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RANDOM WALKS THAT AVOID THEIR PAST CONVEX HULL

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Abstract

We explore planar random walk conditioned to avoid its past convex hull. We prove that it escapes at a positive lim sup speed. Experimental results show that fluctuations from a limiting direction are on the order of $n^{3/4}$. This behavior is also observed for the extremal investor, a natural financial model related to the planar walk.

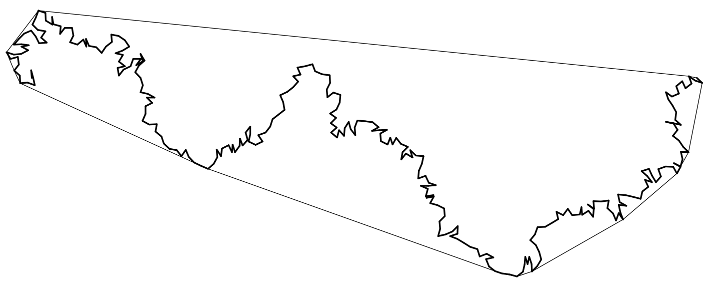

Figure 1: 300 steps of the rancher

1 Introduction

Consider the following walk in $\mathbb{R}^d$. Let $x_0 = 0$, and given the past $x_0, \ldots, x_n$ let x_{n+1} to be uniformly distributed on the sphere of radius 1 around x_n but conditioned so that the step segment $\overline{x_n x_{n+1}}$ does not intersect the interior of the convex hull of $\{x_0, \ldots, x_n\}$. We will call this process the **rancher's walk** or simply the rancher.

The name comes form the planar case: a frontier rancher who is walking about and at each step increases his ranch by dragging with him the fence that defines it, so that the ranch at any time is the convex hull of the path traced until that time. This paper studies the planar case of the process.

Since the model provides some sort of "repulsion" of the rancher from his past, it can be expected that the rancher will escape faster than a regular random walk. In fact, he has positive lim sup speed.

Theorem 1 *There exists a constant* $s > 0$ *such that* $\limsup \|x_n\|/n \geq s$ *a.s.*

Simulations suggest that in fact $\|x_n\|/n$ converges a.s. to some fixed $s \approx 0.314$.

In Section 2 we discuss simulations of the model. In particular, we consider how far is the ranch after n steps from a straight line segment. The experiments suggest that the farthest point in the path from the line ox_n connecting its end-points is at a distance of order $n^{3/4}$.

In Section 3 we discuss a related one-dimensional model that we call the **extremal investor**. This models the fluctuations in a stock's price, when it is subject to market forces that depend on the stock's best and worst past performance in a certain simple way. As a result, the relation between the stock's history and its drift is similar to the same relation for the rancher. Simulations for the critical case of this process yield the same exponent 3/4, distinguishing it from one-dimensional Brownian motion where the exponent equals 1/2.

The rancher's walk falls into the large category of self-interacting random walks, such as reinforced, self-avoiding, or self-repelling walks. These models are difficult to analyze in general. The reader should consult [1], [2], [6], [4], and especially the survey papers [5], [3] for examples.

2 Simulations: scaling limit and the exponent 3/4

Unlike the self-avoiding walk, the rancher is not difficult to simulate in nearly linear time. At any given time we only need to keep track of the convex hull of the random walk's trace so far. If the points on the boundary of the convex hull are kept in cyclic order, updating the convex hull is a matter of finding the largest and smallest elements in a cyclic array, which is monotone on each of the two arcs connecting the extreme values.

With at most n point on the hull, one can update it in order $\log n$ time, giving a running time of order $n \log n$ for n steps of the walk. In fact, the number of points defining the convex hull is much smaller than n, and the extremal elements tend to be very close in the array to the previous point of the walk. This and the actual running times suggest that the theoretical running time is close to linear.

