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ON THE OCCUPATION TIME OF BROWNIAN EXCURSION

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Abstract

Recently, Kalvin M. Jansons derived in an elegant way the Laplace transform of the time spent by an excursion above a given level $a > 0$. This result can also be derived from previous work of the author on the occupation time of the excursion in the interval $(a, a+b]$, by sending $b \to \infty$. Several alternative derivations are included.

1 Introduction

In [5], the author derives in an elegant way the Laplace transform of the time spent by an excursion above a given level $a > 0$. This result can also be derived from the occupation time of the excursion in the interval $(a, a+b]$, by sending $b \to \infty$ (cf. [2] or [4]).

2 Occupation times

Introduce for α, β complex and $a \geq 0$,

$$\psi(\alpha, \beta, a) = \left[\frac{\alpha\cosh(a\beta) + \beta\sinh(a\beta)}{\alpha\sinh(a\beta) + \beta\cosh(a\beta)}\right].$$

Denote by W_0^+, Brownian excursion with time parameter $t \in [0,1]$, see [4], I.2 for a precise definition. According to p. 117 and p. 120 of [4], or Theorem 5.1 of [2], the Laplace transform of the occupation time $T(a, a+b) = \int_0^1 1_{(a,a+b]}(W_0^+(t))\, dt$, is given by:

$$Ee^{-\beta T(a,a+b)} = \frac{\sqrt{2\pi}}{a^3}\sum_{k=1}^{\infty} k^2\pi^2 e^{-k^2\pi^2/2a^2} + \frac{1}{i\sqrt{\pi}}\int_S \frac{\alpha e^{\alpha}}{\sinh\{a\sqrt{2\alpha}\}} \tag{1}$$
$$\times[\sqrt{\alpha}\cosh\{a\sqrt{2\alpha}\} + (\alpha+\beta)^{1/2}\psi(\sqrt{\alpha}, \sqrt{\alpha+\beta}, b\sqrt{2})\sinh\{a\sqrt{2\alpha}\}]^{-1}\, d\alpha,$$

where the path S is defined by

$$S = \{\alpha : \alpha = iy, |y| \geq \xi\} \cup \{\alpha : \alpha = \xi e^{i\eta}, -\pi/2 \leq \eta \leq \pi/2\},$$

for some $\xi > 0$.

In order to write the first term on the right side of (1), which term is equal to the distribution function of the supremum of Brownian excursion, [1] as a complex integral we introduce the path:

$$\Gamma = \{\alpha : \alpha = ye^{\pm i\phi}, y \geq \xi\} \cup \{\alpha : \alpha = \xi e^{i\eta}, -\phi \leq \eta \leq \phi\},$$

with $\pi/2 < \phi < \pi$, $\xi > 0$ and the orientation counterclockwise. We choose the angle ϕ in such a way that all sigularities of the integrand in (1) remain on the left of the path Γ. Then

$$\frac{\sqrt{2\pi}}{a^3} \sum_{k=1}^{\infty} k^2\pi^2 e^{-k^2\pi^2/2a^2} = -\frac{1}{i\sqrt{\pi}} \int_\Gamma \sqrt{\alpha} e^\alpha \frac{\cosh\{a\sqrt{2\alpha}\}}{\sinh\{a\sqrt{2\alpha}\}}\, d\alpha, \tag{2}$$

since the integrand has only simple poles at $\alpha_k = -k^2\pi^2/2a^2$, $k \geq 1$. Combining (1) and (2) and deforming the path S into the path Γ (again using Cauchy's theorem), yields

$$\begin{aligned} Ee^{-\beta T(a,a+b)} &= -\frac{1}{i\sqrt{\pi}} \int_\Gamma \sqrt{\alpha} e^\alpha \, d\alpha \\ &\times \left[\frac{\sqrt{\alpha}\sinh\{a\sqrt{2\alpha}\} + (\alpha+\beta)^{1/2}\psi(\sqrt{\alpha}, \sqrt{\alpha+\beta}, b\sqrt{2})\cosh\{a\sqrt{2\alpha}\}}{\sqrt{\alpha}\cosh\{a\sqrt{2\alpha}\} + (\alpha+\beta)^{1/2}\psi(\sqrt{\alpha}, \sqrt{\alpha+\beta}, b\sqrt{2})\sinh\{a\sqrt{2\alpha}\}} \right]. \end{aligned} \tag{3}$$

By taking the limit for $b \to \infty$, ($\psi(.,.,b\sqrt{2}) \to 1$, uniformly on compacta of Γ) we obtain for the Laplace transform of the occupation time $T(a) = T(a, \infty)$,

$$Ee^{-\beta T(a)} = -\frac{1}{i\sqrt{\pi}} \int_\Gamma \sqrt{\alpha} e^\alpha \psi(\sqrt{\alpha+\beta}, \sqrt{\alpha}, a\sqrt{2})\, d\alpha. \tag{4}$$

