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A NECESSARY AND SUFFICIENT CONDITION FOR THE Λ-COALESCENT TO COME DOWN FROM INFINITY.

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Abstract

Let Π_∞ be the standard Λ-coalescent of Pitman, which is defined so that $\Pi_\infty(0)$ is the partition of the positive integers into singletons, and, if Π_n denotes the restriction of Π_∞ to $\{1, \dots, n\}$, then whenever $\Pi_n(t)$ has b blocks, each k-tuple of blocks is merging to form a single block at the rate $\lambda_{b,k}$, where

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\,\Lambda(dx)$$

for some finite measure Λ. We give a necessary and sufficient condition for the Λ-coalescent to "come down from infinity", which means that the partition $\Pi_\infty(t)$ almost surely consists of only finitely many blocks for all $t > 0$. We then show how this result applies to some particular families of Λ-coalescents.

1 Introduction

Let Λ be a finite measure on the Borel subsets of $[0, 1]$. Let Π_∞ be the standard Λ-coalescent, which is defined in [4] and also studied in [5]. Then Π_∞ is a Markov process whose state space is the set of partitions of the positive integers. For each positive integer n, let Π_n denote the restriction of Π_∞ to $\{1, \dots, n\}$. When $\Pi_n(t)$ has b blocks, each k-tuple of blocks is merging to form a single block at the rate $\lambda_{b,k}$, where

$$\lambda_{b,k} = \int_0^1 x^{k-2}(1-x)^{b-k}\,\Lambda(dx). \tag{1}$$

Note that this rate does not depend on n or the sizes of the blocks. For $b = 2, 3, \dots$, define

$$\lambda_b = \sum_{k=2}^{b} \binom{b}{k} \lambda_{b,k},$$

which is the total rate at which mergers are occurring. Also define

$$\gamma_b = \sum_{k=2}^{b} (k-1) \binom{b}{k} \lambda_{b,k}, \tag{2}$$

which is the rate at which the number of blocks is decreasing because merging k blocks into one decreases the number of blocks by $k-1$. For $n = 1, 2, \dots, \infty$, let $\#\Pi_n(t)$ denote the number of blocks in the partition $\Pi_n(t)$. Then let $T_n = \inf\{t : \#\Pi_n(t) = 1\}$. As stated in (31) of [4], we have

$$0 = T_1 < T_2 \leq T_3 \leq \ldots \uparrow T_\infty \leq \infty. \tag{3}$$

We say the Λ-coalescent *comes down from infinity* if $P(\#\Pi_\infty(t) < \infty) = 1$ for all $t > 0$, and we say it *stays infinite* if $P(\#\Pi_\infty(t) = \infty) = 1$ for all $t > 0$. If Λ has no atom at 1, then Proposition 23 of [4] states that the Λ-coalescent must either come down from infinity, in which case $T_\infty < \infty$ almost surely, or stay infinite, in which case $T_\infty = \infty$ almost surely. We assume hereafter, without further mention, that Λ has no atom at 1. Example 20 of [4] provides a simple description of a Λ-coalescent in which Λ has an atom at 1 in terms of the coalescent with the atom at 1 removed.

In section 3.6 of [4], Pitman shows that the Λ-coalescent comes down from infinity if Λ has an atom at zero. It follows from Lemma 25 of [4] that the Λ-coalescent stays infinite if $\int_0^1 x^{-1} \, \Lambda(dx) < \infty$. Results in [1] imply that the Λ-coalescent stays infinite if Λ is the uniform distribution on $[0,1]$. Also, results in section 5 of [5] imply that if $\Lambda(dx) = (1-\alpha)x^{-\alpha} dx$ for some $\alpha \in (0,1)$, then the Λ-coalescent comes down from infinity.

Proposition 23 of [4] gives a necessary and sufficient condition, involving a recursion, for the Λ-coalescent to come down from infinity. The main goal of this paper is to give a simpler necessary and sufficient condition, which is stated in Theorem 1 below. This condition is much easier to check in examples than the condition given in [4].

Theorem 1 *The Λ-coalescent comes down from infinity if and only if*

$$\sum_{b=2}^{\infty} \gamma_b^{-1} < \infty. \tag{4}$$

We will prove this theorem in section 2.

The condition (4) can be expressed in other ways. For example, let

$$\eta_b = \sum_{k=2}^{b} k \binom{b}{k} \lambda_{b,k}. \tag{5}$$

Clearly $1 \leq k/(k-1) \leq 2$ for all $k \geq 2$, so $\gamma_b \leq \eta_b \leq 2\gamma_b$ for all $b \geq 2$. Therefore, we obtain the following corollary.