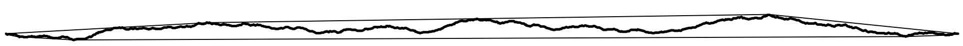

Figure 2: A million step sample of the rancher

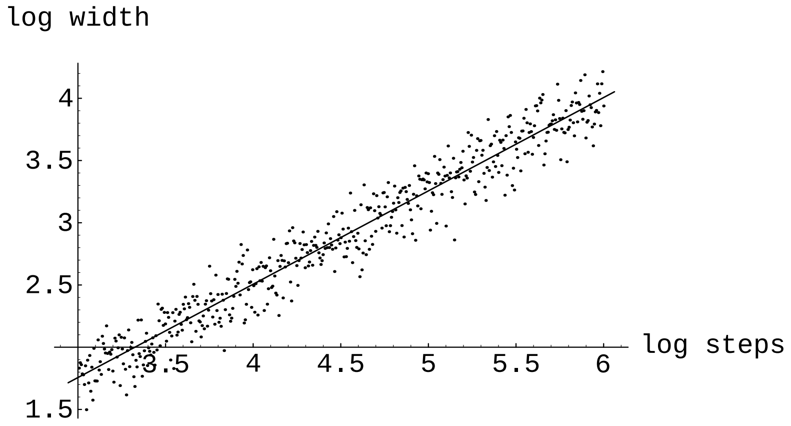

Figure 3: The dimensions of the ranch

In our simulations $\|x_n\|/n$ appears to converge to some fixed $s \approx 0.314$. Assuming this is the case, the rancher's walk is similar to the random walk in the plane conditioned to always increase its distance from the origin. Since the distance is linear in n and the step size is fixed, the angular change is of order n^{-1}. If the signs of the angular change were independent this would imply the following.

Conjecture 1 (Angular convergence) *The process $x_n/\|x_n\|$ converges a.s.*

The difficulty in our case is that the angular movements are positively correlated: if a step has a positive angular component, then subsequent steps have a drift in the same direction. Our simulations suggest that these correlations are not strong enough to prevent angular convergence, and we conjecture that this is in fact the case. This is observed in Figure 2, showing a million-step sample of the rancher's walk. Still larger simulations yield a picture indistinguishable from a straight line segment.

The scaled path of the rancher's walk appears to converge to a straight line segment, and it is natural to ask how quickly this happens. If we assume positive speed and angular convergence, then each step has a component in the eventual (say horizontal) direction and a component in the perpendicular (vertical) direction.

If the vertical components were independent, the vertical movement would essentially be a simple one-dimensional random walk. Since the horizontal component increase linearly, the path is then roughly the graph of a one-dimensional random walk.

To test this, we measured a related quantity, the width w_n of the path at time n, defined as the distance of the farthest point on the path from the line ox_n. Under the above assumptions, one would guess that w_n should behave as the maximum up to time n of the absolute value of a one-dimensional random walk with bounded steps, and have a typical value of order $n^{1/2}$.

Our simulations, however, show an entirely different picture. Figure 3 is a log base 10 plot of 500 realizations of w_n on independent processes. n ranges from a thousand to a million steps equally spaced on the log scale. The slope of the regression line is 0.746 (SE 0.008). A regression line on the medians of 1000 measurements of walks of length $10^3, 10^4, 10^5, 10^6$ gave a value of .75002 (SE 0.002). Based on these simulations, we conjecture that w_n behaves like $n^{3/4}$. To put it rigorously in a weak form:

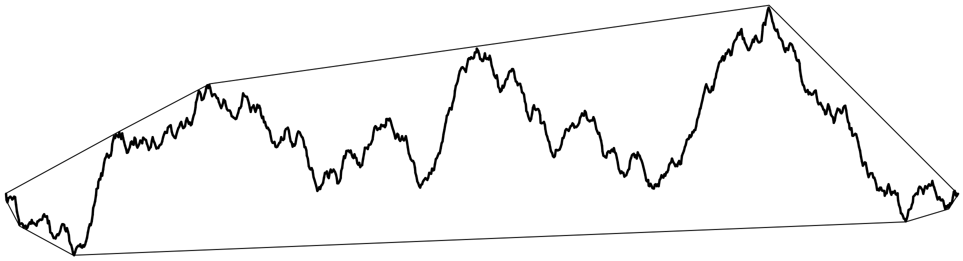

Figure 4: The rancher's path of Figure 2 rescaled vertically

Conjecture 2 (The exponent 3/4) *For every $\varepsilon > 0$, as $n \to \infty$ we have*

$$\mathbf{P}[n^{3/4-\varepsilon} < w_n < n^{3/4+\varepsilon}] \to 1.$$

It is also feasible that if the path is scaled by a factor of $n^{3/4}$ in the vertical axis and by n in the horizontal axis (parallel to the segment ox_n) then the law of the path would converge to some random function. The result of such asymmetric scaling is seen in Figure 4. In the next section we introduce a model that appears closely related.

