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# UNIQUENESS OF MULTI-DIMENSIONAL INFINITE VOLUME SELF-ORGANIZED CRITICAL FOREST-FIRE MODELS

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*Abstract*
In a forest-fire model, each site of the square lattice is either vacant or occupied by a tree. Vacant sites get occupied according to independent rate 1 Poisson processes. Independently at each site ignition occurs according to independent rate lambda Poisson processes. When a site is hit by ignition, then its whole occupied cluster becomes vacant instantaneously.
The article studies whether a multi-dimensional infinite volume forest-fire process with given parameter is unique. Under an assumption on the decay of the cluster size distribution, a process that dominates the forest-fire process is used to show uniqueness. If lambda is big enough, then subcritical site percolation shows the correctness of the assumption.

## 1 Introduction

Let $d \geq 1$. In a forest-fire model on $\mathbb{Z}^d$, each site of the square lattice $\mathbb{Z}^d$ is either vacant or occupied by a tree. Vacant sites become occupied according to independent rate 1 Poisson processes, the growth processes. Independently, at each site lightning strikes according to independent rate $\lambda$ Poisson processes, the ignition processes. When an occupied site is hit by ignition, its entire occupied cluster burns down, that is, becomes vacant instantaneously. Here $\lambda > 0$ is the parameter of the model.
In the physics literature, within the study of self-organized criticality, usually a closely related forest-fire model is studied. In this model time is discrete, space is large but finite, and the fire spreads at finite speed. See [6] for current insights. A general overview of and introduction to self-organized criticality can be found in [5].
In [3], J. van den Berg and A. A. Jàrai study the asymptotic density in a forest-fire model on $\mathbb{Z}$. They show that regardless of the initial configuration, already after time of order $\log(1/\lambda)$ the density of vacant sites is of order $1/\log(1/\lambda)$. They also obtain bounds on the cluster size distribution. In [2], J. van den Berg and R. Brouwer let start forest-fire processes on $\mathbb{Z}^2$ with all sites vacant and study, for positive but small $\lambda$, the behavior near the 'critical time'

$t_c$; that is, the time after which in the modified system without lightning an infinite occupied cluster would emerge. They show that under a percolation-like assumption, that if, for fixed $t > t_c$, they let simultaneously $\lambda$ tend to 0 and $m$ to infinity, the probability that some tree at distance smaller than $m$ from 0 is burnt before time $t$, does not go to 1. In [4] the author shows that for all $d \in \mathbb{N}$, all real numbers $\lambda > 0$ and all initial configurations $\zeta$ that do not contain an infinite connected set of occupied sites, there exists a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda$ and initial configuration $\zeta$. The construction uses a diagonal sequence argument and in that paper, it was not examined whether the constructed process depends on the choice of the sequence of forest-fire processes on finite boxes, which is used within the construction.
Here the more general question is studied, whether a forest-fire process with given parameter that starts with all sites vacant is unique. In Section 4.1, we show that under an assumption on the decay of the cluster size distribution of a process that dominates the forest-fire process, the forest-fire process with given parameter and vacant initial configuration is unique, adapted to the filtration generated by its driving growth and ignition processes, and can be constructed in a very direct way. In Section 4.2 it is shown that the assumption is satisfied, provided that $\lambda$ is big enough. But simulations on the cluster size distribution support the belief that the assumption holds on a much greater parameter range, perhaps even for all $\lambda > 0$.
The central definitions and the main results can be found in Sections 2 and 3.

## 2 Definition of a forest-fire process

**Definition 1.** *For all $F \subseteq \mathbb{Z}^d$ and all $x, y \in \mathbb{Z}^d$, the relation $x \leftrightarrow_F y$ holds, if $x$ and $y$ are connected by a path in $F$, that is, if there exists a sequence $x = x_0, x_1, \ldots, x_n = y$ of distinct sites in $F$ s.t. for all $1 \le i \le n$, the relation $\|x_i - x_{i-1}\|_1 = 1$ holds.*
*Let $S \subseteq \mathbb{Z}^d$ and $(\eta_{t,x})_{t\ge 0, x\in S}$ be a process with values in $\{0,1\}^S$ whose left limits $(\eta_{t^-,x})_{t>0} := (\lim_{s\uparrow t} \eta_{s,x})_{t>0}$, $x \in S$, exist. For all $t \in \mathbb{R}^+$, we define $F_{t^-} := \{y \in S : \eta_{t^-,y} = 1\}$, and for all $x \in S$, the left limit of the cluster at $x$ at time $t$ to be*

$$C_{t^-,x} := \Big\{ y \in S : \; x \leftrightarrow_{F_{t^-}} y \Big\}.$$

We consider the following forest-fire model where the possible locations of trees are the sites of a subset of the lattice $\mathbb{Z}^d$. Each site has two possible states: 'vacant' or 'occupied'. Vacant sites become occupied (growth of a tree) according to independent rate 1 Poisson processes. Independently, ignition (by lightning) occurs at each site according to independent rate $\lambda$ Poisson processes. When a site is hit by ignition, its entire occupied cluster burns down, that is, becomes vacant instantaneously.

