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POSITIVE CORRELATION FOR INCREASING EVENTS WITH DISJOINT DEPENDENCIES DOES NOT IMPLY POSITIVE CORRELATION FOR ALL INCREASING EVENTS

NICHOLAS WEININGER[1]
Department of Mathematics, Rutgers University, Piscataway, NJ 08854
email: `nweining@math.rutgers.edu`

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Abstract
A probability measure μ on the lattice $2^{[n]}$ is said to be positively associated if any two increasing functions on the lattice are positively correlated with respect to μ. Pemantle [3] asked whether, in order to establish positive association for a given μ, it might be sufficient to show positive correlation only for pairs of functions which depend on disjoint subsets of the ground set $[n]$. We answer Pemantle's question in the negative, by exhibiting a measure which gives positive correlation for pairs satisfying Pemantle's condition but not for general pairs of increasing functions.

A fundamental problem in discrete probability is: if μ is a measure on the lattice $2^{[n]}$ of subsets of an n-element ground set, under what conditions does μ have positive association? Here μ is said to have positive association if any two increasing functions are positively correlated with respect to μ– that is, if whenever $f, g : X \to \mathbb{R}^+$ are increasing functions, we have

$$\int fgd\mu \geq \int fd\mu \int gd\mu \tag{1}$$

This property is often called the FKG property, after the celebrated FKG inequality [2] which establishes a sufficient condition for μ to have positive association. (Note that, following a standard abuse, we use "increasing" to mean weakly increasing, i.e. nondecreasing).

Pemantle asked the following question in [3]: to establish that (1) holds for *all* pairs f, g, is it sufficient to assume (1) only for pairs f, g such that the values of f and g depend on disjoint subsets of the elements in the ground set? Here f and g are said to depend on disjoint subsets of the ground set if there exist $S, T \subseteq [n]$ such that $S \cap T = \emptyset$, and such that for any $A \subseteq [n]$, $f(A) = f(A \cap S)$ and $g(A) = g(A \cap T)$.

In this note we show that the answer to Pemantle's question is no. We exhibit a measure μ on a 4-element lattice such that (1) holds whenever f, g depend on disjoint subsets of the ground set, but there exists a more general pair of increasing f, g such that (1) fails.

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We first observe that, following a standard reduction technique (see e.g. [1]), it suffices to consider pairs f, g which are 0-1 valued, i.e. take f, g to be indicator functions of increasing events $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$. To see this, note that any increasing functions f, g can be decomposed into positive linear combinations of such indicator functions, and if (1) holds for all pairs of indicator functions in the decomposition then it holds for f, g as a whole. Furthermore, if f, g are dependent on disjoint subsets of the ground set, so are the indicator functions in their decompositions.

Thus it suffices to exhibit μ such that, first, we have

$$\mu(\mathcal{A} \cap \mathcal{B}) \geq \mu(\mathcal{A})\mu(\mathcal{B}) \tag{2}$$

whenever $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are increasing events and there exist $S, T \subseteq 2^{[n]}$ such that $S \cap T = \emptyset$, and membership of a subset $A \in 2^{[n]}$ in $\mathcal{A}$ (resp. $\mathcal{B}$) depends only on $A \cap S$ (resp. $A \cap T$); and, second, there exists a more general pair of increasing events $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ such that (2) fails to hold.

Let $\mu : 2^{[4]} \to [0, 1]$ be defined as follows:

$$\begin{aligned}
\mu(\{1,2,3,4\}) &= 0.6667 \\
\mu(\{1,2\}) &= 0.1 \\
\mu(\{1,3\}) &= 0.08 \\
\mu(\{2,4\}) &= 0.07 \\
\mu(\{3,4\}) &= 0.05 \\
\mu(\emptyset) &= 0.0333
\end{aligned}$$

The above probabilities add up to 1, so we set the probabilities of all other subsets of $\{1, 2, 3, 4\}$ to zero.

To verify that μ works as promised, first take $\mathcal{A}_0 = \{S \subseteq 2^{[n]} : \{1, 2\} \subseteq S \vee \{3, 4\} \subseteq S\}$, $\mathcal{B}_0 = \{S \subseteq 2^{[n]} : \{1, 3\} \subseteq S \vee \{2, 4\} \subseteq S\}$ and observe that $\mu(\mathcal{A}_0) = \mu(\mathcal{B}_0) = 0.8167$, so $\mu(\mathcal{A}_0)\mu(\mathcal{B}_0) \approx 0.667$ while $\mu(\mathcal{A}_0 \cap \mathcal{B}_0) = 0.6667$, so this pair $\mathcal{A}_0, \mathcal{B}_0$ violates (2).

It remains only to check (2) for pairs $\mathcal{A}, \mathcal{B}$ depending on disjoint subsets of $\{1, 2, 3, 4\}$. There are 185 distinct such pairs. We wrote a computer program that runs through all such pairs and checks that (2) holds for each. The minimum degree of positive correlation (apart from trivial pairs for which (2) holds with equality) is achieved by the events $\mathcal{A}_1 = \{S \subseteq 2^{[n]} : 2 \in S\}$ and $\mathcal{B}_1 = \{S \subseteq 2^{[n]} : 3 \in S\}$; here $\mu(\mathcal{A}_1)\mu(\mathcal{B}_1) = 0.8367 \cdot 0.7967 \approx 0.6666$ while $\mu(\mathcal{A}_1 \cap \mathcal{B}_1) = 0.6667$.

Finally, we note that many other choices of measure would likely have worked; indeed, the weights of the four two-element subsets above can probably be varied over a considerable range without affecting the required properties. If we denote these weights by b_1, b_2, b_3, b_4 in the order they are listed above, then requiring that $\mathcal{A}_0, \mathcal{B}_0$ above be negatively correlated while $\mathcal{A}_1, \mathcal{B}_1$ are positively correlated yields the constraint $(b_1 - b_2)(b_3 - b_4) > 0$. Requiring positive correlation for other pairs $\mathcal{A}, \mathcal{B}$ dependent on disjoint subsets of the variables yields different constraints on the b_i, all of which are satisfied when $b_1 > b_2 > b_3 > b_4$. We arrived at the measure μ above by more or less arbitrarily picking values for the b_i in that order, choosing the weight of $\{1, 2, 3, 4\}$ to give negative correlation for $\mathcal{A}_0, \mathcal{B}_0$ and positive correlation for several other pairs of events, and then checking by computer that all other required pairs were also positively correlated.

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References

[1] Doyle, P., Fishburn, P., and Shepp, L. (1988). "The Match Set of a Random Permutation has the FKG Property." *Annals of Probability* **16** 1194-1214.

[2] Fortuin, C., Kasteleyn, P., and Ginibre, J. (1971). "Correlation inequalities on some partially ordered sets." *Comm. Math. Phys.* **22**, 89-103.

[3] Pemantle, R. (2000). "Towards a theory of negative dependence." *J. Math. Phys.* **41**, 1371-1390.