \text{and} \quad P_\omega(X_{n+1} = j | X_n = i) = \begin{cases} \omega_i & \text{if } j = i+1, \\ 1 - \omega_i & \text{if } j = i-1. \end{cases}$$

Let $\mathcal{G}$ be the cylinder $\sigma-$algebra on $\mathbb{Z}^{\mathbb{N}}$, the path space of the walk. *The random walk in random environment* (RWRE) is the process (ω, X) on the measurable space $\left(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathcal{F} \times \mathcal{G}\right)$ with the *annealed* law $\mathbb{P} = P \otimes P_\omega$ defined by

$$\mathbb{P}(F \times G) = \int_F P_\omega(G) P(d\omega) = E_P\left(P_\omega(G); F\right), \quad F \in \mathcal{F},\ G \in \mathcal{G}.$$

We refer the reader to [Ze04, Section 2] for a detailed exposition of the one-dimensional model. Let $\rho_n = (1-\omega_n)/\omega_n$, and

$$R(\omega) = 1 + \sum_{n=0}^{+\infty} \rho_0\rho_{-1}\cdots\rho_{-n}. \tag{1}$$

If $E_P(\log\rho_0) < 0$ then (Solomon [So75] for i.i.d. environments and Alili [Al99] in the general case) $\lim_{n\to\infty} \mathbb{P}(X_n = +\infty) = 1$ and

$$\mathrm{v}_P := \lim_{n\to+\infty} \frac{X_n}{n} = \frac{1}{2E_P(R)-1}, \ \mathbb{P}-\text{a.s.} \tag{2}$$

The role played by the process $V_n = \sum_{i=0}^{n} \log\rho_{-i} = \log\big(\rho_0\rho_{-1}\cdots\rho_{-n}\big)$ in the theory of one-dimensional RWRE stems from the explicit form of harmonic functions (cf. [Ze04, Section 2.1]), which allows one to relate hitting times of the RWRE to those associated with the random walk V_n.
The assumptions we impose below will imply that

$$E_P(\log\rho_0) < 0 \quad \text{and} \quad E_P(R) = \infty, \tag{3}$$

that is the random walk is transient to the right but $\mathrm{v}_P = 0$.
The limit theorem of Kesten, Kozlov, and Spitzer [KKS75] states that if P is a product measure and $E_P(\rho_0^\kappa) = 1$ for some positive κ, then, under some additional technical conditions, an appropriately normalized sequence X_n converges to a non-degenerate (scaled stable) limit law. In particular, if $\kappa \in (0,1)$, then $n^{-\kappa}X_n$ converges in law and thus $\log X_n/\log n$ converges to κ in probability.
The limit laws established in [KKS75] were extended to Markov-dependent environments in [MwRZ04], where the condition $E_P(\rho_0^\kappa) = 1$ is replaced by:

$$\lim_{n\to\infty} \frac{1}{n} \log E_P\left(\prod_{i=0}^{n-1} \rho_i^\kappa\right) = 0 \quad \text{for some } \kappa > 0. \tag{4}$$

The key element of the proofs of the limit laws in [KKS75] and [MwRZ04] is that the distribution tail of the random variable R is slowly varying. In fact, under assumptions imposed on the environment in [KKS75] and [MwRZ04], the limit $\lim_{t\to\infty} t^\kappa P(R>t)$ exists and is strictly positive for κ given by (4). The existence of the limit is closely related to a renewal theory for the random walk $V_n = \sum_{i=0}^{n-1} \log\rho_{-i}$ (cf. [Ke73], see also [Go91] for an alternative proof and [Sa04] and [MwRZ04, Section 2.3] for Markovian extensions), and thus the lack of an adequate renewal theory for dependent random variables puts limitation on the scope of this approach to limit laws for RWRE.
In this paper, assuming that the environment is strongly mixing, the sequence V_n/n satisfies the Large Deviation Principle (LDP), and (4) holds for some $\kappa \in (0,1)$, we prove that $\log X_n$ is asymptotic to $\kappa\log n$ by using a different method, based on a study of the generating function of an associated branching process and a large deviation analysis of the tail.
We assume that the following holds for the environment ω :

Assumption 1.1. *$\omega = (\omega_n)_{n\in\mathbb{Z}}$ is a stationary sequence such that:*