**Definition 2** (Definition of a forest-fire process)**.** *Let $S \subseteq \mathbb{Z}^d$ and $\lambda \in \mathbb{R}^+$. A forest-fire process on $S$ with parameter $\lambda$ is a process $\bar{\eta}_t = \big(\bar{\eta}_{t,x}\big)_{x\in S} = \big(\eta_{t,x},\ G_{t,x},\ I_{t,x}\big)_{x\in S}$ with values in $\big(\{0,1\} \times \mathbb{N}_0 \times \mathbb{N}_0\big)^S$, $t \ge 0$, that has the following properties:*

*(a) The processes $(G_{t,x})_{t\ge 0}$ and $(I_{t,x})_{t\ge 0}$, $x \in S$, are independent Poisson processes with parameter 1 and $\lambda$, respectively;*

*(b) For all $x \in S$, the process $(\eta_{t,x},\ G_{t,x},\ I_{t,x})_{t\ge 0}$ is càdlàg, i.e., right-continuous with left limits.*

(c) *For all $t \in \mathbb{R}_0^+$, the increments of the growth and ignition processes after time $t$, $(G_{t+s,x} - G_{t,x}, I_{t+s,x} - I_{t,x})_{s\geq 0, x\in S}$, are independent of the forest-fire process up to time $t$, $(\bar{\eta}_s)_{0\leq s\leq t}$;*

(d) *For all $x \in S$ and all $t > 0$,*

- $\lim_{s\uparrow t} G_{s,x} < G_{t,x} \Rightarrow \eta_{t,x} = 1$;
  *(Growth of a tree at the site $x$ at time $t \Rightarrow$ The site $x$ is occupied at time $t$)*
- $\lim_{s\uparrow t} \eta_{s,x} < \eta_{t,x} \Rightarrow \lim_{s\uparrow t} G_{s,x} < G_{t,x}$;
  *(The site $x$ gets occupied at time $t \Rightarrow$ Growth of a tree at the site $x$ at time $t$)*
- $\lim_{s\uparrow t} I_{s,x} < I_{t,x} \Rightarrow \forall y \in C_{t^-,x} : \ \eta_{t,y} = 0$;
  *(Ignition at the site $x$ at time $t \Rightarrow$ All sites of the cluster at $x$ get vacant at time $t$)*
- $\lim_{s\uparrow t} \eta_{s,x} > \eta_{t,x} \Rightarrow \exists y \in C_{t^-,x} : \ \lim_{s\uparrow t} I_{s,x} < I_{t,y}$.
  *(The site $x$ gets vacant at time $t \Rightarrow$ The cluster at $x$ must be hit by ignition at time $t$)*

*We call $(G_{t,x})_{t\geq 0}$ the growth process, $(I_{t,x})_{t\geq 0}$ the ignition process and $(\eta_{t,x})_{t\geq 0}$ the forest-fire process at the site $x \in S$. We say that the site $x \in S$ is occupied at time $t$, if $\eta_{t,x} = 1$ holds, and vacant, if $\eta_{t,x} = 0$ holds.*
*$\zeta := (\eta_{0,x})_{x\in S}$ is called the initial configuration of the process. We say that the process has vacant initial configuration, if for all $x \in S$, the relation $\eta_{0,x} = 0$ holds. I.e., if at time 0 all sites are vacant.*

**Remark 1.** In [4] it is shown that for all $d \in \mathbb{N}$, all real numbers $\lambda > 0$ and all initial configurations $\zeta$ that do not contain an infinite connected set of occupied sites, there exists a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda$ and initial configuration $\zeta$.

**Definition 3** (Definition of the dominating processes). *Let $d \in \mathbb{N}$ and let the process $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda > 0$ and vacant initial configuration. For all $\epsilon > 0$, we define for all $k \in \mathbb{N}_0$ and all $x \in \mathbb{Z}^d$, a process by*

$$\lceil \eta \rceil_{k,x,\epsilon} := \begin{cases} 1, & \text{if } \eta_{k\epsilon,x} = 1 \text{ or if } G_{k\epsilon,x} < G_{(k+1)\epsilon,x}; \\ 0, & \text{else.} \end{cases}$$

Informally, the process $(\lceil \eta \rceil_{k,x,\epsilon})_{k\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ behaves as follows: At the $k$'th time step, the process takes over the configuration of the forest-fire process at time $k\epsilon$, and then the process additionally occupies all sites at which there is the growth of a tree in between time $k\epsilon$ and $(k+1)\epsilon$.
For all $\epsilon > 0$ and all $k \in \mathbb{N}_0$, we define the set of sites that are occupied in $(\lceil \eta \rceil_{l,x,\epsilon})_{l\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ at the $k$'th time step by $F_{\epsilon,k} := \big\{x \in \mathbb{Z}^d : \ \lceil \eta \rceil_{k,x,\epsilon} = 1\big\}$. For all $x \in \mathbb{Z}^d$, we define its cluster at $x$ at the $k$'th time step to be

$$\lceil C_{k,x,\epsilon} \rceil := \Big\{ y \in \mathbb{Z}^d : \ x \leftrightarrow_{F_{\epsilon,k}} y \Big\}.$$

Finally, we define

$$\operatorname{diam} \lceil C_{k,x,\epsilon} \rceil := \sup \Big\{ \|y - z\|_\infty \Big| y, z \in \lceil C_{k,x,\epsilon} \rceil \Big\}$$

to denote its diameter.

Given events $(A_i)_{1\le i\le n}$, we sometimes write $\{A_1, A_2, \dots, A_n\} := \cap_{1\le i\le n} A_n$ to denote the intersection of the events.
For all $n \in \mathbb{N}$, let $B_n^d := \{x \in \mathbb{Z}^d : \|x\|_\infty \le n\}$ be the square box with center 0 and side-length $2n$.