3 The extremal investor

Stock or portfolio prices are often modeled by exponentiated random walk or Brownian motion. In the simplest discrete-time model, the log stock price, denoted x_n, changes every time by an independent standard Gaussian random variable.

Ones decision whether to invest in, say, a mutual fund is often based on past performance of the fund. Mutual fund companies report past performance for periods ending at present; the periods are often hand-picked to show the best possible performance. The simplest such statistic is the overall best performance over periods ending in the present. In terms of log interest rate it is given by

$$r_n^{\max} = \max_{m<n} \frac{x_n - x_m}{n-m}, \tag{1}$$

that is the maximal slope of lines intersecting the graph of x_n in both a past point and the present point.

A more cautious investor also looks at the worst performance $r_n^{\min}$, given by (1) with a min, and makes a decision to buy, sell or hold accordingly, influencing the fund price. In the simplest model, which we call the **extremal investor model**, the change in the log fund price given the present is simply a Gaussian with standard deviation 1 and expected value given by a fixed influence parameter α times the average of $r^{\max}$ and $r^{\min}$:

$$x_{n+1} = x_n + \alpha \frac{r_n^{\max} + r_n^{\min}}{2} + \text{ standard Gaussian.}$$

This process is related to the rancher in two dimensions, since the future behavior of x_n is influenced through the shape of the convex hull of the graph of x_n at the tip. For $\alpha = 1$ the drift of the rancher starting with the convex hull has the same direction as the expected next step for the stock value.

Figure 5: The extremal investor process for $\alpha = 1$ and its convex hull

Let w_n denote the greatest distance between x_n and the linear interpolation from time zero to the present (assume $x_0 = 0$):

$$w_n = \max_{m \le n} \left| x_m - \frac{m}{n} x_n \right| .$$

The following analogue of Conjecture 2 is consistent with our simulations:

Conjecture 3 (The exponent 3/4 for the extremal investor) *Let $\alpha = 1$. For every $\varepsilon > 0$ as $n \to \infty$ we have*

$$\mathbf{P}[n^{3/4-\varepsilon} < w_n < n^{3/4+\varepsilon}] \to 1.$$

A moment of thought shows that for $\alpha > 1$, x_n will blow up exponentially, so $\alpha_c = 1$ is the critical parameter. For $\alpha < 1$ the behavior of w_n seems to be governed by an exponent between $1/2$ and $3/4$ depending on α. For $\alpha < 1$ the x_n/n seems to converge to 0, but in the case that $\alpha = 1$, it appears that x_n/n converges a.s. to a random limit.

4 Proof of Theorem 1

Denote $\{x_n;\, n \ge 0\}$ the rancher's walk. Define the **ranch** R_n as the convex hull of $\{x_0, \dots, x_n\}$. Since x_n is always on the boundary and R_n is convex, the angle of R_n at x_n is always in $[0, \pi]$. Denote this angle by γ_n (as in Figure 6).

The idea of the proof is to find a set of times of positive upper density in which the expected gain in distance is bounded away from 0. There are two cases where the expected gain in distance can be small. First, if γ_n is close to 0, the distribution of the next step is close to uniform on the unit circle. Second, when γ_n is close to π, the next step is uniformly distributed on roughly a semicircle. If in addition the direction to the origin is near one of the end-points of the semicircle then the expected gain in distance is small.

We now introduce further notation used in the proof. Set $s_n = \|x_{n+1}\| - \|x_n\|$. Note that since the direction of the nth step is uniformly distributed on an arc not containing the direction of origin, $\mathbf{E}s_n \ge 0$. For three points x, y, z, let xyz denote the angle in the range $(0, 2\pi]$. The