Corollary 2 *The Λ-coalescent comes down from infinity if and only if*

$$\sum_{b=2}^{\infty} \eta_b^{-1} < \infty. \tag{6}$$

The formulation of the condition given in Theorem 1 seems more natural conceptually, because of the interpretation of γ_b as the rate at which the number of blocks is decreasing, and is easier to use for the proof. However, the formulation in Corollary 2 is more convenient for the calculations in section 3, where we give examples of measures Λ for which the Λ-coalescent comes down from infinity and other examples of measures Λ for which the Λ-coalescent stays infinite.

2 Proof of the necessary and sufficient condition

In this section, we prove Theorem 1, which follows immediately from Lemmas 6 and 9 below. We begin by collecting facts about the γ_b and the η_b.

Lemma 3 *We have*

$$\gamma_b = \int_0^1 (bx - 1 + (1-x)^b)x^{-2}\, \Lambda(dx) \tag{7}$$

and

$$\eta_b = b\int_0^1 (1-(1-x)^{b-1})x^{-1}\, \Lambda(dx) = b\sum_{k=0}^{b-2}\int_0^1 (1-x)^k\, \Lambda(dx). \tag{8}$$

Also, the sequence $(\gamma_b)_{b=2}^\infty$ is increasing.

Proof. From the identities

$$\sum_{k=0}^{b}\binom{b}{k} x^k(1-x)^{b-k} = 1$$

and

$$\sum_{k=0}^{b} k\binom{b}{k} x^k(1-x)^{b-k} = bx,$$

it follows that

$$\sum_{k=2}^{b}(k-1)\binom{b}{k} x^{k-2}(1-x)^{b-k} = (bx-1+(1-x)^b)x^{-2} \tag{9}$$

and

$$\sum_{k=2}^{b} k\binom{b}{k} x^{k-2}(1-x)^{b-k} = b(1-(1-x)^{b-1})x^{-1} = b\sum_{k=0}^{b-2}(1-x)^k. \tag{10}$$

Then (7) and (8) follow by integrating (9) and (10) with respect to $\Lambda(dx)$. Therefore,

$$\gamma_{b+1} - \gamma_b = \int_0^1 (x + (1-x)^{b+1} - (1-x)^b)x^{-2}\, \Lambda(dx) = \int_0^1 (1-(1-x)^b)x^{-1}\, \Lambda(dx) \geq 0,$$

which implies that $(\gamma_b)_{b=2}^\infty$ is increasing. □

The next step is to show that if the Λ-coalescent comes down from infinity, then it does so in finite expected time. We will need the lemma below, which we take from page 78 of [3].

Lemma 4 (Kochen-Stone Lemma). *Let $(A_n)_{n=1}^{\infty}$ be events such that $\sum_{n=1}^{\infty} P(A_n) = \infty$. Let A be the event that infinitely many of the A_n occur. Then,*

$$P(A) \geq \limsup_{n\to\infty} \frac{[\sum_{m=1}^{n} P(A_m)]^2}{\sum_{k=1}^{n}\sum_{m=1}^{n} P(A_k \cap A_m)}.$$

Proposition 5 *The Λ-coalescent comes down from infinity if and only if $E[T_\infty] < \infty$.*

Proof. If $E[T_\infty] < \infty$, then clearly $T_\infty < \infty$ almost surely, which means the Λ-coalescent comes down from infinity. We now prove the converse. For $m \geq 2$, let A_m be the event that m is not in same block as 1 at time T_{m-1}, which, up to a null set, is the same as the event $\{T_m > T_{m-1}\}$. On the event A_m, the partition $\Pi_m(T_{m-1})$ has two blocks, one of which is $\{1,\dots,m-1\}$ and the other of which is $\{m\}$. The expected time, after T_{m-1}, that it takes for these two blocks to merge is $\lambda_{2,2}^{-1}$. Therefore, using (3) and the Monotone Convergence Theorem to get the first equality, we have

$$E[T_\infty] = \lim_{n\to\infty} E[T_n] = \lim_{n\to\infty} \sum_{m=2}^{n} E[T_m - T_{m-1}] = \lim_{n\to\infty} \sum_{m=2}^{n} \lambda_{2,2}^{-1} P(A_m) = \lambda_{2,2}^{-1} \sum_{m=2}^{\infty} P(A_m). \tag{11}$$

