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# STRONG LAW OF LARGE NUMBERS UNDER A GENERAL MOMENT CONDITION

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*Abstract*

We use our maximum inequality for $p$-th order random variables ($p > 1$) to prove a strong law of large numbers (SLLN) for sequences of $p$-th order random variables. In particular, in the case $p = 2$ our result shows that $\sum f(k)/k < \infty$ is a sufficient condition for SLLN for $f$-quasi-stationary sequences to hold. It was known that the above condition, under the additional assumption of monotonicity of $f$, implies SLLN (Erdös (1949), Gal and Koksma (1950), Gaposhkin (1977), Moricz (1977)). Besides getting rid of the monotonicity condition, the inequality enables us to extend the general result to $p$-th order random variables, as well as to the case of Banach-space-valued random variables.

## Notations

$\mathbf{N}$ stands for the set of positive integers, $\mathbf{N_0} = \mathbf{N} \cup \{0\}$. $X$ denotes a Banach space, real or complex. Let $(\Omega, \mathcal{A}, P)$ be an underlying probability space. By an $X$-valued random variable we mean a Bochner measurable mapping $\xi : \Omega \to X$.
Given a sequence $(\xi_n), n \in \mathbf{N_0}$ of $X$-valued random variables denote

$$S_{a,b} = \sum_{k=a}^{a+b-1} \xi_k \ , \quad M_{a,b} = \max_{k \leq b} \|S_{a,k}\| \ , \quad a, b \in \mathbf{N_0} \ .$$

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We say that for a sequence $(\xi_n), n \in \mathbf{N_0}$ *the strong law of large numbers* (SLLN) *holds*, if $S_{0,n}/n \to 0$ a.s. as $n \to \infty$ .

## Main Results

The main objective of this note is to prove the following theorem and some of its consequences.

**Theorem 1** *Let* $1 < p < \infty$ *. If for a sequence* $(\xi_n) \subset L_p(X)$

$$\sum_{n=0}^{\infty} \sup_{k \in \mathbf{N_0}} \mathbf{E} \| \frac{S_{k,2^n}}{2^n} \|^p < \infty , \tag{1}$$

*then* SLLN *holds for* $(\xi_n)$.

We apply Theorem 1 to quasi-stationary sequences.

**Corollary 1** *Let* $(\xi_n), n \in \mathbf{N_0}$ *be a sequence of* $X$*-valued random variables such that for some* $1 < p < \infty$ *and each* $k, n \in \mathbf{N_0}$

$$\mathbf{E}\|S_{k,n}\|^p \le g(n) ,$$

*for a numerical function* $g$*. Then*

*(i) If*

$$\sum_{n=1}^{\infty} \frac{g(2^n)}{2^{np}} < \infty ,$$

*then* SLLN *holds for* $(\xi_n)$ *.*

*(ii) If* $g(n)/n^{p+1}$ *is monotone, and*

$$\sum_{n=0}^{\infty} \frac{g(n)}{n^{p+1}} < \infty ,$$

*then* SLLN *holds for* $(\xi_n)$.

Part (ii) of Corollary 1 has been proved earlier for the case $p = 2$, and 1-dimensional $X$ (see Gal and Koksma, 1950 and Gaposhkin, 1977). Below we also discuss Moricz's, 1977 further contribution.

Let $f(n), n \in \mathbf{N_0}$ be a non-negative function. We say that a real or complex-valued sequence $(\xi_n), n \in \mathbf{N_0}$ is *f-quasi-stationary*, if $\mathbf{E}|\xi_k|^2 < \infty,\ k \in \mathbf{N_0}$ , and

$$|\mathbf{E}\xi_l \overline{\xi}_{l+m}| \le f(m) , \quad l, m \in \mathbf{N_0} .$$

The following proposition is a consequence of Theorem 1.