(A1) The series $\sum_{n=1}^{\infty} n^\xi \cdot \eta(n)$ converges for any $\xi > 0$, where the strong mixing coefficients $\eta(n)$ are defined by

$$\eta(n) := \sup\{|P(A\cap B) - P(A)P(B)| : A \in \sigma(\omega_i : i \le 0),\ B \in \sigma(\omega_i : i \ge n)\}.$$

(A2) The process $\overline{V}_n = n^{-1}\sum_{i=0}^{n-1}\log\rho_i$ *satisfies the LDP with a good rate function* $J : \mathbb{R} \to [0,\infty]$, *that is the level sets* $J^{-1}[0,a]$ *are compact for any* $a > 0$, *and for any Borel set* A,

$$-J(A^o) \le \liminf_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\overline{V}_n \in A) \le \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}(\overline{V}_n \in A) \le -J(A^c),$$

where A^o *denotes the interior of* A, A^c *the closure of* A, *and* $J(A) := \inf_{x\in A} J(x)$.

(A3) There exist constants $\lambda_1 \in (0,1)$ *and* $\lambda_2 \in (\lambda_1, 1]$ *such that* $\Lambda^*(\lambda_1) < 0$ *and* $\Lambda^*(\lambda_2) > 0$, *where we denote* $\Lambda^*(\lambda) := \limsup_{n\to\infty}\frac{1}{n}\log E_P\left(\prod_{i=0}^{n-1}\rho_i^\lambda\right)$.

Assumption 1.1 is a modification of a condition which was introduced in the context of RWRE in [Ze04, Sect. 2.4]. For conditions on Markov and general stationary processes implying Assumption 1.1 we refer to [BD96, DZ98, GV93] and references therein. In particular, all bounded, stationary and strongly mixing sequences $(\rho_n)_{n\in\mathbb{Z}}$ with a fast mixing rate (namely $\frac{\log\eta(n)}{n(\log n)^{1+\delta}} \to -\infty$ for some $\delta > 0$) do satisfy *(A2)* (cf. [BD96]). Note that by Jensen's inequality, $E_P(\log\rho_0) \le \Lambda(\lambda_1)/\lambda_1 < 0$ and $\Lambda^*(1) \ge \Lambda^*(\lambda_2) > 0$, and hence (3) holds.
Assumptions *(A2)* and *(A3)* ensure by Varadhan's lemma [DZ98, p. 137] that for any $\lambda \in (0,\lambda_2)$, the limit $\Lambda(\lambda) := \lim_{n\to\infty}\frac{1}{n}\log E_P\left(\prod_{i=0}^{n-1}\rho_i^\lambda\right)$ exists and satisfies

$$\Lambda(\lambda) = \sup_x\{\lambda x - J(x)\}, \qquad \lambda \in (0,\lambda_2). \tag{5}$$

Moreover, since $\Lambda^*(\lambda)$ is a convex function with $\Lambda^*(0) = 0, \Lambda^*(\lambda_2) > 0$, and taking both positive and negative values, there exists $\kappa \in (\lambda_1,\lambda_2)$ such that

$$\Lambda(\lambda) \text{ has the same sign as } \lambda - \kappa \text{ for each } \lambda \in (0,\lambda_2). \tag{6}$$

Following [Ze04, Sect 2.4], the value of the parameter κ is identified in terms of the rate function J in the following lemma, whose proof is deferred to the appendix.

Lemma 1.2. *Suppose that Assumption 1.1 holds. Then*

$$\kappa = \min_{y>0} J(y)/y. \tag{7}$$

Our main result is:

Theorem 1.3. *Suppose that Assumption 1.1 holds. Then,*

$$n^{-\alpha}X_n \overset{\mathbb{P}}{\Rightarrow} \begin{cases}\infty & \text{if } \alpha \in (0,\kappa),\\ 0 & \text{if } \alpha > \kappa,\end{cases}$$

where κ *is defined in* (7).

The following corollary is immediate from the theorem.

Corollary 1.4. $\log X_n/\log n \overset{\mathbb{P}}{\Rightarrow} \kappa$.