Suppose $E[T_\infty] = \infty$. Then by (11), $\sum_{m=2}^{\infty} P(A_m) = \infty$. Let $\{B_{1,k}, B_{2,k}, \dots,\}$ be the blocks of $\Pi_\infty(T_k)$ in order of their smallest elements. Let $l_{i,k}$ be the smallest element of $B_{i,k}$. Note that $B_{i,k}$ and $l_{i,k}$ are undefined if $\Pi_\infty(T_k)$ has fewer than i blocks. Also note that if $m > k$, then unless $m = l_{i,k}$ for some $i \geq 2$, the event A_m can not occur. If $m = l_{i,k}$, then the event A_m only occurs if, at time T_{m-1}, the block $B_{i,k}$ is separate from the cluster containing the blocks $B_{1,k}, \dots, B_{i-1,k}$. Let $\mathcal{F}_{T_k} = \{A \in \mathcal{F}_\infty : A \cap \{T_k \leq t\} \in \mathcal{F}_t\}$, where $(\mathcal{F}_t)_{t\geq 0}$ is the smallest complete, right-continuous filtration with respect to which $(\Pi_\infty(t))_{t\geq 0}$ is adapted and $\mathcal{F}_\infty = \sigma(\bigcup_{t\geq 0} \mathcal{F}_t)$. Conditionally on $\mathcal{F}_{T_k}$, if $m = l_{i,k}$ then the probability that $B_{i,k}$ is separate from $B_{1,k}, \dots, B_{i-1,k}$ at time T_{m-1} is the same as the unconditional probability that $\{i\}$ is separate from the block containing $\{1, 2, \dots, i-1\}$ at time T_{i-1}, which is $P(A_i)$. Note that here we are using the strong Markov property of $(\Pi_\infty(t))_{t\geq 0}$, which is asserted in Theorem 1 of [4]. We have

$$\begin{aligned}
\sum_{m=k+1}^{n} P(A_k \cap A_m) &= E\left[\sum_{m=k+1}^{n} P(A_k \cap A_m | \mathcal{F}_{T_k})\right] = E\left[\sum_{m=k+1}^{n} 1_{A_k} P(A_m | \mathcal{F}_{T_k})\right] \\
&= E\left[1_{A_k} \sum_{i=2}^{\#\Pi_n(T_k)} P(A_{l_{i,k}} | \mathcal{F}_{T_k})\right] \leq E\left[1_{A_k} \sum_{i=2}^{n} P(A_i)\right] = P(A_k) \sum_{i=2}^{n} P(A_i).
\end{aligned}$$

Thus, for all n,

$$\begin{aligned}
\sum_{k=2}^{n}\sum_{m=2}^{n} P(A_k \cap A_m) &= 2\sum_{k=2}^{n}\sum_{m=k+1}^{n} P(A_k \cap A_m) + \sum_{m=2}^{n} P(A_m) \\
&\leq 2\sum_{k=2}^{n}\left(P(A_k)\sum_{i=2}^{n} P(A_i)\right) + \sum_{m=2}^{n} P(A_m) \\
&= 2\left(\sum_{m=2}^{n} P(A_m)\right)^2 + \sum_{m=2}^{n} P(A_m).
\end{aligned}$$

Since $\sum_{m=2}^{\infty} P(A_m) = \infty$, we have $\left(\sum_{m=2}^{n} P(A_m)\right)/\left(\sum_{m=2}^{n} P(A_m)\right)^2 \to 0$ as $n \to \infty$. Thus,

$$\limsup_{n\to\infty} \frac{[\sum_{m=2}^{n} P(A_m)]^2}{\sum_{k=2}^{n}\sum_{m=2}^{n} P(A_k \cap A_m)} \geq \limsup_{n\to\infty} \frac{[\sum_{m=2}^{n} P(A_m)]^2}{2[\sum_{m=2}^{n} P(A_m)]^2 + \sum_{m=2}^{n} P(A_m)} = \frac{1}{2}.$$

By the Kochen-Stone Lemma, with probability at least $1/2$ infinitely many of the A_n occur. If infinitely many of the A_n occur, then $\#\Pi_\infty(T_2) = \infty$. We have $T_2 > 0$ by (3). Therefore, $P(\#\Pi_\infty(t) = \infty) > 0$ for some $t > 0$, which means $P(\#\Pi_\infty(t) = \infty) = 1$ for all $t > 0$. Hence, the Λ-coalescent stays infinite. □

Thus, to determine whether the Λ-coalescent comes down from infinity, it suffices to determine whether $E[T_\infty] < \infty$. Since $(E[T_n])_{n=1}^{\infty} \uparrow E[T_\infty]$ by (3) and the Monotone Convergence Theorem, the Λ-coalescent comes down from infinity if and only if $(E[T_n])_{n=1}^{\infty}$ is bounded.