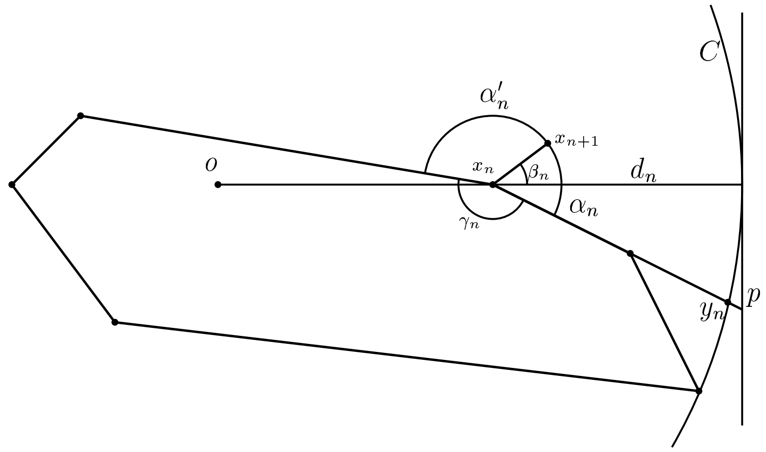

Figure 6: Notation used in the proof

angle $ox_nx_{n+1} - \pi$ is denoted by β_n, so that $\beta_n \in [-\pi, \pi)$. Thus $\beta = 0$ means that the walker moved directly away from o, $\beta > 0$ means that the walker moved counterclockwise.
Let C be the boundary of the smallest closed disk centered at o containing the ranch R_n. Consider the half-line starting from x_n that contains the edge of R_n incident to and clockwise from x_n. Let y_n denote the intersection of this half line and C. Let α_n denote the angle $\pi - ox_ny_n$, and let α'_n denote the analogous angle in the counterclockwise direction. Let d_n be the distance between C and x_n. It follows from these definitions that $\alpha_n + \alpha'_n + \gamma_n = 2\pi$ and that β_n has uniform distribution on $[-\alpha_n, \alpha'_n]$.

Proof of Theorem 1. We find a set of times of positive upper density in which $\mathbf{E}s_n$ is positive and bounded away from 0. If $\gamma_n \in [\varepsilon, \pi-\varepsilon]$, then $\mathbf{E}s_n$ is bounded from below by some function of ε. Thus we need only consider the times when $\gamma_n < \varepsilon$ or when $\gamma_n > \pi - \varepsilon$, where $\varepsilon > 0$ will be chosen later to satisfy further constraints.
In the case $\gamma_n < \varepsilon$, the rancher is at the tip of a thin ranch, so a single step can make a large change in γ, thus we look at two consecutive steps. With probability at least a quarter $\beta_n \in [\pi/4, 3\pi/4]$. In that case γ_{n+1} is bounded away from both 0 and π, and then $\mathbf{E}s_{n+1}$ is bounded away from 0. If β_n is not in $[\pi/4, 3\pi/4]$, we use the bound $\mathbf{E}s_{n+1} > 0$. Combining these gives a uniform positive lower bound on $\mathbf{E}s_{n+1}$.
If γ_n is close to π, then we are in a tighter spot: it could stay large for several steps, and $\mathbf{E}s_n$ may remain small. The rest of the proof consists of showing that at a positive fraction of time the angle γ_n is not close to π.
If $d_n < D$, where D is some large bound to be determined later, then with probability at least half $\gamma_{n+1} < \pi - 1/(2D)$, thus if we take $\varepsilon \leq 1/(2D)$, it suffices to show that a.s. the Markov process $\{(R_n, x_n)\}$ returns to the set $A = \{(R, x)|d < D\}$ at a set of times with positive upper density.
To show this, we use a martingale argument; it suffices to exhibit a function $f(R_n, x_n)$ bounded from below, so that the expected increase in f given the present is negative and bounded away from zero when $(R_n, x_n) \notin A$, and is bounded from above when $(R_n, x_n) \in A$. The sufficiency of the above is proved in Lemma 1 below; there take A_n to be the event $(R_n, x_n) \in A$, and $X_n = \|x_n\|$. We now proceed to construct a function f with the above properties.