Theorem 1.3 is deduced from a similar result for the hitting time T_n, defined by

$$T_n = \min\{i : X_i = n\}, \quad n \in \mathbb{N}. \tag{8}$$

We have:

Proposition 1.5. *Suppose that Assumption 1.1 holds. Then,*

$$n^{-1/\alpha}T_n \overset{\mathbb{P}}{\Rightarrow} \begin{cases} 0 & \text{if } \alpha \in (0,\kappa), \\ \infty & \text{if } \alpha > \kappa, \end{cases}$$

where κ is defined in (7).

The proof of Proposition 1.5, included in Section 2, is based on the study of generating functions of a branching process Z_n, defined therein. Once this proposition is obtained, Theorem 1.3 is derived from it as follows (cf. [KKS75]). For any positive integers η, ζ, n,

$$\{T_\zeta \geq n\} \subset \{X_n \leq \zeta\} \subset \{T_{\zeta+\eta} \geq n\} \bigcup \{\inf_{i \geq T_{\zeta+\eta}} X_i - (\zeta+\eta) \leq -\eta\}.$$

Since the random variables $\inf_{i\geq T_{\zeta+\eta}} X_i - (\zeta+\eta)$ and $\inf_{i\geq 0} X_k$ have the same annealed distribution,

$$\mathbb{P}(T_\zeta \geq n) \leq \mathbb{P}(X_n \leq \zeta) \leq P(T_{\zeta+\eta} \geq n) + \mathbb{P}(\inf_{i\geq 0} X_i \leq -\eta). \tag{9}$$

Fix any numbers $\eta \in \mathbb{N}$, $x > 0, \alpha \in (0,\kappa) \cup (\kappa, \infty)$, and let $\zeta(n) = [xn^\alpha]$, where $[t]$ denotes the integer part of t. It follows from (9) that

$$\begin{aligned} \limsup_{n\to\infty} P(n^{-\alpha}X_n \leq x) &\leq \limsup_{n\to\infty} P(X_n \leq \zeta(n)+1) \leq \liminf_{n\to\infty} P(T_{\zeta(n)+1+\eta} \geq n) \\ &+\mathbb{P}(\inf_{i\geq 0} X_i \leq -\eta) = \lim_{\zeta\to\infty} P(T_\zeta \geq \zeta^{1/\alpha}x^{-1/\alpha}) + \mathbb{P}(\inf_{i\geq 0} X_i \leq -\eta). \end{aligned}$$

Similarly,

$$\begin{aligned} \liminf_{n\to\infty} P(n^{-\alpha}X_n \leq x) &\geq \liminf_{n\to\infty} P(X_n \leq \zeta(n)) \geq \limsup_{n\to\infty} P(T_{\zeta(n)} \geq n) \\ &= \lim_{\zeta\to\infty} P(T_\zeta \geq \zeta^{1/\alpha}x^{-1/\alpha}). \end{aligned}$$

Since X_n is transient to the right, $\mathbb{P}(\inf_{i\geq 0} X_i \leq -\eta)$ can be made arbitrary small by fixing η large. Thus,

$$\lim_{n\to\infty} P(n^{-\alpha}X_n \leq x) = \lim_{\zeta\to\infty} P(T_\zeta \geq \zeta^{1/\alpha}x^{-1/\alpha}) = \begin{cases} 0 & \text{if } \alpha < \kappa \\ 1 & \text{if } \alpha > \kappa. \end{cases}$$

The next section is devoted to the proof of Proposition 1.5.

2 Proof of Proposition 1.5

First, we consider a branching process Z_n in random environment, closely related to the RWRE. The hitting times T_n of the random walk are associated by (10) and (11) below to the partial sums of the branching process and Proposition 1.5 is then derived from a corresponding scale result for $\sum_{i=1}^n Z_i$ (see Proposition 2.2).
The branching process was introduced in the context of the RWRE in [Ko73], and has been exploited in several works including [KKS75] (a detailed description of its construction can be found e.g. in [GS02]). Let

$$U_i^n = \#\{t < T_n : X_t = i,\ X_{t+1} = i-1\}, \quad n \in \mathbb{N},\ i \in \mathbb{Z},$$