Lemma 6 *If $\sum_{b=2}^{\infty} \gamma_b^{-1} < \infty$, then the Λ-coalescent comes down from infinity.*

Proof. Fix $n < \infty$, and recursively define times $R_0, R_1, \ldots, R_{n-1}$ by:

$$\begin{aligned}
&R_0 = 0 && \\
&R_i = \inf\{t : \#\Pi_n(t) < \#\Pi_n(R_{i-1})\} && \text{if } i \geq 1 \text{ and } \#\Pi_n(R_{i-1}) > 1. \\
&R_i = R_{i-1} && \text{if } i \geq 1 \text{ and } \#\Pi_n(R_{i-1}) = 1.
\end{aligned}$$

Note that $R_{n-1} = T_n$. For $i = 0, 1, \ldots, n-1$, let $N_i = \#\Pi_n(R_i)$. For $i = 1, 2, \ldots, n-1$, define $L_i = R_i - R_{i-1}$ and $J_i = N_{i-1} - N_i$. We have $E[L_i|N_{i-1}] = \lambda_{N_{i-1}}^{-1}$ on the set $\{N_{i-1} > 1\}$. Also, $E[J_i|N_{i-1}] = \gamma_{N_{i-1}}\lambda_{N_{i-1}}^{-1}$ on $\{N_{i-1} > 1\}$ because

$$P(J_i = k-1|N_{i-1} = b) = \binom{b}{k}\frac{\lambda_{b,k}}{\lambda_b}$$

for all $b > 1$. Thus,

$$\begin{aligned}
E[T_n] &= E[R_{n-1}] = E\left[\sum_{i=1}^{n-1} L_i\right] = \sum_{i=1}^{n-1} E[E[L_i|N_{i-1}]] = \sum_{i=1}^{n-1} E[\lambda_{N_{i-1}}^{-1} 1_{\{N_{i-1}>1\}}] \\
&= \sum_{i=1}^{n-1} E[\gamma_{N_{i-1}}^{-1} E[J_i|N_{i-1}] 1_{\{N_{i-1}>1\}}] = \sum_{i=1}^{n-1} E[E[\gamma_{N_{i-1}}^{-1} J_i 1_{\{N_{i-1}>1\}}|N_{i-1}]].
\end{aligned}$$

Since $J_i = 0$ on $\{N_{i-1} = 1\}$, we have

$$E[T_n] = \sum_{i=1}^{n-1} E[E[\gamma_{N_{i-1}}^{-1} J_i|N_{i-1}]] = \sum_{i=1}^{n-1} E[\gamma_{N_{i-1}}^{-1} J_i] = E\left[\sum_{i=1}^{n-1} \gamma_{N_{i-1}}^{-1} J_i\right] = E\left[\sum_{i=1}^{n-1}\sum_{j=0}^{J_i-1} \gamma_{N_{i-1}}^{-1}\right]. \tag{12}$$

Since $(\gamma_b)_{b=2}^{\infty}$ is increasing by Lemma 3, we have

$$E[T_n] \leq E\left[\sum_{i=1}^{n-1}\sum_{j=0}^{J_i-1} \gamma_{N_{i-1}-j}^{-1}\right] = E\left[\sum_{b=2}^{n} \gamma_b^{-1}\right] < \sum_{b=2}^{\infty} \gamma_b^{-1}.$$

Thus, if $\sum_{b=2}^{\infty} \gamma_b^{-1} < \infty$, then $(E[T_n])_{n=1}^{\infty}$ is bounded, which proves the lemma. □

We now work towards the converse of Lemma 6, which we first prove in the special case that Λ has no mass in $(1/2, 1]$.

Lemma 7 *Suppose Λ is concentrated on $[0,1/2]$, and suppose $\sum_{b=2}^{\infty}\gamma_b^{-1} = \infty$. Then, the Λ-coalescent stays infinite.*

Proof. Fix positive integers b and l such that $b > 2^l$. Consider a Λ-coalescent with b blocks. Let $R_{b,1}$ be the total rate of all collisions that would take the coalescent down to 2^l or fewer blocks. Let $R_{b,2}$ be the total rate of all collisions that would take the coalescent down to between $2^{l-1}+1$ and 2^l blocks. We have

$$R_{b,1} = \sum_{k=b-2^l+1}^{b} \binom{b}{k} \lambda_{b,k} = \sum_{i=0}^{2^l-1} \binom{b}{b-i} \lambda_{b,b-i} = \sum_{i=0}^{2^l-1} \binom{b}{i} \int_0^{1/2} x^{b-i-2}(1-x)^i \, \Lambda(dx). \quad (13)$$

Likewise,

$$R_{b,2} = \sum_{k=b-2^l+1}^{b-2^{l-1}} \binom{b}{k} \lambda_{b,k} = \sum_{i=2^{l-1}}^{2^l-1} \binom{b}{i} \int_0^{1/2} x^{b-i-2}(1-x)^i \, \Lambda(dx). \quad (14)$$

If $0 \le j \le 2^{l-1}-1$, then

$$\binom{b}{j} \le \binom{b}{2^l-1-j}. \quad (15)$$

Also, we have

$$x^{b-j-2}(1-x)^j \le x^{b-(2^l-1-j)-2}(1-x)^{2^l-1-j} \quad (16)$$

for all $x \in [0,1/2]$, because the ratio of the right-hand side to the left-hand side in (16) is $((1-x)/x)^{2^l-2j-1} \ge 1$. Equations (13)-(16) imply that $R_{b,1} - R_{b,2} \le R_{b,2}$, and so $R_{b,2}/R_{b,1} \ge 1/2$.

