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DYNAMICAL PROPERTIES AND CHARACTERIZATION OF GRADIENT DRIFT DIFFUSION

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Abstract

We study the dynamical properties of the Brownian diffusions having $\sigma\,\mathrm{Id}$ as diffusion coefficient matrix and $b = \nabla U$ as drift vector. We characterize this class through the equality $D_+^2 = D_-^2$, where D_+ (resp. D_-) denotes the forward (resp. backward) stochastic derivative of Nelson's type. Our proof is based on a remarkable identity for $D_+^2 - D_-^2$ and on the use of the martingale problem.

1 Introduction

In the current note, we are interested in the dynamical properties of gradient drift diffusions with a constant diffusion coefficient, also known as Langevin diffusions or Kolmogorov processes. Precisely, we characterize the class of gradient drift diffusions by means of Nelson stochastic derivatives of second order. In the sixties, Nelson introduced the notion of backward and forward stochastic derivatives in his seminal work [7]. Namely, on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$ endowed with an increasing (resp. decreasing) filtration $(\mathscr{P}_t)$ (resp. $(\mathscr{F}_t)$), he considered the processes $Y = (Y_t)_{t\in[0,T]}$ such that $\lim_{h\downarrow 0} h^{-1}E[Y_{t+h} - Y_t|\mathscr{P}_t]$ and $\lim_{h\downarrow 0} h^{-1}E[Y_t - Y_{t-h}|\mathscr{F}_t]$ exist in $L^2(\Omega)$. On the other hand, for a given process $Z = (Z_t)_{t\in[0,T]}$, these quantities may not exist neither for the fixed filtrations $(\mathscr{P}_t)$ and $(\mathscr{F}_t)$ nor for some filtrations generated by the process. Thus, the following generalization (introduced in [2]) is natural: A sub-σ-field $\mathscr{A}^t$ of $\mathscr{F}$ *differentiates* (resp. *forward differentiates*, *backward differentiates*) Z at time t if the quantity $h^{-1}\mathrm{E}\,[Z_{t+h} - Z_t|\mathscr{A}^t]$ converges in probability (or for another topology) when $h \to 0$ (resp. $h \downarrow 0$, $h \uparrow 0$); the limit being called the stochastic derivatives of Z at t w.r.t. $\mathscr{A}^t$.

When we consider Brownian diffusions of the form

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad t \in [0, T], \tag{1}$$

then, under suitable conditions, the σ-field $\mathscr{T}_t^X$ generated by X_t is both a forward and backward differentiating σ-field for X at t. We call the associated derivatives *Nelson derivatives*, due to the Markov property of the diffusion and of its time reversal which allows to take the conditional expectation with respect to the past $\mathscr{P}_t^X$ of X when $h \downarrow 0$ and with respect to the future $\mathscr{F}_t^X$ when $h \uparrow 0$. For simplicity, we respectively denote them by D_+ and D_- in the sequel. Notice that these derivatives are relevant and natural quantities for Brownian diffusions. When they exist, they are indeed respectively equals to the forward and the backward (up to sign) drift of X. Moreover, they exist under the rather mild conditions given by Millet, Nualart and Sanz in [6].

We shall see that these stochastic derivatives turn out to have remarkable properties when we work with diffusions of the type

$$X_t = X_0 + \int_0^t b(s, X_s)ds + \sigma\, W_t, \quad t \in [0, T]. \tag{2}$$

Here, $\sigma \in \mathbb{R}$ is assumed to be constant. For instance, the equality $D_+X_t = -D_-X_t$, $t \in (0, T)$, characterizes the class of stationary diffusions of the type (2) having moreover an homogeneous gradient drift (see Proposition 4). This statement, which is substantially contained in [3, 13], is actually a straightforward consequence of well known formulas for Nelson derivatives. A more difficult one, which is the main result of this paper, states that a Brownian diffusion of the type (2) is a gradient diffusion - that is whose drift coefficient has the form $b = \nabla_x U$ for a certain U - if and only if $D_+^2 X_t = D_-^2 X_t$ for almost all $t \in (0, T)$. See Theorem 5 for a precise statement. Notice that this result was conjectured at the end of the note [1]. Our proof is based on the discovery of a remarkable identity (Lemma 7): we can write the quantity $p_t(X_t)(D_+^2 X_t - D_-^2 X_t)$ as the divergence of a certain vector field, where p_t denotes the density of the law of X_t. Combined with the expression of the adjoint of the infinitesimal generator, we can then conclude using probabilistic arguments, especially the martingale problem. Let us moreover stress the fact that we were able to solve our problem with probabilistic tools, whereas its analytic transcription with the help of partial differential equations seemed more difficult to treat.