the number of moves to the left at site i up to time T_n. Then

$$T_n = n + 2 \sum_{i=-\infty}^{n} U_i^n. \tag{10}$$

When $U_n^n = 0, U_{n-1}^n, \ldots, U_{n-i+1}^n$ and $\omega_n, \omega_{n-1} \ldots, \omega_{n-i}$ are given, U_{n-i}^n is the sum of $U_{n-i+1}^n +$ 1 i.i.d. geometric random variables that take the value k with probability $\omega_{n-i}(1-\omega_{n-i})^k$, $k = 0, 1, \ldots$ Since X_n is transient to the right we have:

$$\sum_{i \le 0} U_i^n < \infty, \quad \mathbb{P} - \text{a.s.} \tag{11}$$

Therefore, since $\kappa < 1$, in order to prove Proposition 1.5 it is sufficient to show that the corresponding result holds for the sums $\sum_{i=1}^n U_i^n$. These sums have the same distribution as

$$\sum_{i=0}^{n-1} Z_i,$$

where $Z_0 = 0, Z_1, Z_2, \ldots$ forms a branching process in random environment with one immigrant at each unit of time. Without loss of generality, we shall assume that the underlying probability space is extended to fit not only the random walk but also the branching process. Thus, when ω and $Z_0, \ldots, Z_n$ are given, Z_{n+1} is the sum of $Z_n + 1$ independent variables $V_{n,0}, V_{n,1}, \ldots, V_{n,Z_n}$ each having the geometric distribution

$$P_\omega(V_{n,i} = j) = \omega_{-n}(1-\omega_{-n})^j, \quad j = 0, 1, 2, \ldots \tag{12}$$

For $s \in [0,1]$ define random variables dependent on the environment ω :

$$\begin{aligned} n \in \mathbb{Z}: \quad f_n(s) &= E_\omega\left(s^{V_{n,0}}\right) = \frac{1}{1+\rho_{-n}(1-s)}, \\ n \ge 1: \quad \psi_n(s) &= E_\omega\left(s^{\sum_{i=1}^n Z_i}\right). \end{aligned}$$

Iterating, and taking in account that the variables $V_{n,i}$ are conditionally independent given Z_n and ω, we obtain:

$$\begin{aligned} \psi_{n+1}(s) &= E_\omega\left(E_\omega\left(s^{\sum_{i=1}^{n+1} Z_i} | Z_1, Z_2, \ldots, Z_n\right)\right) = E_\omega\left(s^{\sum_{i=1}^n Z_i} E_\omega\left(s^{\sum_{j=0}^{Z_n} V_{n,j}} | Z_n\right)\right) = \\ &= E_\omega\left(s^{\sum_{i=1}^n Z_i} f_n(s)^{Z_n+1}\right) = f_n(s) E_\omega\left(s^{\sum_{i=1}^{n-1} Z_i} \left(s f_n(s)\right)^{Z_n}\right) = \\ &= f_n(s) \cdot f_{n-1}(s f_n(s)) \cdot \ldots \cdot f_0(s f_1(s \ldots f_{n-1}(s f_n(s)) \ldots)). \end{aligned}$$

Thus the random variable $\psi_{n+1}(s)$ is distributed the same as

$$\varphi_{n+1}(s) = f_0(s) \cdot f_{-1}(s f_0(s)) \cdot \ldots \cdot f_{-n}(s f_{-n+1}(s \ldots f_{-1}(s f_0(s)) \ldots)). \tag{13}$$

Lemma 2.1. *For $n \ge 1$, $\varphi_n(s) = 1/B_n(s)$, where $B_n(s)$ are polynomials of s satisfying the following recursion formula*

$$B_{n+1}(s) = (1+\rho_n) B_n(s) - s\rho_n B_{n-1}(s) \tag{14}$$

and the initial conditions $B_0(s) = 1,\ B_1(s) = 1 + \rho_0(1-s)$.