Let Π_n be a standard Λ-coalescent restricted to $\{1,\ldots,n\}$. For l such that $2^l \le n$, let D_l be the event that $2^{l-1}+1 \le \#\Pi_n(t) \le 2^l$ for some t. By conditioning on the value of N_{K-1}, where $K = \inf\{i : N_i \le 2^l\}$, we see from the above calculation that $P(D_l) \ge 1/2$.

Suppose $n = 2^m$. For $j = 2,3,\ldots,n$, let $L_n(j) = \min\{s \ge j : \#\Pi_n(t) = s \text{ for some } t\}$. If $N_{i-1} \ge j > N_i$, or equivalently if $N_i + J_i \ge j > N_i$, then $L_n(j) = N_{i-1}$. Therefore, using (12) for the first equality, we have

$$E[T_n] = \sum_{i=1}^{n-1} E[\gamma_{N_{i-1}}^{-1} J_i] = \sum_{j=2}^{n} E[\gamma_{L_n(j)}^{-1}] = \sum_{l=1}^{m} \sum_{j=2^{l-1}+1}^{2^l} E[\gamma_{L_n(j)}^{-1}].$$

Since $(\gamma_b)_{b=2}^{\infty}$ is increasing by Lemma 3 and $L_n(j) \le 2^{l+1}$ on D_{l+1} when $j \le 2^l$, we have

$$\begin{aligned} E[T_n] &\ge \sum_{l=1}^{m-1} \sum_{j=2^{l-1}+1}^{2^l} E[\gamma_{L_n(j)}^{-1}] \ge \sum_{l=1}^{m-1} \sum_{j=2^{l-1}+1}^{2^l} P(D_{l+1})\gamma_{2^{l+1}}^{-1} \\ &\ge \frac{1}{2}\sum_{l=1}^{m-1} 2^{l-1}\gamma_{2^{l+1}}^{-1} = \frac{1}{8}\sum_{l=1}^{m-1} 2^{l+1}\gamma_{2^{l+1}}^{-1}. \end{aligned}$$

Therefore, using the monotonicity of the sequence $(T_n)_{n=1}^{\infty}$ for the first equality, we have

$$\lim_{n\to\infty} E[T_n] = \lim_{m\to\infty} E[T_{2^m}] \ge \lim_{m\to\infty} \frac{1}{8}\sum_{l=1}^{m-1} 2^{l+1}\gamma_{2^{l+1}}^{-1} \ge \frac{1}{8}\sum_{l=4}^{\infty} \gamma_l^{-1} = \infty.$$

Hence, the Λ-coalescent stays infinite. $\square$

Lemma 8 *Fix $a > 0$. Let Λ_1 be the restriction of Λ to $[0, a]$. Suppose the Λ_1-coalescent stays infinite. Then, the Λ-coalescent stays infinite.*

Proof. Let Λ_2 be the restriction of Λ to $(a, 1]$. Then $\Lambda = \Lambda_1 + \Lambda_2$. We consider a Poisson process construction of the Λ-coalescent, as given in the discussion preceding Corollary 3 of [4]. This construction is valid as long as Λ has no atom at zero. Here, Λ_2 clearly has no atom at zero, and Λ_1 has no atom at zero because, as stated in the discussion following Proposition 23 of [4], the Λ_1-coalescent comes down from infinity if Λ_1 has an atom at zero. Let N_1 and N_2 be independent Poisson point processes on $(0, \infty) \times \{0, 1\}^{\infty}$ such that N_i has intensity $dt\, L_i(d\xi)$ for $i = 1, 2$, where

$$L_i(A) = \int_0^1 x^{-2} P_x(A)\, \Lambda_i(dx)$$

for all product measurable $A \subset \{0, 1\}^{\infty}$ and P_x is the law of a sequence $\xi = (\xi_i)_{i=1}^{\infty}$ of independent Bernoulli random variables, each of which takes on the value 1 with probability x. Let N be the Poisson point process consisting of all of the points of N_1 and N_2, so that N has intensity $dt\, L(d\xi)$, where