**Corollary 2** *Let* $(\xi_n), n \in \mathbf{N_0}$ *be an* $f$*-qusi-stationary sequence. If*

$$f(0) + \sum_{m=1}^{\infty} \frac{f(m)}{m} < \infty , \tag{2}$$

*then* SLLN *holds for* $(\xi_n)$ *.*

Corollary 2 was known earlier under the additional condition of monotonicity of $f$ . It has been established first by Erdös, 1949 for monotone $f(m) = O(\log^{-\alpha} m), \alpha > 1$. In Gal and Koksma, 1950 it was extended to monotone sequences $f(m)$ satisfying (2). Gaposhkin, 1975 has shown that condition (2) for monotone $f$ is in a sense necessary: If

$$\sum_{m=1}^{\infty} \frac{f(m)}{m} = \infty,$$

then there is an $f$-quasi-stationary sequence $(\xi_n), n \in \mathbf{N_0}$ for which SLLN fails.
Regarding a general norming in SLLN for an $f$-quasi-stationary sequence, the reader is referred to the papers by Moricz,1977 and Serfling, 1978. In the case of classical norming ($\lambda_n = 1/n$) Moricz has proved Theorem 1 above for real valued random variables in the case $p = 2$, and our Corollary 2 (see Moricz, 1977, Theorem $2'$, p.228 and Theorem 2, p.227 respectively), both under some additional conditions (see (1.16) and (1.17), respectively, p.227). His main condition (1.16) is in fact equivalent to

$$\sum_m \overline{\varphi}(2^m) < \infty \quad \text{and} \quad \sum_m \frac{\overline{f}(m)}{m} < \infty,$$

where

$$\varphi(m) = \sup_{k \in \mathbf{N_0}} [\mathbf{E}|\frac{S_{k,m}}{m}|^2] \text{ , and } \quad \overline{a}_m = \max_{n \geq m}\{a_n\}.$$

Moricz's second condition (1.17) is not relevant for the purpose of comparison with our paper so we do not discuss it.

**Example**. Let us show that $\sum_m f(m)/m$ might be finite, whereas $\sum_m \overline{f}(m)/m$ is infinite. This would show that Moricz's condition (1.16) is restrictive. Notice first that for every $f, \quad 0 \leq f(m) \leq 1, \ m \in \mathbf{N_0}$ there is a sequence $(\xi_k)$ of real random variables so that

$$\mathbf{E}\xi_k^2 = 1, \ \mathbf{E}\xi_k) = 0 \quad \text{and} \quad f(m) = \sup_k |\mathbf{E}\xi_k \xi_{k+m}|.$$

Then we put $f(m) = 1/\log m$, if $m = n^2, \ n \in \mathbf{N}$, and $f(m) = 0$ otherwise. It is worthy to note that for weakly stationary sequences condition (2) can be replaced by a weaker condition of convergence (conditional) of the series

$$\sum_{m=1}^{\infty} \frac{R(m)}{m \log m} \log\log m \ ,$$

where $R$ is the correlation function of the sequence (Gaposhkin, 1977).

## Proofs

The proof of Theorem 1 is based on the following proposition proved in Chobanyan, Levental and Salehi, 2004.

**Theorem 2** *Let* $1 < p < \infty$ . *For any sequence* $(\xi_n) \subset L_p(X)$ *we have*

$$\sum_{n=0}^{\infty} \mathbf{E} \frac{M_{2^n,2^n}^p - \|S_{2^n,2^n}\|^p}{2^{np}} \leq \frac{2^{p+1}}{2^p - 2} \sum_{n=0}^{\infty} G_n \ ,$$

*where*

$$G_n = \sup_{k\in\mathbf{N_0}} (\frac{1}{2}\mathbf{E}\|\frac{S_{k,2^n}}{2^n}\|^p + \frac{1}{2}\mathbf{E}\|\frac{S_{k+2^n,2^n}}{2^n}\|^p - \mathbf{E}\|\frac{S_{k,2^{n+1}}}{2^{n+1}}\|^p).$$

For the sake of completeness we outline the proof of Theorem 2. We have for any $k \in \mathbf{N}_0$ $n \in \mathbf{N}_0$

$$M_{k,2^{n+1}} \le \max\{M_{k,2^n},\ \|S_{k,2^n}\| + M_{k+2^n,2^n}\} \quad .$$

Making use of the following elementary inequality $|a+b|^p \le 2^{p-1}(|a|^p + |b|^p)$, we get

$$\begin{aligned} M^p_{k,2^{n+1}} &\le \max\{M^p_{k,2^n},\ 2^{p-1}(\|S_{k,2^n}\|^p + M^p_{k+2^n,2^n})\} \le \\ &(2^{p-1}-1)\|S_{k,2^n}\|^p + M^p_{k,2^n} + 2^{p-1}M^p_{k+2^n,2^n} \quad . \end{aligned} \tag{3}$$