Proof. For $n \geq 0$ let $q_n(s) = f_{-n}(sf_{-n+1}(s \dots f_{-1}(sf_0(s)) \dots))$. In particular, we have $q_0(s) = f_0(s) = B_0(s)/B_1(s)$. By induction, $q_n(s) = B_n(s)/B_{n+1}(s) \in (0,1]$ for all $n \in \mathbb{N}$. Indeed, assuming that for some $n \geq 0$, $q_{n-1} = B_{n-1}/B_n$, we obtain:

$$q_n = f_{-n}(sq_{n-1}) = \frac{1}{1+\rho_n\left(1 - s\dfrac{B_{n-1}}{B_n}\right)} = \frac{B_n}{(1+\rho_n)B_n - s\rho_n B_{n-1}} = \frac{B_n}{B_{n+1}}.$$

It follows from (13) that

$$\varphi_n(s) = \prod_{j=0}^{n-1} q_j(s) = \frac{1}{B_1(s)} \cdot \frac{B_1(s)}{B_2(s)} \cdot \ldots \cdot \frac{B_{n-1}(s)}{B_n(s)} = \frac{1}{B_n(s)},$$

completing the proof. □

Rewriting (14) in the form $B_{n+1} - B_n = \rho_n(B_n - B_{n-1}) + \rho_n(1-s)B_{n-1}$ and iterating, we obtain another, useful in the sequel, form of the recursion:

$$B_{n+1}(s) = B_n(s) + (1-s) \cdot \sum_{i=0}^{n} B_{i-1}(s) \prod_{j=i}^{n} \rho_j \quad \text{for } n \geq 0, \tag{15}$$

with the convention that $B_{-1}(s) = 1$ and a summation over an empty set is null. In particular, $\big(B_n(s)\big)_{n\geq 0}$ is a strictly increasing sequence for a fixed $s \in [0,1)$.

Proposition 2.2. *Suppose that Assumption 1.1 holds. Then,*

$$n^{-1/\alpha} \sum_{i=1}^{n} Z_i \overset{\mathbb{P}}{\Rightarrow} \begin{cases} 0 & \text{if } \alpha \in (0,\kappa), \\ \infty & \text{if } \alpha > \kappa, \end{cases} \tag{16}$$

where κ is defined in (7).

Proof. First, suppose that $0 < \alpha < \kappa$. For $m \geq 0$ let

$$\widetilde{H}_m(\omega) \quad = \quad \rho_{-m} + \rho_{-m}\rho_{-m+1} + \ldots + \rho_{-m}\rho_{-m+1} \cdots \rho_0.$$

Choose any $\beta \in (\alpha, \kappa)$. By Chebyshev's inequality, we obtain that for any $x > 0$,

$$\mathbb{P}\left(\frac{1}{n^{1/\alpha}} \sum_{i=1}^{n} Z_i > x\right) \leq \frac{1}{x^\beta n^{\beta/\alpha}} \mathbb{E}\left(\left(\sum_{i=1}^{n} Z_i\right)^\beta\right). \tag{17}$$

Using the inequality

$$\left(\sum_{i=1}^{n} a_i\right)^\beta \leq \sum_{i=1}^{n} a_i^\beta, \qquad a_i \geq 0,\ \beta \in [0,1],\ n \in \mathbb{N},$$

and Jensen's inequality $E_\omega\big(Z_i^\beta\big) \le \big(E_\omega(Z_i)\big)^\beta$, we obtain

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{i=1}^n Z_i\right)^\beta\right) &\le \mathbb{E}\left(\sum_{i=1}^n Z_i^\beta\right) = \sum_{i=1}^n E_P\left(E_\omega\left(Z_i^\beta\right)\right) \le \sum_{i=1}^n E_P\left(\big(E_\omega(Z_i)\big)^\beta\right) \\
&= \sum_{i=1}^n E_P\big(\widetilde{H}_{i-1}^\beta\big) \le \sum_{i=1}^n \sum_{j=0}^{i-1} E_P\left(\rho_{-i+1}^\beta \rho_{-i+2}^\beta \cdots \rho_{-i+1+j}^\beta\right) \\
&\le \sum_{i=1}^\infty \sum_{j=0}^{i-1} E_P\left(\rho_{-i+1}^\beta \rho_{-i+2}^\beta \cdots \rho_{-i+1+j}^\beta\right) < \infty,
\end{aligned}$$

where the last step is due to (6). Thus the claim for $\alpha \in (0,\kappa)$ follows from (17).