$$L(A) = L_1(A) + L_2(A) = \int_0^1 x^{-2} P_x(A)\, \Lambda(dx)$$

for all product measurable A.
We now define, for each n, a coalescent Markov chain Π_n. We define $\Pi_n(0)$ to be the partition of $\{1, \dots, n\}$ consisting of n singletons. We allow Π_n possibly to jump at the times t of points (t, ξ) of N such that $\sum_{i=1}^{n} \xi_i \geq 2$. For such t, if $\Pi_n(t-)$ consists of the blocks $B_1, \dots, B_b$, then $\Pi_n(t)$ is defined by merging all of the blocks B_i such that $\xi_i = 1$. By Corollary 3 of [4], these processes Π_n determine a unique coalescent process Π_{∞} whose restriction to $\{1, \dots, n\}$ is Π_n for all n, and Π_{∞} is a standard Λ-coalescent. For $i = 1, 2$, define $\Pi_n^{(i)}$ analogously, only allowing jumps at times t of points (t, ξ) of N_i. These processes give rise to a Λ_1-coalescent $\Pi_{\infty}^{(1)}$ and a Λ_2-coalescent $\Pi_{\infty}^{(2)}$.
Note that

$$\int_0^1 x^{-2} \Lambda_2(dx) = \int_a^1 x^{-2} \Lambda_2(dx) \leq a^{-2} \Lambda_2([0, 1]) < \infty,$$

which, as stated in section 2.1 of [4], means that the Λ_2-coalescent holds in its initial state for an exponential time of rate at most $a^{-2}\Lambda_2([0, 1])$. Therefore, given $t > 0$, there is some probability $p > 0$ that there are no points (s, ξ) in N_2 with $s \leq t$. Therefore, with probability at least p, we have $\Pi_{\infty}(t) = \Pi_{\infty}^{(1)}(t)$. However, since the Λ_1-coalescent stays infinite, we have $\#\Pi_{\infty}^{(1)}(t) = \infty$ almost surely. Thus, $\#\Pi_{\infty}(t) = \infty$ with probability at least p, which by Proposition 23 of [4] implies that the Λ-coalescent stays infinite. $\square$

Lemma 9 *If $\sum_{b=2}^{\infty} \gamma_b^{-1} = \infty$, then the Λ-coalescent stays infinite.*

Proof. Let Λ_1 be the restriction of Λ to $[0, 1/2]$, and let Λ_2 be the restriction of Λ to $(1/2, 1]$. Then, $\Lambda = \Lambda_1 + \Lambda_2$. For $i = 1, 2$, let $\gamma_b^{(i)}$ be the quantity for the Λ_i-coalescent analogous to that defined by (2) for the Λ-coalescent. From (1) and (2), we see that $\gamma_b^{(1)} \leq \gamma_b$ for all b, so $\sum_{b=2}^{\infty} (\gamma_b^{(1)})^{-1} = \infty$. By Lemma 7, the Λ_1-coalescent stays infinite. It now follows from Lemma 8 that the Λ-coalescent stays infinite. $\square$

3 Consequences for some families of Λ-coalescents

In this section, we use Corollary 2 to determine whether the Λ-coalescent comes down from infinity for particular families of measures Λ. We begin with the following lemma. Note that if Λ_1 and Λ_2 are probability measures, then the hypothesis is equivalent to the condition that a random variable with distribution Λ_1 is stochastically smaller than a random variable with distribution Λ_2.

Lemma 10 *Suppose $\Lambda_1([0,x]) \geq \Lambda_2([0,x])$ for all $x \in [0,1]$. If the Λ_1-coalescent stays infinite, then the Λ_2-coalescent stays infinite. If the Λ_2-coalescent comes down from infinity, then the Λ_1-coalescent comes down from infinity.*

Proof. For $i = 1, 2$, define $\eta_b^{(i)}$ for the Λ_i-coalescent as in (5). For $x \in [0,1]$, let

$$g(x) = b \sum_{k=0}^{b-2} (1-x)^k.$$

Then $g'(x) < 0$ for all $x \in (0,1)$. Following a similar derivation on page 43 of [2], we apply Fubini's Theorem and Lemma 3 to get

$$\begin{aligned}
\int_0^1 g'(y)\Lambda_i([0,y])\,dy &= \int_0^1 g'(y)\left(\int_0^1 1_{[0,y]}(x)\,\Lambda_i(dx)\right)\,dy \\
&= \int_0^1 \left(\int_0^1 g'(y)1_{[0,y]}(x)\,dy\right)\,\Lambda_i(dx) = \int_0^1 \left(\int_x^1 g'(y)\,dy\right)\,\Lambda_i(dx) \\
&= \int_0^1 (g(1)-g(x))\,\Lambda_i(dx) = b\Lambda_i([0,1]) - \eta_b^{(i)}.
\end{aligned}$$