The standard function that has this property is the expected hitting time of A. We will try to guess this. The motivation for our guess is the following heuristic picture. When the angle α is small, it has a tendency to increase by a quantity of order roughly $1/d$, and d tends to decrease by a quantity of order α. This means that d performs a random walk with downward drift at least $1/d$, but this is not enough for positive recurrence. So we have to wait for a few steps for α to increase enough to provide sufficient drift for d; the catch is that in every step α has a chance of order α to decrease, and the same order of chance to decrease to a fraction of its size. So α tends to grow steadily and collapse suddenly. If the typical size is α_*, then it takes order $1/\alpha_*$ time to collapse. During this time it grows by about $1/(d\alpha_*)$, which should be on the order of the typical size α_*, giving $\alpha_* = d^{-1/2}$. This suggests that the process d has drift of this order, so the expected hitting time of 0 is of order $d^{3/2}$. A more accurate guess takes into account the fact that if α is large, the hitting time is smaller.

Define the functions $f_1(d) = d^{3/2}$, and $f_2(d,\alpha) = -((cd^{1/2}) \wedge (\alpha d))$, where is a constant to be chosen later. Define $f(d,\alpha,\alpha') = f_1(d) + f_2(d,\alpha) + f_2(d,\alpha')$. Since A is defined by some bound on d is clear that if $(R_n, x_n) \in A$, then $f(d_n, \alpha_n, \alpha'_n)$ can only increase by a bounded amount (this is true for each of the terms). Since $\alpha, \alpha' \le \pi$, f is bounded from below. To conclude the proof we need to show that if $(R_n, x_n) \notin A$, then the expected change in $f(d_n, \alpha_n, \alpha'_n)$ is negative and bounded away from zero.

First we consider the expected change in f_1. All expected values will be conditional on the information available at time n. To simplify notation, assume x_n is on the positive X-axis. This may be done since the process is invariant to rotations of the plane. We first bound the expected decrease $d_n - d_{n+1}$.

$$-\mathbf{E}\Delta d = \mathbf{E}\|x_{n+1}\| - \|x_n\| \ge \mathbf{E}x_{n+1,1} - x_{n,1} \tag{2}$$

Recall that $\beta_n = ox_nx_{n+1} - \pi$ is uniformly distributed in $[-\alpha_n, \alpha'_n]$. Thus the RHS is simply

$$\frac{1}{\alpha_n + \alpha'_n}\int_{-\alpha_n}^{\alpha'_n} \cos\beta \, d\beta \ge \frac{\sin\alpha_n + \sin\alpha'_n}{2\pi}$$

Since $|\Delta d| \le 1$ and outside A we have $d > D$, we have

$$(d + \Delta d)^{3/2} = d^{3/2} + \frac{3}{2}d^{1/2}\Delta d + O(d^{-1/2}).$$

We can therefore bound

$$\mathbf{E}\Delta f_1 \le -\frac{3}{4\pi}\left(\sin\alpha_n + \sin\alpha'_n\right) d_n^{1/2} + o(1), \tag{3}$$

where here (and later) $o(1)$ denotes a quantity that is arbitrarily small if D is taken sufficiently large (D will be fixed later so that these term is small enough).

We now proceed to bound the expected change in $f_2(d_n, \alpha_n)$; denote this change by Δf_2. We break up Δf_2 into important and unimportant parts:

$$\begin{aligned}
\Delta f_2 &= (cd_n^{1/2} \wedge \alpha_n d_n - cd_n^{1/2} \wedge \alpha_{n+1} d_n) \\
&+ (cd_n^{1/2} \wedge \alpha_{n+1} d_n - cd_{n+1}^{1/2} \wedge \alpha_{n+1} d_n) \\
&+ (cd_{n+1}^{1/2} \wedge \alpha_{n+1} d_n - cd_{n+1}^{1/2} \wedge \alpha_{n+1} d_{n+1}).
\end{aligned}$$

The second term is bounded above by $c|d_{n+1}^{1/2} - d_n^{1/2}| = o(1)$. The third term is non-positive unless $cd_{n+1}^{1/2} > \alpha_{n+1} d_n$, implying that $\alpha_{n+1} < cd_n^{-1/2}$, and then this term is at most $\alpha_{n+1}|\Delta d| = o(1)$. Thus important increase can only come from the first term. We therefore denote

$$z = (cd_n^{1/2} \wedge \alpha_n d_n - cd_n^{1/2} \wedge \alpha_{n+1} d_n),$$

and consider three cases given by the following events, which depend on the value of $\beta = \beta_n$:

$$B_1 = \{\beta \in [-\alpha_n, 0)\}, \quad B_2 = \{\beta \in [0, \pi - \alpha_n]\}, \quad B_3 = \{\beta \in (\pi - \alpha_n, \alpha_n']\}.$$

As we will see in detail, the contribution of the first and last cases is small. If α_n is small enough then the second also has a small contribution, while if α_n is large the negative expected change of f_1 offsets any positive change in f_2.