# 3 Main results

**Theorem 1.** *Let $d \in \mathbb{N}$ and let $p_c^d$ be the critical probability of site percolation on $\mathbb{Z}^d$. Let $\lambda > \frac{1-p_c^d}{p_c^d}$ and let $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda$ and vacant initial configuration. Then*

*(a) There exists a null set s.t. restricted to the complement of the null set, all forest-fire processes on $\mathbb{Z}^d$ with vacant initial configuration that are driven by the growth and ignition processes $(G_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$ and $(I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$ must be equal to the forest-fire process $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$.*

*(b) The forest-fire process $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$ is adapted to the filtration generated by its driving growth and ignition processes $(G_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$ and $(I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$;*

*(c) Uniformly on a bounded time interval on a finite set of sites, the finite volume forest-fire processes on the square boxes $(B_n^d)_{n\in\mathbb{N}}$ with parameter $\lambda$ and vacant initial configuration converge a.s. to the forest-fire process $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$. Formally, for all $T \in \mathbb{R}_0^+$ and all $n_2 \in \mathbb{N}$, the relation*

$$\lim_{n_1\to\infty} \mu\Bigg( \sup_{n\ge n_1} \sup_{0\le t\le T} \sup_{x\in B_{n_2}^d} |\bar\eta_{t,x} - \bar\eta_{t,x}^n| > 0 \Bigg) = 0$$

*holds. Here for all $n \in \mathbb{N}$, the process $(\bar\eta_{t,x}^n)_{t\ge 0, x\in B_n^d}$ denotes the finite volume forest-fire process on the square box $B_n^d$ with vacant initial configuration that is driven by the growth and ignition processes $(G_{t,x})_{t\ge 0, x\in B_n^d}$ and $(I_{t,x})_{t\ge 0, x\in B_n^d}$. (The above events are measurable since the processes are right continuous with values in $\{0,1\}$.)*

Theorem 2 states that under an assumption on the distribution of the cluster size in the dominating processes defined in Definition 3, the statements *(a)*, *(b)* and *(c)* of Theorem 1 hold. In particular, we use Theorem 2 to show Theorem 1.

**Theorem 2.** *Let $d \in \mathbb{N}$ and let $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\ge 0, x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda > 0$ and vacant initial configuration. Suppose that there exist $\epsilon > 0$ and a function $f : \mathbb{N} \to \mathbb{R}_0^+$ with $f(m) = o(m^{-d})$ as $m$ tends to $\infty$ s.t. for all $k \in \mathbb{N}_0$, all $x \in \mathbb{Z}^d$, and all $m \in \mathbb{N}$, the relation*

$$\mu\Big( \operatorname{diam}\lceil C_{k,x,\epsilon}\rceil \ge m \Big) \le f(m)$$

*holds. Then the statements (a), (b) and (c) of Theorem 1 hold.*

# 4 Proof of the main results

In [4] the author shows the existence of multi-dimensional infinite volume forest-fire processes for all parameters $\lambda > 0$. The construction uses a diagonal sequence argument and in that paper, it was not examined whether the constructed process depends on the choice of the diagonal sequence. This implies the question whether a multi-dimensional infinite volume forest-fire process with given parameter $\lambda$ and given initial configuration is unique.
Intuitively, the situation is as follows: Suppose that two forest-fire processes on $\mathbb{Z}^d$ with the same initial configuration and the same driving growth and ignition processes differ on the square box $B_n^d$. The set $B_n^d$ is finite and by the definition of a forest-fire process, at each site, the forest-fire process is right continuous with values in $\{0,1\}$. That is, there must exist a first point in time $t_n > 0$, at which the two processes differ at a site $x \in B_n^d$. The definition of a forest-fire process implies that a vacant site gets occupied, if and only if there is the growth of a tree at the site. Therefore it is impossible that a vacant site gets occupied in the one forest-fire process, but not in the other. It follows that at time $t_n$, the occupied site $x$ must have become vacant in the one forest-fire process, but not in the other. In a forest-fire process an occupied site gets vacant, if and only if the cluster at the site is hit by ignition. Thus at time $t_n$, the cluster at the site $x$ must be hit by ignition in the one forest-fire process, but not in the other. That is, in the two forest-fire processes, the cluster at the site $x$ must have been different before time $t_n$; the two processes must have been different at a site outside of $B_n^d$ before time $t_n$. It follows that if the two processes differ at a given site $x$, then the set of sites the two processes differ must have spread from infinity to the site $x$.
Given $n_1 > n_2 \geq 1$, we use the dominating processes defined in Definition 3 to estimate the time the set of differing sites needs to spread from outside $B_{n_1}^d$ to $B_{n_2}^d$.

## 4.1 Proof of Theorem 2

**Definition 4.** *The set of all finite and non-empty connected subsets of $\mathbb{Z}^d$ is*

$$C^f := \left\{ C \subset \mathbb{Z}^d \middle| 1 \leq |C| < \infty,\ \forall x, y \in C:\ x \leftrightarrow_C y \right\}.$$

*For all $S \subseteq \mathbb{Z}^d$, we define*

$$\partial S := \left\{ x \in \mathbb{Z}^d \setminus S \middle| \exists y \in S:\ \|x - y\|_1 = 1 \right\},$$

*that is, the set of sites next to $S$.*

Let $d \in \mathbb{N}$, $\lambda > 0$ and let $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with parameter $\lambda$ and vacant initial configuration. We restrict the process to the complement of a null set s.t. there must not be two growth and ignition events at the same time. For all $\epsilon > 0$, let $(\lceil\eta\rceil_{l,x,\epsilon})_{l\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ be the dominating process defined in Definition 3.