Alternatively, one could take the limit for $a \downarrow 0$ in (3), resulting in the transform: $Ee^{-\beta(1-T(b))}$. For the occcupation time $T_t(a)$ of the excursion straddling t, we have

$$T_t(a) \stackrel{d}{=} (L_t)^{1/2}\, T(a\,(L_t)^{-1/2}), \tag{5}$$

with $T(a)$ and L_t independent, and where L_t denotes the length of the excursion. It is readily verified from the density of L_t, see [1], (4.4), that for integrable φ,

$$\int_0^\infty e^{-\alpha t} E\varphi(L_t)\, dt = \frac{1}{2\sqrt{\pi\alpha^3}} \int_0^\infty \varphi(y)(1 - e^{-\alpha y})\, dy. \tag{6}$$

[1] According to the Poisson-summation formula

$$\frac{\sqrt{2\pi}}{a^3} \sum_{k=1}^{\infty} k^2\pi^2 e^{-k^2\pi^2/2a^2} = 1 + 2\sum_{k=1}^{\infty}(1 - 4k^2a^2)e^{-2k^2a^2},$$

which is the more familiar form of this distribution function.

Hence, using (5) and (6), the Laplace transform (4) yields the double Laplace transform:

$$\begin{aligned}
&\int_0^\infty e^{-\alpha t} E e^{-\beta T_t(a)}\,dt \qquad (7)\\
&= \frac{1}{\alpha}\psi(\sqrt{\alpha+\beta},\sqrt{\alpha},a\sqrt{2}) - \frac{1}{\alpha^{3/2}}\lim_{\alpha\downarrow 0}\sqrt{\alpha}\psi(\sqrt{\alpha+\beta},\sqrt{\alpha},a\sqrt{2})\\
&= \frac{1}{\alpha}\left[\frac{\sqrt{\alpha}\sinh\{a\sqrt{2\alpha}\}+\sqrt{\alpha+\beta}\cosh\{a\sqrt{2\alpha}\}}{\sqrt{\alpha}\cosh\{a\sqrt{2\alpha}\}+\sqrt{\alpha+\beta}\sinh\{a\sqrt{2\alpha}\}}\right] - \frac{1}{\alpha^{3/2}}\frac{\sqrt{\beta}}{1+a\sqrt{2\beta}}.
\end{aligned}$$

This result can also be derived starting from reflected Brownian motion $|W|$ (cf. [3], p. 92, Remark (3.20)).

Perhaps the most elegant formulation of the Laplace transform of the occupation time is that for β strictly positive

$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{e^{-\alpha x}}{x^{3/2}}\left[1-Ee^{-\beta x T(x^{-1/2})}\right]dx \qquad (8)\\
&= \frac{2\sqrt{2\alpha}(\sqrt{\alpha+\beta}-\sqrt{\alpha})}{(\sqrt{\alpha}+\sqrt{\alpha+\beta})e^{2\sqrt{2\alpha}}+(\sqrt{\alpha}-\sqrt{\alpha+\beta})}.
\end{aligned}$$

Equation (8) can be derived as follows. On the path Γ we have:

$$\begin{aligned}
1-Ee^{-\beta T(a)} &= -\frac{1}{i\sqrt{\pi}}\int_\Gamma \sqrt{\alpha}e^{\alpha}\,d\alpha + \frac{1}{i\sqrt{\pi}}\int_\Gamma \sqrt{\alpha}e^{\alpha}\psi(\sqrt{\alpha+\beta},\sqrt{\alpha},a\sqrt{2})\,d\alpha\\
&= \frac{1}{i\sqrt{\pi}}\int_\Gamma \sqrt{\alpha}e^{\alpha}\left[\frac{2(\sqrt{\alpha+\beta}-\sqrt{\alpha})e^{-a\sqrt{2\alpha}}}{(\sqrt{\alpha}+\sqrt{\alpha+\beta})e^{a\sqrt{2\alpha}}+(\sqrt{\alpha}-\sqrt{\alpha+\beta})e^{-a\sqrt{2\alpha}}}\right]d\alpha.
\end{aligned}$$

Now for $a>0$ the integral over the path Γ may be replaced by integration over the line $(c-i\infty,c+i\infty)$, where $c>0$ is arbitrary. Hence after the substitution $\alpha = xz$, with x positive and replacement of the path $(c/x-i\infty,c/x+i\infty)$ by the path $(c-i\infty,c+i\infty)$, we obtain

$$\begin{aligned}
&x^{-3/2}\left(1-Ee^{-\beta x T(x^{-1/2})}\right)\\
&= \frac{1}{i\sqrt{\pi}}\int_{c-i\infty}^{c+i\infty}\sqrt{z}e^{xz}\left[\frac{2(\sqrt{z+\beta}-\sqrt{z})e^{-\sqrt{2z}}}{(\sqrt{z}+\sqrt{z+\beta})e^{\sqrt{2z}}+(\sqrt{z}-\sqrt{z+\beta})e^{-\sqrt{2z}}}\right]dz.
\end{aligned}$$

Taking Laplace transforms on both sides gives (8).

Each of the representations (4), (7) or (8) is equivalent to Theorem 1 of [5], where the duration of the excursion was scaled with a gamma$(\frac{1}{2},\frac{1}{2}\nu^2)$ density. In particular, Theorem 1 of [5] can be obtained from (8) by differentiating both sides with respect to α and using that

$$\int_0^\infty x^{-1/2}e^{-\alpha x}\,dx = \sqrt{\pi/\alpha}.$$

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