Event B_2: $\beta \in [0, \pi - \alpha_n]$ (equivalently, x_{n+1} is on the side opposite of R_n for the lines ox_n and $x_n y_n$). In this case the rancher moves sufficiently away from the ranch, so that α increases:

$$\begin{aligned} \Delta\alpha &= ox_n y_n - ox_{n+1} y_{n+1} \geq ox_n y_n - ox_{n+1} y_n \\ &= x_n o x_{n+1} + x_{n+1} y_n x_n \geq x_{n+1} y_n x_n \geq 0. \end{aligned} \tag{4}$$

All inequalities follow from our assumption B_2. The equality follows from the fact that the angles in the quadrangle $ox_n y_n x_{n+1}$ add up to 2π.
We now compute the last angle in (4) using a simple identity in the triangle $x_n y_n x_{n+1}$, and the value of the angle $y_n x_n x_{n+1}$:

$$\|x_{n+1} - y_n\| \sin(x_{n+1} y_n x_n) = \|x_n - x_{n+1}\| \sin(y_n x_n x_{n+1}) = \sin(\beta_n + \alpha_n). \tag{5}$$

A byproduct of (4) is that B_2 implies $z \leq 0$. If $\alpha_n < cd_n^{-1/2}/2$, then a better bound is possible:

$$\|x_{n+1} - y_n\| \leq 1 + \|x_n - y_n\| \leq 1 + \|x_n - p\| = 1 + d_n(\cos\alpha_n)^{-1} = d_n(1 + o(1)),$$

where the point p is the intersection of the line $x_n y_n$ and the tangent to C perpendicular to the ray ox_n. We can then conclude from (4) and (5) that

$$\Delta\alpha \geq x_{n+1} y_n x_n \geq \frac{\sin(\beta_n + \alpha_n)}{d_n}(1 - o(1)). \tag{6}$$

The criterion $\alpha_n < cd_n^{-1/2}/2$ guarantees that the cutoff at $cd^{-1/2}$ does not apply, and so (6) implies $z \leq -\sin(\beta + \alpha_n)(1 - o(1))$. Therefore

$$\mathbf{E}[z;\ B_2] \leq -(\alpha_n + \alpha_n')^{-1} \int_0^{\pi - \alpha_n} \sin(\beta - \alpha_n) d\beta + o(1) \leq -2\pi^{-1} + o(1),$$

since we assumed $\alpha_n < cd_n^{-1/2}/2$. For larger α_n we only need $z \leq 0$.

Event B_3: $\beta_n > \pi - \alpha_n$. If $\alpha_{n+1} \geq cd_n^{-1/2}$, then $z \leq 0$ because of the cutoff at $cd^{1/2}$. However, R_{n+1} has an edge $x_{n+1} x_n$, and clearly $\alpha_{n+1} > \pi - \beta$. Thus the probability of B_3 and $\{z > 0\}$ is at most

$$\mathbf{P}[0 < z \text{ and } B_3] \leq \mathbf{P}[0 \leq \pi - \beta < cd_n^{-1/2}] \leq cd_n^{-1/2}\pi^{-1}.$$

Since $z \leq cd^{1/2}$ always holds, this gives

$$\mathbf{E}[z;\ B_3] \leq c^2/\pi.$$

Event B_1: $\beta < 0$. We can bound α_{n+1} below by $\beta + \alpha_n$ as follows. First, note that $\alpha_{n+1} = \pi - ox_{n+1}y_{n+1} \geq \pi - ox_{n+1}y_n$. Also $\beta + \alpha_n = y_n x_n x_{n+1} = \pi - x_n x_{n+1} y_n - x_{n+1} y_n x_n$, since the angles of a triangle add to π. We can split $x_n x_{n+1} y_n = x_n x_{n+1} o + o x_{n+1} y_n$. Putting these together we get $\alpha_{n+1} = \beta + \alpha_n + x_n x_{n+1} o + x_{n+1} y_n x_n$, and since the latter two angles are small and positive, $\alpha_{n+1} > \beta + \alpha_n$. Therefore