**Lemma 1.** Let $\epsilon > 0$, $k \in \mathbb{N}_0$ and $S \in C^f$. Let for all $n \in \mathbb{N}$, the set $F_n \subseteq \mathbb{Z}^d$ s.t. the relation $S \cup \partial S \subseteq F_n$ holds. For all $n \in \mathbb{N}$, let $(\eta^n_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in F_n}$ be a forest-fire process on $F_n$ with vacant initial configuration that is driven by the growth and ignition processes $(G_{t,x})_{t\geq 0, x\in F_n}$ and $(I_{t,x})_{t\geq 0, x\in F_n}$.
Suppose that in the dominating process $(\lceil\eta\rceil_{l,x,\epsilon})_{l\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ all sites next to $S$ are vacant at the $k$'th time step, and that the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ are equal on the set $S \cup \partial S$ at

time $k\epsilon$. Then they must be equal on the set $S$ within the whole time interval $[k\epsilon,(k+1)\epsilon]$. Formally, for all $\epsilon > 0$, we have for all $k \in \mathbb{N}_0$ and all $S \in C^f$,

$$\left\{\forall y \in \partial S:\ \lceil \eta \rceil_{k,y,\epsilon} = 0,\ \forall n \in \mathbb{N}\ \forall x \in S \cup \partial S:\ \eta_{k\epsilon,x} = \eta^n_{k\epsilon,x}\right\}$$
$$\subseteq \left\{\forall n \in \mathbb{N}\ \forall x \in S\ \forall t \in [k\epsilon,(k+1)\epsilon]:\ \eta_{t,x} = \eta^n_{t,x}\right\}.$$

*Proof.* Let $\epsilon > 0$, $k \in \mathbb{N}_0$ and $S \in C^f$. Suppose that for all $y \in \partial S$, the relation $\lceil \eta \rceil_{k,y,\epsilon} = 0$ holds, and that the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ are equal on the set $\partial S$ at time $k\epsilon$. Then the definition of the dominating process implies that for all $n \in \mathbb{N}$ and all $y \in \partial S$, the relation $\eta_{k\epsilon,y} = \eta^n_{k\epsilon,y} = 0$ must hold, and that there must not be the growth of a tree at $\partial S$ in the time between $k\epsilon$ and $(k+1)\epsilon$. The definition of a forest-fire process implies that a vacant site must remain vacant, if there is not the growth of a tree at the site. Thus in the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ the set $\partial S$ must remain vacant up to time $(k+1)\epsilon$. Formally, we have

$$\left\{\forall y \in \partial S:\ \lceil \eta \rceil_{k,y,\epsilon} = 0,\ \forall n \in \mathbb{N}\ \forall x \in S \cup \partial S:\ \eta_{k\epsilon,x} = \eta^n_{k\epsilon,x}\right\}$$
$$\subseteq \left\{\forall n \in \mathbb{N}\ \forall x \in S:\ \eta_{k\epsilon,x} = \eta^n_{k\epsilon,x},\ \forall y \in \partial S\ \forall t \in [k\epsilon,(k+1)\epsilon]:\ \eta_{t,y} = \eta^n_{t,y} = 0\right\} =: (*).$$

Suppose that the event $(*)$ holds. Then in the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ on the time interval $[k\epsilon,(k+1)\epsilon]$, the sites next to $S$ are vacant, that is, for all $x \in S$, the cluster at the site $x$ must be a subset of the set $S$. By the definition of a forest-fire process an occupied site gets vacant, if and only if the cluster at the site is hit by ignition; a vacant site gets occupied, if and only if there is the growth of a tree at the site. It follows that within the time interval $[k\epsilon,(k+1)\epsilon]$, if there neither an ignition nor the growth of a tree at $S$, then the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ must remain unchanged on $S$. (Only if $(*)$ holds.)
Let $t_n$, $1 \le n \le N$, be the time of the $n$'th of the finitely many growth and ignition events that occur on $S$ within the time interval $[k\epsilon,(k+1)\epsilon]$. By $(*)$ the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ are equal on $S$ at time $k\epsilon$. As discussed above, in the time between $t_0 := k\epsilon$ and $t_1$ on the set $S$, the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ must remain unchanged, that is, equal. The definition of a forest-fire process implies that they must behave equal at time $t_1$: If the first growth or ignition event is an ignition at a site $x \in S$, then the cluster at the site $x$ must get vacant completely. If the first growth or ignition event is the growth of a tree at a site $x \in S$, then the site $x$ must be occupied. Recursively, it follows for all $1 \le n \le N$, that the forest-fire processes must remain equal on the set $S$ in the time between $t_{n-1}$ and $t_n$, and that they must behave equal at time $t_n$. That is, if $(*)$ holds, then the forest-fire processes $\eta$ and $(\eta^n)_{n\in\mathbb{N}}$ must be equal on the set $S$ within the time interval $[k\epsilon,(k+1)\epsilon]$. This shows the result.
In the proof it is elementary that the growth and ignition events that occur on $S$ within the time interval $[k\epsilon,(k+1)\epsilon]$ are well-ordered. First, they are finitely many since the set $S$ is finite and the definition of a forest-fire process requires the growth and ignition processes to be càdlàg. Thus they are well-ordered since there must not be two growth and ignition events at the same time. (We restricted to the complement of a null set s.t. this must hold.) $\square$

**Remark 2.** In the dominating processes for all $n, m \in \mathbb{N}$, if for all $x \in \partial B^d_n$, the diameter of the cluster at $x$ is smaller than $m$, then the set $B^d_n$ must be separated by vacant sites from the sites outside $B^d_{n+m}$. Formally, for all $\epsilon > 0$, all $k \in \mathbb{N}_0$, for all $n \in \mathbb{N}$ and all $m \in \mathbb{N}$, the relation

$$\Big\{\forall x \in \partial B_n^d :\ \mathrm{diam}\lceil C_{k,x,\epsilon}\rceil < m\Big\}$$
$$\subseteq \Big\{\exists S \in C^f : B_n^d \subseteq S \subseteq B_{n+m-1}^d, \forall x \in \partial S :\ \lceil \eta \rceil_{k,x,\epsilon} = 0\Big\}$$

holds.