(3) can be rewritten as

$$\begin{aligned} M^p_{k,2^{n+1}} - \|S_{k,2^{n+1}}\|^p &\le M^p_{k,2^n} - \|S_{k,2^n}\|^p + 2^{p-1}(M^p_{k+2^n,2^n} - \|S_{k+2^n,2^n}\|^p) \\ &-\|S_{k,2^{n+1}}\|^p + 2^{p-1}\|S_{k,2^n}\|^p + 2^{p-1}\|S_{k+2^n,2^n}\|^p \quad . \end{aligned}$$

Dividing both sides by $2^{(n+1)p}$, taking expectations, and then maximums over all $k'$s, we get

$$F_{n+1} \le \frac{1}{2^p}F_n + \frac{1}{2}F_n + G_n\ , \quad n \in \mathbf{N}_0, \tag{4}$$

where

$$F_n = \sup_{k\in\mathbf{N}_0} \mathbf{E}\big(\frac{M^p_{k,2^n} - \|S_{k,2^n}\|^p}{2^{np}}\big) \quad ;$$

$$G_n = \sup_{k\in\mathbf{N}_0} \Big( \frac{1}{2}\,\mathbf{E}\,\|\frac{S_{k,2^n}}{2^n}\|^p + \frac{1}{2}\,\mathbf{E}\,\|\frac{S_{k+2^n,2^n}}{2^n}\|^p - \mathbf{E}\,\|\frac{S_{k,2^{n+1}}}{2^{n+1}}\|^p \Big)\ .$$

It is easy to make sure by induction in $n$ that

$$F_{n+1} \le \sum_{k=0}^{n} c^{n-k} G_k\ , \quad n \in \mathbf{N}_0\ ,$$

where $c = \frac{1}{2} + \frac{1}{2^p}$ . Summing up (4) from $n=0$ to $n=N$, we come to Theorem 2.

Proof of Theorem 1. Assuming (1) holds we get

$$\sum_{n=0}^{\infty} G_n \le \sum_{n=0}^{\infty} \sup_{k\in\mathbf{N_0}} \mathbf{E}\|\frac{S_{k,2^n}}{2^n}\|^p < \infty\ .$$

Therefore, by Theorem 2,

$$\frac{M^p_{2^n,2^n} - \|S_{2^n,2^n}\|^p}{2^{np}} \to 0 \quad \text{a.s.} \tag{5}$$

But (1) also implies that

$$\frac{\|S_{2^n,2^n}\|^p}{2^{np}} \to 0 \quad \text{a.s.}$$

This convergence along with (5) implies

$$\frac{\|M_{2^n,2^n}\|}{2^n} \to 0 \quad \text{a.s.,}$$

which is equivalent to SLLN (Chobanyan, Levental and Mandrekar, 2004). $\square$

Proof of Corollary 2. Assume that $(\xi_n), n \in \mathbf{N_0}$ is an $f$-quasi-stationary sequence. Then we have for any $k \in \mathbf{N_0}$ and any $n \in \mathbf{N_0}$

$$\mathbf{E}|\frac{S_{k,2^n}}{2^n}|^2 \leq \sum_{m=0}^{2^n-1} \frac{f(m)(2^n-m)}{2^{2n}} \leq \frac{1}{2^n} \sum_{m=0}^{2^n-1} f(m) \; .$$

This implies

$$\sum_{n=0}^{\infty} \sup_k \mathbf{E}|\frac{S_{k,2^n}}{2^n}|^2 \leq \sum_{n=0}^{\infty} \sum_{m=0}^{2^n} \frac{f(m)}{2^n} \leq 2f(0) + \sum_{m=1}^{\infty} f(m) \sum_{n=[\log_2 m]}^{\infty} \frac{1}{2^n} \leq 2f(0) + 2\sum_{m=1}^{\infty} \frac{f(m)}{m}.$$

Corollary 2 is proved. $\square$

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