$$\mathbf{P}[0 < z \text{ and } B_1] \leq \mathbf{P}[\beta + \alpha_n < cd_n^{-1/2}] \leq cd_n^{-1/2}\pi^{-1},$$

and as in case B_3:

$$\mathbf{E}[z;\ B_1] \leq c^2/\pi.$$

Summarizing the cases B_1, B_2, B_3 we get the bound

$$\mathbf{E}\Delta f_2(d,\alpha) \leq 2c^2\pi^{-1} + o(1),$$

and if $\alpha < cd^{-1/2}/2$, then

$$\mathbf{E}\Delta f_2(d,\alpha) \leq 2c^2\pi^{-1} - 2\pi^{-1} + o(1).$$

Of course, the same bounds hold for $f_2(d,\alpha')$.
We now summarize our estimates on all the components of Δf.

- If $\alpha_n < cd^{-1/2}/2$, then

$$\begin{aligned} \mathbf{E}\Delta f &= \mathbf{E}\Delta f_1 + \mathbf{E}\Delta f_2(d,\alpha) + \mathbf{E}\Delta f_2(d,\alpha') \\ &\leq 0 + (2c^2-2)\pi^{-1} + 2c^2\pi^{-1} + o(1) = \frac{4c^2-2}{\pi} + o(1), \end{aligned}$$

  which is negative for large D if $c < 2^{-1/2}$. The same bound holds if $\alpha'_n < cd^{-1/2}/2$.

- If at least one of α_n, α'_n is in $[cd^{-1/2}/2, \pi - cd_n^{-1/2}/2]$, then using (3), we get

$$\mathbf{E}\Delta f \leq -3/8\pi^{-1}c + o(1) + 4c^2\pi^{-1} + o(1) = \frac{4c^2 - 3c/8}{\pi} + o(1),$$

  which is negative for large D if $c < 3/32$.

- If $\alpha_n, \alpha'_n > \pi - cd_n^{-1/2}/2$, and $d_n > D$ is large enough, then we have seen that the two step drift $\mathbf{E}d_{n+2} - d_n < -c_1$ for some $c_1 > 0$. Thus in this case $\mathbf{E}\Delta f_1 \leq -c_1 d_n^{1/2} + o(1)$, while the drift of f_2 is uniformly bounded.

Putting the three cases together shows that if we take $0 < c < 3/32$ then for large enough D the function f satisfies the requirements of Lemma 1. □

For the following technical lemma, we use the notation $\Delta_m a_n = a_{n+m} - a_n$, and $\Delta a_n = \Delta_1 a_n$.

Lemma 1 *Let $\{(X_n, f_n, A_n)\}$ be a sequence of triples adapted to the increasing filtration $\{\mathcal{F}_n\}$ (with $\mathcal{F}_0$ trivial) so that X_n, f_n are random variables and A_n are events satisfying the following.*

There exist positive constants c_1, c_2, c_3, c_4, and a positive integer m, so we have a.s. for all n

$$
\begin{aligned}
|\Delta X_n| &\leq 1, \\
\mathbf{E}[\Delta X_n \mid \mathcal{F}_n] &\geq 0, &(7)\\
\mathbf{E}[\Delta_m X_n \mid \mathcal{F}_n, A_n] &> c_1, &(8)\\
f_n &> -c_2, \\
\Delta f_n \mathbf{1}(A_n) &< c_3, &(9)\\
\mathbf{E}[\Delta f_n \mid \mathcal{F}_n, A_n^c] &< -c_4. &(10)
\end{aligned}
$$

Then for some positive constant c_5 we have

$$\limsup X_n/n > c_5 \quad a.s. \tag{11}$$

PROOF. Let $G_n = \sum_{i=0}^{n-1} \mathbf{1}_{A_i}$, and let $G_{n,k} = \sum_{i=0}^{n-1} \mathbf{1}_{A_{mi+k}}$, $0 \leq k < n$. First we show that the $m+1$ processes