**Lemma 2.** Let $(\eta'_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with vacant initial configuration that is driven by the growth and ignition processes $(G_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ and $(I_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$. Suppose that within a given time interval $[0, k\epsilon]$, the processes $(\eta_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ and $(\eta'_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ differ on a given square box $B_n^d$. Then for all $m \in \mathbb{N}$, there must exist an $i$'th time step, $1 \leq i \leq k$, at which in the dominating process $(\lceil\eta\rceil_{l,x,\epsilon})_{l\in\mathbb{N}_0,x\in\mathbb{Z}^d}$ there exists a cluster at a site in $\partial B^d_{n+(k-i)m}$ whose diameter is bigger than or equal to $m$. Formally, for all $\epsilon > 0$, all $n, k \in \mathbb{N}$, for all $m \in \mathbb{N}$, the relation

$$\Big\{\exists t \in [0, k\epsilon]\ \exists x \in B_n^d :\ \eta_{t,x} \neq \eta'_{t,x}\Big\}$$
$$\subseteq \Big\{\exists 1 \leq i \leq k\ \exists y \in \partial B^d_{n+(k-i)m} :\ \mathrm{diam}\lceil C_{i-1,y,\epsilon}\rceil \geq m\Big\} =: \mathcal{N}_{\epsilon,n,k,m}$$

holds.

*Proof.* Let $\epsilon > 0$ and $n, k, m \in \mathbb{N}$.
In the first step, we suppose that in the dominating process $(\lceil\eta\rceil_{l,x,\epsilon})_{l\in\mathbb{N}_0,x\in\mathbb{Z}^d}$ at time step 0 for all $x \in \partial B_{n+(k-1)m}$, the diameter of the cluster at $x$ is smaller than $m$. Then Remark 2 implies that the set $B_{n+(k-1)m}$ must be separated by vacant sites from the sites outside $B_{n+km}$. Both forest-fire processes have vacant initial configuration, i.e., are equal at time 0. Lemma 1 implies that the two forest-fire processes must remain equal on the set $B^d_{n+(k-1)m}$ on the time interval $[0, \epsilon]$. In the second step, we additionally suppose that in the dominating process at the first time step for all $x \in \partial B_{n+(k-2)m}$, the diameter of the cluster at $x$ is smaller than $m$. Again Remark 2 implies that $B_{n+(k-2)m}$ must be separated by vacant sites from the sites outside $B_{n+(k-1)m}$. In the first step, we showed that the two forest-fire processes must be equal on the set $B^d_{n+(k-1)m}$ on the time interval $[0, \epsilon]$. Thus Lemma 1 implies that the two forest-fire processes must be equal on the time interval $[0, 2\epsilon]$ on the set $B^d_{n+(k-2)m}$.
Formally, we prove by induction that for all $1 \leq j \leq k$, the relation

$$\Big\{\forall 1 \leq i \leq j\ \forall y \in \partial B^d_{n+(k-i)m} :\ \mathrm{diam}\lceil C_{i-1,y,\epsilon}\rceil < m\Big\}$$
$$\subseteq \Big\{\forall t \in [0, j\epsilon]\ \forall x \in B^d_{n+(k-j)m} :\ \eta_{t,x} = \eta'_{t,x}\Big\} \tag{*}$$

holds. Then the result follows by taking the complements in (*) for $j = k$.
To show that the relation (*) holds for $j = 1$, we note that both forest-fire processes have

vacant initial configuration. Thus Lemma 1 and Remark 2 imply that the relation

$$\begin{aligned}
&\Big\{\forall y \in \partial B^d_{n+(k-1)m} :\ \mathrm{diam}\lceil C_{0,y,\epsilon}\rceil < m\Big\} \\
\subseteq &\Big\{\exists S \in C^f :\ B^d_{n+(k-1)m} \subseteq S \subseteq B^d_{n+km-1},\ \forall z \in \partial S :\ \lceil \eta \rceil_{0,z,\epsilon} = 0\Big\} \\
\subseteq &\Big\{\exists S \in C^f :\ B^d_{n+(k-1)m} \subseteq S \subseteq B^d_{n+km-1},\ \forall x \in S\ \forall t \in [0,\epsilon] :\ \eta_{t,x} = \eta'_{t,x}\Big\} \\
\subseteq &\Big\{\forall t \in [0,\epsilon]\ \forall x \in B^d_{n+(k-1)m} :\ \eta_{t,x} = \eta'_{t,x}\Big\}
\end{aligned}$$

holds. That is, the relation (*) holds for $j = 1$. In the induction step $l \to l+1$, we suppose that the relation (*) holds for $j = l$. Then the insertion of Lemma 1 and Remark 2 yields to

$$\begin{aligned}
&\Big\{\forall 1 \le i \le l+1\ \forall y \in \partial B^d_{n+(k-i)m} :\ \mathrm{diam}\lceil C_{i-1,y,\epsilon}\rceil < m\Big\} \\
\subseteq &\Big\{\forall t \in [0,l\epsilon]\ \forall x \in B^d_{n+(k-l)m} :\ \eta_{t,x} = \eta'_{t,x},\ \exists S \in C^f : B^d_{n+(k-(l+1))m} \subseteq S \subseteq B^d_{n+(k-l)m-1}, \\
&\qquad \forall y \in \partial S :\ \lceil \eta \rceil_{l,y,\epsilon} = 0\Big\} \\
\subseteq &\Big\{\forall t \in [0,l\epsilon]\ \forall x \in B^d_{n+(k-l)m} :\ \eta_{t,x} = \eta'_{t,x},\ \exists S \in C^f : B^d_{n+(k-(l+1))m} \subseteq S \subseteq B^d_{n+(k-l)m-1}, \\
&\qquad \forall x' \in S\ \forall t' \in [l\epsilon,(l+1)\epsilon] :\ \eta_{t',x'} = \eta'_{t',x'}\Big\} \\
\subseteq &\Big\{\forall t \in [0,(l+1)\epsilon]\ \forall x \in B^d_{n+(k-(l+1))m} :\ \eta_{t,x} = \eta'_{t,x}\Big\},
\end{aligned}$$