Now suppose that $\alpha > \kappa$. Fix a real number $\lambda > 0$ and let $s_n = e^{-\lambda/n^{1/\alpha}}$. In order to prove that (16) holds for $\alpha > \kappa$, it is sufficient to show that

$$P\left(\lim_{n\to\infty} B_n(s_n) = \infty\right) = 1. \tag{18}$$

Indeed, using Chebyshev's inequality and Lemma 2.1 we obtain for any $M > 0$,

$$\begin{aligned}
\mathbb{P}\left(n^{-1/\alpha}\sum_{i=1}^n Z_i \le M\right) &\le e^{\lambda M}\mathbb{E}\left(e^{-\lambda n^{-1/\alpha}\sum_{i=1}^n Z_i}\right) = e^{\lambda M} E_P\big(\psi(s_n)\big) \\
&= e^{\lambda M} E_P\big(1/B_n(s_n)\big).
\end{aligned}$$

Therefore (16) follows from (18) by the bounded convergence theorem.
We will next prove that for any $M \in \mathbb{R}$, $P\big(B_n(s_n) < M \text{ i.o.}\big) = 0$, implying that (18) holds. For $m \ge 0$ let

$$H_m(\omega) = \rho_m + \rho_m\rho_{m-1} + \ldots + \rho_m\rho_{m-1}\cdots\rho_0.$$

If $B_n(s_n) < M$, then $1 \le B_0(s_n) \le B_1(s_n) \le \ldots \le B_n(s_n) < M$. It follows from (15) that for $m \le n$,

$$1 = \frac{B_{m-1}(s_n)}{B_m(s_n)} + (1-s_n)\sum_{i=0}^{m-1}\frac{B_{i-1}(s_n)}{B_m(s_n)}\prod_{j=i}^{m-1}\rho_j \ge \frac{B_{m-1}(s_n)}{B_m(s_n)} + \frac{1}{M}(1-s_n)H_{m-1}.$$

For all n large enough, $1 - s_n = 1 - e^{-\lambda/n^{1/\alpha}} \ge \lambda/2 \cdot 1/n^{1/\alpha}$ and hence $B_n(s_n) < M$ implies that $H_m \le 2M\lambda^{-1}n^{1/\alpha}$ for $m = 0,1,\ldots,n-1$. We conclude that if $\kappa < \gamma < \alpha$ and n large enough, then $B_n(s_n) < M$ implies

$$H_m \le n^{1/\gamma}, \qquad m = 0,1,\ldots,n-1. \tag{19}$$

Let $K_n = [1/\delta \cdot \log n]$, where $[\,\cdot\,]$ denotes the integer part of a number and $\delta > 0$ is a parameter whose precise value will be determined later. Further, let $\varepsilon = \frac{1}{3}\left(1 - \kappa/\gamma\right)$, $L_n = K_n + n^\varepsilon$ and

$$G_{j,n} = \prod_{i=(j-1)L_n+1}^{(j-1)L_n+K_n} \rho_i, \quad j = 1,2,\ldots[n/L_n].$$

That is $G_{j,n}$ is a product of variables ρ_i with indexes i coming from a block of the length K_n, and two successive such blocks are separated by the distance n^ε. From the LDP for $\overline{V}_n = \frac{1}{n}\sum_{i=0}^{n-1} \log \rho_i$ we obtain that there is $N_{\varepsilon,\delta} > 0$ such that $n > N_{\varepsilon,\delta}$ implies

$$P\big(G_{1,n} > n^{1/\gamma}\big) > n^{-1/\delta \cdot J(\delta/\gamma)-\varepsilon}. \tag{20}$$

Since $G_{j,n} \le H_{(j-1)L_n+K_n}$ it follows from (19) and (20) that for n large enough,

$$\begin{aligned} P\big(B_n(s_n) < M\big) &\le P\Big(G_{j,n} < (n+1)^{1/\gamma}, \quad j = 1,2,\ldots[n/L_n]\Big) \le \\ &\le \Big(1 - n^{-1/\delta \cdot J(\delta/\gamma)-\varepsilon}\Big)^{[n/L_n]} + [n/L_n]\cdot \eta\left(n^\varepsilon\right), \end{aligned}$$

where $\eta(l)$ are the strong mixing coefficients. That is, for n large enough,

$$P(B_n(s_n) < M) \le \exp\left\{-\frac{1}{2}n^{1-1/\delta\cdot J(\delta/\gamma)-2\varepsilon}\right\} + n^{1-\varepsilon}\cdot \eta\left(n^\varepsilon\right). \tag{21}$$