$$\{c_1 G_{n,k} - X_{mn+k}\}_{n\geq 0}, \quad 0 \leq k < m, \tag{12}$$

$$\{f_n - c_3 G_n + c_4(n - G_n)\}_{n\geq 0} \tag{13}$$

are supermartingales adapted to $\{\mathcal{F}_{mn+k}\}_{n\geq 0}$, $0 \leq k < m$, $\{\mathcal{F}_n\}_{n\geq 0}$, respectively. For the first m processes fix k, and note that $\mathbf{E}[c_1(G_{n+1,k} - G_{n,k}) \,|\, F_{mn+k}] = c_1 \mathbf{1}(A_{mn+k})$. Consider

$$\mathbf{E}[c_1(G_{n+1,k} - G_{n,k}) \,|\, \mathcal{F}_{mn+k}] + \mathbf{E}[-(X_{m(n+1)+k} - X_{mn+k}) \,|\, \mathcal{F}_{mn+k}]$$

If A_{mn+k} happens, then the first term equals c_1, and the second is less than $-c_1$ by (8). If A_{mn+k} does not happen, then the first term equals 0 and the second is non-positive by (7). Putting these two together shows that (12) are supermartingales. For the last process, consider

$$\mathbf{E}[\Delta f_n \,|\, \mathcal{F}_n] + \mathbf{E}[-c_3 \Delta G_n \,|\, \mathcal{F}_n] + \mathbf{E}[c_4(1 - \Delta G_n) \,|\, \mathcal{F}_n].$$

If A_n happens, then the first term is less than c_3 by (9), the second term equals $-c_3$, and the last equals 0. If A_n does not happen, then the first term is less than $-c_4$ by (10), the second term equals 0, and the third equals c_4. In both cases we get that the process (13) is a supermartingale.

It follows from the supermartingale property that for some $c > 0$ and all $n \geq 0$ we have

$$
\begin{aligned}
\mathbf{E}X_{mn+k} &\geq c_1 \mathbf{E}G_{n,k} - c, \quad 0 \leq k < m, &(14)\\
\mathbf{E}G_n &\geq c_4/(c_3 + c_4)n - c. &(15)
\end{aligned}
$$

Since $G_{mn} = G_{n,0} + \ldots + G_{n,m-1}$, it follows from (15) that for some $c_6 > 0$ and all large n there is $k = k(n)$, so that $\mathbf{E}G_{n,k} > c_6 n$. Then for some $c_7 > 0$ we have $\mathbf{E}X_{nm+k} > c_7 n$ by (14). As a consequence, for $Y_n = \max\{X_{mn}, \ldots, X_{mn+m-1}\}$ we have $\mathbf{E}Y_n > c_7 n$.

Thus for some $c_8 < 1$ we have $\mathbf{E}(1 - Y_n/(mn)) < c_8$ for all large n. Since $X_n \leq X_0 + n$, we have $Y_n \leq X_0 + mn + m - 1 = mn + c_9$ and therefore $1 - (Y_n - c_9)/(mn) \geq 0$. Fatou's lemma then implies

$$\mathbf{E} \liminf(1 - Y_n/(mn)) = \mathbf{E} \liminf(1 - (Y_n - c_9)/(mn)) \leq c_8,$$

for some $c_{10} \in (c_8, 1)$ Markov's inequality gives $\mathbf{P}(\liminf(1 - Y_n/(mn)) < c_8/c_{10}) > 1 - c_{10}$. So for some $c_5 > 0$,

$$\mathbf{P}(\limsup X_n/n > c_5) > 1 - c_{10},$$

but we can repeat this argument while conditioning on the σ-field $\mathcal{F}_t$ to get

$$\mathbf{P}(\limsup X_n/n > c_5 \mid \mathcal{F}_t) > 1 - c_{10}$$

so letting $t \to \infty$ by Lévy's 0-1 law we get (11). □

5 Further open questions and conjectures

Conjecture 4 *Theorem 1 holds with* $\liminf$ *instead of* $\limsup$.

Conjecture 5 *The speed* $\lim \|x_n\|/n$ *exists and is constant a.s. This could follow from some super-linearity result on the rancher's travels.*

Question 6 *What is the scaling limit of the asymmetrically normalized path?*

Question 7 *What is the behavior in higher dimensions? Is the lim sup (or even lim inf) speed still positive? If not, is* $\|x_n\| = O(\sqrt{n})$ *or is it significantly faster then a simple random walk? What about convergence of direction?*

Question 8 *If longer step sizes are allowed what happens when the tail is thickened? Are there distributions which give positive speed without convergence of direction?*

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