that is, the relation (*) holds for $j = l+1$. □

*Proof of Theorem 2 (a).* Suppose that there exist $\epsilon > 0$ and a function $f : \mathbb{N} \to \mathbb{R}^+_0$ with $f(m) = o(m^{-d})$ as $m$ tends to $\infty$ s.t. for all $k \in \mathbb{N}_0$, all $x \in \mathbb{Z}^d$, and all $m \in \mathbb{N}$, the relation

$$\mu\Big(\mathrm{diam}\lceil C_{k,x,\epsilon}\rceil \ge m\Big) \le f(m)$$

holds. Then for all $n, k \in \mathbb{N}$, the relation

$$\begin{aligned}
0 &\le \limsup_{m\to\infty} \mu\Big(\mathcal{N}_{\epsilon,n,k,m}\Big) \le \limsup_{m\to\infty} \sum_{i=1}^{k} \sum_{y \in \partial B^d_{n+(k-i)m}} \mu\Big(\mathrm{diam}\lceil C_{i-1,y,\epsilon}\rceil \ge m\Big) \\
&\le \lim_{m\to\infty} k \cdot \big(2(n+km)+1\big)^d \cdot f(m) = 0
\end{aligned}$$

must hold, that is, the set

$$\bigcup_{k\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} \bigcap_{m\in\mathbb{N}} \mathcal{N}_{\epsilon,n,k,m} =: \mathcal{N}_\epsilon$$

is a null set. We show that restricted to the complement of the null set $\mathcal{N}_\epsilon$, all forest-fire processes on $\mathbb{Z}^d$ with vacant initial configuration that are driven by the growth and ignition processes $(I_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ and $(G_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ must be equal to the forest-fire process $(\eta_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$:

Let $(\eta'_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ be a forest-fire process on $\mathbb{Z}^d$ with vacant initial configuration that is driven by the growth and ignition processes $(G_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$ and $(I_{t,x})_{t\geq 0,x\in\mathbb{Z}^d}$. Lemma 2 implies that for all $n, k \in \mathbb{N}$, the relation

$$\left\{\exists t \in [0, k\epsilon]\ \exists x \in B_n^d :\ \eta_{t,x} \neq \eta'_{t,x}\right\} \subseteq \bigcap_{m\in\mathbb{N}} \mathcal{N}_{\epsilon,n,k,m}$$

must hold. Thus the event that the two forest-fire processes differ satisfies

$$\left\{\exists x \in \mathbb{Z}^d\ \exists t \in \mathbb{R}_0^+ :\ \eta_{t,x} \neq \eta'_{t,x}\right\} = \bigcup_{k\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}\left\{\exists t \in [0, \epsilon k]\ \exists x \in B_n^d :\ \eta_{t,x} \neq \eta'_{t,x}\right\} \subseteq \mathcal{N}_\epsilon.$$

That is, the null set $N_\epsilon$ has the property described in Theorem 2 statement *(a)*. □

For all $n \in \mathbb{N}$, let $(\eta^n_{t,x}, G_{t,x}, I_{t,x})_{t\in\mathbb{R}_0^+,x\in B_n^d}$ be the finite volume forest-fire process on the square box $B_n^d$ with vacant initial configuration that is driven by the growth and ignition processes $(G_{t,x})_{t\geq 0,x\in B_n^d}$ and $(I_{t,x})_{t\geq 0,x\in B_n^d}$.

In analogy to Lemma 2, we have

**Lemma 3.** For all $k \in \mathbb{N}$, all $\epsilon > 0$, all $n_2 \in \mathbb{N}$, and all $n_1 \geq n_2 + k$, the relation

$$\left\{\sup_{n\geq n_1}\sup_{0\leq t\leq k\epsilon}\sup_{x\in B^d_{n_2}}\left|\eta_{t,x} - \eta^n_{t,x}\right| > 0\right\} \subseteq \left\{\exists 1 \leq i \leq k \exists y \in B^d_{n_1} : \mathrm{diam}\lceil C_{i-1,y,\epsilon}\rceil \geq \left\lfloor \frac{n_1 - n_2}{k}\right\rfloor\right\}$$

holds.

*Proof.* Let $k \in \mathbb{N}$,$\epsilon > 0$, let $n_2 \in \mathbb{N}$ and $n_1 \geq n_2 + k$. We define $m = m(n_1, n_2, k) := \lfloor(n_1 - n_2)/k\rfloor$. We prove by induction that for all $1 \leq j \leq k$, the relation

$$\begin{aligned}&\left\{\forall 1 \leq i \leq j,\ \forall y \in \partial B^d_{n_1-im} :\ \mathrm{diam}\lceil C_{i-1,y,\epsilon}\rceil < m\right\}\\ \subseteq&\left\{\forall n \geq n_1\ \forall t \in [0, j\epsilon]\ \forall x \in B^d_{n_1-jm} :\ \eta_{t,x} = \eta^n_{t,x}\right\}\end{aligned} \qquad (*)$$

holds. We defined $m = m(n_1, n_2, k)$ such that the relation $n_1 - km \geq n_2$ holds, and thus the result follows by taking the complements in (*) for $j = k$.