Therefore,

$$\eta_b^{(i)} = b\Lambda_i([0,1]) + \int_0^1 |g'(y)|\Lambda_i([0,y])\,dy.$$

It follows from the assumptions on Λ_1 and Λ_2 that $\eta_b^{(1)} \geq \eta_b^{(2)}$ for all $b \geq 2$. An application of Corollary 2 completes the proof. □

Corollary 2 can be interpreted to mean that the Λ-coalescent stays infinite whenever the η_b don't grow too rapidly as $b \to \infty$. Lemma 25 of [4] shows that the Λ-coalescent stays infinite when $\int_0^1 x^{-1}\,\Lambda(dx) < \infty$. This condition is equivalent to the condition that the η_b don't grow faster than $O(b)$, because by Lemma 3,

$$\lim_{b\to\infty} b^{-1}\eta_b = \sum_{k=0}^{\infty} \int_0^1 (1-x)^k\,\Lambda(dx) = \int_0^1 x^{-1}\,\Lambda(dx).$$

In Proposition 11 below, we exhibit another collection of measures Λ for which the Λ-coalescent stays infinite. Some of the measures do not satisfy the condition $\int_0^1 x^{-1}\,\Lambda(dx) < \infty$.

Proposition 11 *Suppose there exist $\epsilon > 0$ and $M < \infty$ such that $\Lambda([0,\delta]) \leq M\delta$ for all $\delta \in [0,\epsilon]$. Then the Λ-coalescent stays infinite.*

Proof. Let Λ_1 be the restriction of Λ to $[0,\epsilon]$. By Lemma 8, it suffices to prove that the Λ_1-coalescent stays infinite. Let U be the uniform distribution on $[0,1]$. As mentioned in section 3.6 of [4], it is a consequence of results in [1] that the U-coalescent stays infinite. Multiplying U by the constant M multiplies all of the γ_b by M. Therefore, the MU-coalescent also stays infinite. Since

$$\Lambda_1([0,x]) \leq Mx = (MU)([0,x])$$

for all $x \in [0,1]$, it follows from Lemma 10 that the Λ_1-coalescent stays infinite. □

Remark. Define η_b^u for the MU-coalescent as in (5). We can also show that the MU-coalescent stays infinite by using Lemma 3 to calculate

$$\eta_b^u = Mb \sum_{k=0}^{b-2} \int_0^1 (1-x)^k \, dx = Mb \sum_{k=0}^{b-2} \frac{1}{k+1} \leq Cb \log b$$

for some $C < \infty$ not depending on b. Thus,

$$\sum_{b=2}^{\infty} (\eta_b^u)^{-1} \geq \frac{1}{C} \sum_{b=2}^{\infty} \frac{1}{b \log b} \geq \frac{1}{C} \int_2^{\infty} \frac{1}{x \log x} \, dx = \infty,$$

where the integral diverges because $\log(\log x)$ is an antiderivative of $1/x \log x$.

There also exist measures Λ with densities that approach infinity as $x \to 0$ for which the Λ-coalescent stays infinite, as the following example shows.

Example 12 Suppose, for some $\epsilon < 1/e$, Λ has a Radon-Nikodym derivative f with respect to Lebesgue measure given by $f(x) = \log(\log(1/x))$ when $x \in (0,\epsilon)$ and $f(x) = 0$ otherwise. Then there exists a constant $C_1 < \infty$ such that for all $k > 1/\epsilon$, we have

$$\begin{aligned}
\int_0^1 (1-x)^k \, \Lambda(dx) &= \sum_{n=1}^{\infty} \int_{k^{-(n+1)}}^{k^{-n}} (1-x)^k \log(\log(1/x)) \, dx + \int_{k^{-1}}^{\epsilon} (1-x)^k \log(\log(1/x)) \, dx \\
&\leq \sum_{n=1}^{\infty} k^{-n} \log(\log k^{n+1}) + \log(\log k) \int_0^1 (1-x)^k \, dx \\
&= \sum_{n=1}^{\infty} k^{-n} \log(\log k) + \sum_{n=1}^{\infty} k^{-n} \log(n+1) + (k+1)^{-1} \log(\log k) \\
&\leq C_1 k^{-1}(1 + \log(\log k)).
\end{aligned}$$