Now, we will choose the parameter δ in such a way that $1 - 1/\delta \cdot J(\delta/\gamma) - 2\varepsilon = \varepsilon$. By (7), there exists $x^* > 0$ such that $J(x^*) = \kappa x^*$. Hence, letting $\delta = \gamma x^*$ we obtain,

$$1 - 1/\delta \cdot J(\delta/\gamma) - 2\varepsilon = 1 - \kappa/\gamma - 2\varepsilon = \varepsilon.$$

It follows from (21) that for n large enough,

$$P\big(B_n(s_n) < M\big) \le e^{-n^\varepsilon/2} + n^{1-\varepsilon}\cdot \eta\left(n^\varepsilon\right).$$

Thus, $P\big(B_n(s_n) < M \text{ i.o.}\big) = 0$ by the Borel-Cantelli lemma. □

Proposition 1.5 follows from (10), (11), and Proposition 2.2.

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Appendix. Proof of Lemma 1.2

First observe that by (5) and (6), $\sup_{x\in\mathbb{R}}\{\kappa x - J(x)\} = 0$. We next show that in fact

$$\sup_{x>0}\{\kappa x - J(x)\} = 0. \tag{22}$$

Indeed, since $J(x)$ is a nonnegative function, if there exists a sequence of constants $(x_n)_{n\in\mathbb{N}}$ such that $x_n \le 0$ for all n and $\lim_{n\to\infty}\big(\kappa x_n - J(x_n)\big) = 0$, we have both $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} J(x_n) = 0$. But $J(x)$ is a lower semicontinuous function and hence $\lim_{n\to\infty} J(x_n) \ge J(\lim_{n\to\infty} x_n) = J(0)$. Therefore, in order to prove that such a sequence $(x_n)_{n\ge 0}$ does not exist and thus (22) holds, it suffices to show that $J(0) > 0$. Toward this end observe that in virtue of (5), $J(x) \ge \sup_{\lambda\in(0,\lambda_2)}\{\lambda x - \Lambda(\lambda)\}$ and hence $J(0) \ge -\Lambda(\lambda_1) > 0$.

Let $s := \inf_{x>0} J(x)/x$. It follows from (22) that $s \geq \kappa$. Assume, in order to get a contradiction, that $s = \kappa + \varepsilon$ for some $\varepsilon \in (0, \infty]$. Then $\kappa x - J(x) \leq -\varepsilon x$ for any $x > 0$. Thus (22) is only possible if there exists a sequence $(x_n)_{n\in\mathbb{N}}$ such that $x_n > 0$ for all n, $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} J(x_n) = 0$. Since the rate function $J(x)$ is lower semicontinuous and $J(0) > 0$, such a sequence does not exist and hence $s = \kappa$.
It remains only to show that the infimum in the definition of s is in fact the minimum, i.e. $\kappa = \min_{x>0} J(x)/x$. It is sufficient to check that there exist two constants $\alpha > 0$ and $\beta \in (\alpha, \infty)$ such that $J(x)/x > (\kappa + \lambda_2)/2 > \kappa$ for $x \notin [\alpha, \beta]$. Indeed, in this case $\kappa = \inf_{x\in[\alpha,\beta]} J(x)/x$ and the minimum is achieved by the lower semicontinuous function $J(x)/x$ over the closed interval $[\alpha, \beta]$. By using (5), it follows that

$$J(x)/x \geq \lambda_1 - \Lambda(\lambda_1)/x > (\kappa + \lambda_2)/2 \quad \text{for} \quad x < -2\Lambda(\lambda_1)/(\kappa + \lambda_2 - 2\lambda_1),$$

and, letting $\lambda_3 = \kappa/2 + \lambda_2/2 + \varepsilon$ for some $\varepsilon \in (0, \lambda_2/2 - \kappa/2)$,

$$J(x)/x \geq \lambda_3 - \Lambda(\lambda_3)/x > (\kappa + \lambda_2)/2 \quad \text{for} \quad x > \Lambda(\lambda_3)/\varepsilon.$$

This completes the proof of the lemma. □

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