In the induction step $l \to l + 1$, we suppose that the relation (*) holds for $j = l$. Then as in the proof of Lemma 2, insertion of Lemma 1 and 2 provides

$$\begin{aligned}
&\Big\{\forall 1 \leq i \leq l+1,\ \forall y \in \partial B^d_{n_1-im} :\ \operatorname{diam}\lceil C_{i-1,y,\epsilon}\rceil < m\Big\}\\
\subseteq &\Big\{\forall n \geq n_1\ \forall t \in [0, l\epsilon]\ \forall x \in B^d_{n_1-lm} :\ \eta_{t,x} = \eta^n_{t,x},\ \exists S \in C^f : B^d_{n_1-(l+1)m} \subseteq S \subseteq B^d_{n_1-lm-1},\\
&\qquad\qquad \forall y \in \partial S :\ \lceil \eta \rceil_{l,y,\epsilon} = 0\Big\}\\
\subseteq &\Big\{\forall n \geq n_1\ \forall t \in [0, l\epsilon]\ \forall x \in B^d_{n_1-lm} :\ \eta_{t,x} = \eta^n_{t,x},\ \exists S \in C^f : B^d_{n_1-(l+1)m} \subseteq S \subseteq B^d_{n_1-lm-1},\\
&\qquad\qquad \forall x' \in S\ \forall t' \in [l\epsilon, (l+1)\epsilon] :\ \eta_{t',x'} = \eta^n_{t',x'}\Big\}\\
\subseteq &\Big\{\forall n \geq n_1\ \forall t \in [0, (l+1)\epsilon]\ \forall x \in B^d_{n_1-(l+1)m} :\ \eta_{t,x} = \eta^n_{t,x}\Big\}.
\end{aligned}$$

That is, if the relation (*) holds for $j = l$, then it holds for $j = l + 1$.
All the forest-fire processes have vacant initial configuration. Similar as in Lemma 2, insertion of Lemma 1 and Remark 2 provides that the relation (*) holds for $j = 1$. $\square$

*Proof of Theorem 2 (b) and (c).* Suppose that there exist $\epsilon > 0$ and a function $f : \mathbb{N} \to \mathbb{R}^+_0$ with $f(m) = o(m^{-d})$ as $m$ tends to $\infty$ s.t. for all $k \in \mathbb{N}_0$, all $x \in \mathbb{Z}^d$, and all $m \in \mathbb{N}$, the relation

$$\mu\Big(\operatorname{diam}\lceil C_{k,x,\epsilon}\rceil \geq m\Big) \leq f(m)$$

holds. Then Lemma 3 provides that for all $k \in \mathbb{N}$ and all $n_2 \in \mathbb{N}$, the relation

$$\begin{aligned}
0 &\leq \lim_{n_1\to\infty} \mu\Big(\sup_{n\geq n_1} \sup_{0\leq t\leq k\epsilon} \sup_{x\in B^d_{n_2}} \big|\eta_{t,x} - \eta^n_{t,x}\big| > 0\Big)\\
&\leq \limsup_{n_1\to\infty} \sum_{i=1}^{k} \sum_{y\in B^d_{n_1}} \mu\Big(\operatorname{diam}\lceil C_{i-1,y,\epsilon}\rceil \geq \Big\lfloor \frac{n_1 - n_2}{k} \Big\rfloor\Big)\\
&\leq \lim_{n_1\to\infty} k \cdot \big(2n_1 + 1\big)^d \cdot f\Big(\Big\lfloor \frac{n_1 - n_2}{k} \Big\rfloor^{-d}\Big) = 0
\end{aligned}$$

must hold; the events are measurable since the processes are right continuous with values in $\{0, 1\}$. This is statement Theorem 2 *(c)*.
For all $n \in \mathbb{N}$, the finite volume forest-fire process $(\eta^n_{t,x}, G_{t,x}, I_{t,x})_{t\geq 0, x\in B^d_n}$ on the square box $B^d_n$ is adapted to the filtration generated by the growth and ignition processes $(G_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$ and $(I_{t,x})_{t\geq 0, x\in\mathbb{Z}^d}$: For all $l \in \mathbb{N}$, let $t_l$ be the $l$'th growth or ignition event that occurs on the finite set $B^d_n$. The definition of a forest-fire process implies that the configuration of a forest-fire process on $B^d_n$ does not change, if there is neither the growth of a tree nor an ignition at the set $B^d_n$. That is, for all $l \in \mathbb{N}$, the forest-fire process on $B^d_n$ does not change in the time between $t_{l-1}$ and $t_l$ ($t_0 := 0$). For all $l \in \mathbb{N}$, at time $t_l$, that is at the time of the $l$'th growth or ignition event, the forest-fire process on $B^d_n$ is determined by the definition of a forest-fire process: A vacant site $x \in B^d_n$ gets occupied, if and only if there is the growth of a tree at the site; an occupied site $x \in B^d_n$ gets vacant, if and only if the cluster at $x$ is hit by ignition.

That is, for all $T \in \mathbb{R}_0^+$, all $n_2 \in \mathbb{N}$ and all $n \geq n_2$, the process $(\eta^n_{t,x})_{t \leq T, x \in B^d_{n_2}}$ is measurable by the $\sigma$-field generated by the processes $(G_{t,x})_{0 \leq t \leq T, x \in \mathbb{Z}^d}$ and $(I_{t,x})_{0 \leq t \leq T, x \in \mathbb{Z}^d}$. Thus it follows by the convergence shown in Theorem 2 *(c)* that for all $T \in \mathbb{R}_0^+$ and all $n_2 \in \mathbb{N}$, the process $(\eta_{t,x})_{t \leq T, x \in B^d_{n_2}}$ is measurable by the $\sigma$-field generated by the processes $(G_{t,x})_{0 \leq t \leq T, x \in \mathbb{Z}^d}$ and $(I_{t,x})_{0 \leq t \leq T, x \in \mathbb{Z}^d}$. This shows Theorem 2 *(b)*. □