Let N be the smallest integer such that $N \geq 1 + 1/\epsilon$. Then there exist constants C_2 and C_3 not depending on b such that for $b \geq N+2$, we have

$$\begin{aligned}
b^{-1}\eta_b &\leq C_2 + C_1 \sum_{k=N}^{b-2} k^{-1}(1 + \log(\log k)) \leq C_2 + C_1 \int_e^b x^{-1}(1 + \log(\log x)) \, dx \\
&= C_2 + C_1 (\log b)(\log(\log b)) \leq C_3 (\log b)(\log(\log b)).
\end{aligned}$$

Thus,

$$\sum_{b=2}^{\infty} \eta_b^{-1} \geq \sum_{b=N+2}^{\infty} \eta_b^{-1} \geq \frac{1}{C_3} \int_{N+2}^{\infty} \frac{1}{x(\log x)(\log(\log x))} \, dx = \infty,$$

where the divergence of the integral can be seen after substituting $u = \log(x)$. By Corollary 2, the Λ-coalescent stays infinite.

We now exhibit a family of measures Λ for which the Λ-coalescent comes down from infinity. The family is slightly larger than that studied in section 5 of [5].

Proposition 13 *Suppose there exist $\epsilon > 0$, $M > 0$, and $\alpha \in (0,1)$ such that $\Lambda([0,\delta]) \geq M\delta^\alpha$ for all $\delta \in [0,\epsilon]$. Then the Λ-coalescent comes down from infinity.*

Proof. By Lemma 10, it suffices to prove the result when $\Lambda([0,\delta]) = M\delta^\alpha$ for all $\delta \in [0,\epsilon]$ and $\Lambda((\epsilon,1]) = 0$. We may therefore assume that the Radon-Nikodym derivative of Λ with respect to Lebesgue measure is given by $M\alpha x^{\alpha-1}$ on $[0,\epsilon]$ and 0 on $(\epsilon,1]$. We then have

$$\int_0^1 (1-x)^k \Lambda(dx) = M\alpha \int_0^1 x^{\alpha-1}(1-x)^k \, dx = M\alpha B(\alpha, k+1) = \frac{M\alpha\Gamma(\alpha)\Gamma(k+1)}{\Gamma(k+1+\alpha)},$$

where B denotes the beta function. By Stirling's formula, $\Gamma(k+1)/\Gamma(k+1+\alpha) \sim k^{-\alpha}$, where $\sim$ denotes asymptotic equivalence as $k \rightarrow \infty$. Therefore, there exists a constant $C_1 > 0$ such that $\int_0^1 (1-x)^k \, \Lambda(dx) \geq C_1 k^{-\alpha}$ for all $k \geq 1$. Then, for some $C_2 > 0$, we have

$$\eta_b = b \sum_{k=0}^{b-2} \int_0^1 (1-x)^k \, \Lambda(dx) \geq b\Lambda([0,1]) + C_1 b \sum_{k=1}^{b-2} k^{-\alpha} \geq C_2 b^{2-\alpha}$$

for all $b \geq 2$. Thus, $\sum_{b=1}^\infty \eta_b^{-1} < \infty$, so the Λ-coalescent comes down from infinity. $\square$

The following example shows that the result above is not sharp.

Example 14 Suppose the Radon-Nikodym derivative of Λ with respect to Lebesgue measure on $[0,1]$ is given by $f(x) = \log(1/x)$. For $k \geq 1$, we have

$$\begin{aligned} \int_0^1 (1-x)^k \log(1/x) \, dx &\geq \int_0^{k^{-1}} (1-x)^k \log(1/x) \, dx \\ &\geq \left(1 - \frac{1}{k}\right)^k \int_0^{k^{-1}} \log(1/x) \, dx \\ &= \left(1 - \frac{1}{k}\right)^k k^{-1}(1 - \log(1/k)) \geq \frac{C_1 \log k}{k} \end{aligned}$$

for some constant $C_1 > 0$. It follows that for all $b \geq 2$,

$$\eta_b \geq b + C_1 b \sum_{k=1}^{b-2} \frac{\log k}{k} \geq C_2 b (\log b)^2$$

for some $C_2 > 0$. We can see by substituting $u = \log x$ that

$$\int_2^\infty \frac{1}{x(\log x)^2} \, dx < \infty.$$

Therefore, $\sum_{b=2}^\infty \eta_b^{-1} < \infty$, and the Λ-coalescent comes down from infinity.

Example 15 Suppose Λ has the beta density $f(x) = B(\alpha,\beta)^{-1} x^{\alpha-1}(1-x)^{\beta-1}$ with respect to Lebesgue measure on $[0,1]$, where $\alpha > 0$ and $\beta > 0$. If $\alpha \in (0,1)$, then Λ satisfies the hypotheses of Proposition 13. If $\alpha \geq 1$, then Λ satisfies the hypotheses of Proposition 11. Thus, the Λ-coalescent comes down from infinity if and only if $\alpha < 1$.

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