## 4.2 Proof of Theorem 1

*Proof.* If $\lambda$ is big enough, then we can estimate the diameter of the clusters of the dominating process using subcritical site percolation.
Let $\lambda > \frac{1-p_c^d}{p_c^d}$, $p_c^d$ the critical probability of site percolation on $\mathbb{Z}^d$. Let $\epsilon > 0$ s.t.

$$p_\lambda^\epsilon := \frac{1}{1+\lambda} + (1 - e^{-\epsilon}) < p_c^d$$

holds. For all $k \in \mathbb{N}_0$ and all $x \in \mathbb{Z}^d$, we write

$$\mathrm{GROWTH}_x(k) := \Big\{ G_{k\epsilon,x} < G_{(k+1)\epsilon,x} \Big\}$$

to denote the event that there is the growth of a tree at the site $x$ in the time between $k\epsilon$ and $(k+1)\epsilon$. For all $k \in \mathbb{N}_0$ and all $x \in \mathbb{Z}^d$, we write

$$\mathrm{LASTGROWTH}_x(k) := \Big\{ \exists t < k\epsilon : \quad G_{t,x} < G_{k\epsilon,x},\ I_{t,x} = I_{k\epsilon,x} \Big\}$$

to describe the event that the last growth or ignition event at the site $x$ before time $k\epsilon$ has been the growth of a tree. For all $x \in \mathbb{Z}^d$ and all $k \in \mathbb{N}_0$, we define

$$\chi_{k,x} := \begin{cases} 1, & \text{if the event } \mathrm{LASTGROWTH}_x(k) \text{ or } \mathrm{GROWTH}_x(k) \text{ holds;} \\ 0, & \text{otherwise.} \end{cases}$$

At the $k$'th time step, the process $(\lceil \eta \rceil_{l,x,\epsilon})_{l \in \mathbb{N}_0, x \in \mathbb{Z}^d}$ takes over the configuration of the forest-fire process at time $k\epsilon$, and then the process additionally occupies all sites at which there is the growth of a tree in between time $k\epsilon$ and $(k+1)\epsilon$. In a forest-fire process on $\mathbb{Z}^d$, the site $x \in \mathbb{Z}^d$ must be vacant at time $k\epsilon$, if the last growth or ignition event at the site $x$ has been an ignition. It follows that for all $k \in \mathbb{N}_0$ and all $x \in \mathbb{Z}^d$, the relation $\chi_{k,x} \geq \lceil \eta \rceil_{k,x,\epsilon}$ holds.
The growth and ignition processes at the sites of $\mathbb{Z}^d$ are independent Poisson processes with parameter 1 and $\lambda$, respectively. Thus for all $k \in \mathbb{N}_0$, independently for all $x \in \mathbb{Z}$, we get

$$\begin{aligned}\mu\Big(\lceil \eta \rceil_{k,x,\epsilon} = 1\Big) \leq \mu\Big(\chi_{k,x} = 1\Big) &\leq \mu\Big(\mathrm{LASTGROWTH}_x(k)\Big) + \mu\Big(\mathrm{GROWTH}_x(k)\Big) \\ &\leq \frac{1}{1+\lambda} + (1 - e^{-\epsilon}) = p_\lambda^\epsilon < p_c^d.\end{aligned}$$

In [1] Michael Aizenman and David J. Barsky show that for site percolation on $\mathbb{Z}^d$ the percolation threshold $p_c^d$ equals the point where the cluster size distribution ceases to decay exponentially. This shows that the assumption on the distribution of the cluster size in the dominating process in Theorem 2 holds; Theorem 1 follows. □

**Remark 3.** Suppose that we can choose $\epsilon > 0$ s.t. $p^\epsilon_\lambda < 1/(2d)$ holds, that is, that $\lambda > 2d-1$ holds. Then a Peierls argument provides that for all $k \in \mathbb{N}_0$ and all $x \in \mathbb{Z}^d$, the relation

$$\mu\Big(\operatorname{diam}\lceil C_{k,x,\epsilon}\rceil \geq m\Big) \leq (2d)^m (p^\epsilon_\lambda)^m = \big(2dp^\epsilon_\lambda\big)^m$$

holds. We have $2dp^\epsilon_\lambda < 1$; Theorem 1 follows by Theorem 2.

**Open Problems.**
There are several further natural questions: What about non-empty initial configurations (with very large but finite clusters at time 0)? Does there exist a unique stationary distribution? Is the distribution of a forest-fire process on $\mathbb{Z}^d$ (with empty initial configuration) translation invariant? In particular, is the forest-fire process constructed in [4] translation invariant?

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# References

[1] Michael Aizenman and David J. Barsky. Sharpness of the phase transition in percolation models. *Comm. Math. Phys.*, 108(3):489–526, 1987. MR0874906

[2] J. van den Berg and R. Brouwer. Self-organized forest-fires near the critical time. *Comm. Math. Phys.*, 267(1):265–277, 2006. MR2238911

[3] J. van den Berg and A. A. Járai. On the asymptotic density in a one-dimensional self-organized critical forest-fire model. *Comm. Math. Phys.*, 253(3):633–644, 2005. MR2116731

[4] Maximilian Duerre. Existence of multi-dimensional infinite volume self-organized critical forest-fire models. *Electronic Journal of Probability*, 11:513–539, 2006. MR2242654

[5] Henrik Jeldtoft Jensen. *Self-organized criticality*, volume 10 of *Cambridge Lecture Notes in Physics*. Cambridge University Press, Cambridge, 1998. Emergent complex behavior in physical and biological systems. MR1689042

[6] Klaus Schenk, Barbara Drossel, and Franz Schwabl. Self-organized critical forest-fire model on large scales. *Physical Review E (Statistical, Nonlinear, and Soft Matter Physics)*, 65(2):026135, 2002.