Our note is organized as follows. In Section 2, we introduce some notations and we give the useful expressions of the Nelson derivatives under the conditions given in [6]. In Section 3, we study the above mentioned characterizations and we prove our main result.

2 Recalls on time reversal and stochastic derivatives

2.1 Notations

Let $T > 0$ and $d \in \mathbb{N}^*$. The space $\mathbb{R}^d$ is endowed with its canonical scalar product $\langle \cdot, \cdot \rangle$. Let $|\cdot|$ be the induced norm.

If $f : [0,T] \times \mathbb{R}^d \to \mathbb{R}$ is a smooth function, we set $\partial_j f = \frac{\partial f}{\partial x_j}$. We denote by $\nabla f = (\partial_i f)_i$ the gradient of f and by $\Delta f = \sum_j \partial_j^2 f$ its Laplacian. For a smooth map $\Phi : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$, we denote by Φ^j its j^{th}-component, by $(\partial_x \Phi)$ its differential which we represent into the canonical basis of $\mathbb{R}^d$: $(\partial_x\Phi) = (\partial_j\Phi^i)_{i,j}$, and by $\mathrm{div}\Phi = \sum_j \partial_j\Phi^j$ its divergence. By convention, we denote by $\Delta\Phi$ the vector $(\Delta\Phi^j)_j$. The image of a vector $u \in \mathbb{R}^d$ under a linear map M is simply denoted by Mu, for instance $(\partial_x\phi)u$. The map $a : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ is viewed as $d \times d$ matrices whose columns are denoted by a_k. Finally, we denote by $\mathrm{div}\, a$ the vector $(\mathrm{div}\, a_k)_k$.

Let $(\Omega, \mathscr{A}, \mathbb{P})$ be a probability space on which is defined a d-dimensional Brownian motion W. For a process Z defined on $(\Omega, \mathscr{A}, \mathbb{P})$, we set $\mathscr{P}_t^Z$ the σ-field generated by Z_s for $0 \leqslant s \leqslant t$ and $\mathscr{F}_t^Z$ the σ-field generated by Z_s for $t \leqslant s \leqslant T$. Consider the d-dimensional diffusion process $X = (X_t)_{t\in[0,T]}$ solution of the stochastic differential equation (1) where $X_0 \in \mathrm{L}^2(\Omega)$ is a random vector independent of W, and the functions $\sigma : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \otimes \mathbb{R}^d$ and $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ are Lipschitz with linear growth. More precisely, we assume that σ and b satisfy the two following conditions. There exists a constant $K > 0$ such that, for all $x, y \in \mathbb{R}^d$, we have

$$\sup_{t\in[0,T]} \big[\, |b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \,\big] \leqslant K\, |x - y|$$

and

$$\sup_{t\in[0,T]} \big[\, |b(t,x)| + |\sigma(t,x)| \,\big] \leqslant K(1 + |x|).$$

We moreover assume that b is differentiable w.r.t. x and we set $G = (\partial_x b) - (\partial_x b)^*$, *i.e.* $G_i^j = \partial_i b^j - \partial_j b^i$. Finally, we set $a = \sigma\sigma^*$, *i.e.* $a_i^j = \sum_k \sigma_i^k \sigma_j^k$.

In the sequel, we will work under the following assumption $(H) = (H_1) \cap (H_2)$:

(H_1) For any $t \in (0,T)$, the law of X_t admits a positive density $p_t : \mathbb{R}^d \to (0, +\infty)$ and we have, for any $t_0 \in (0,T)$:

$$\max_{j=1,\dots,n} \int_{t_0}^T \int_{\mathbb{R}^d} |\mathrm{div}(a_j(t,x)p_t(x))| \, dxdt < +\infty. \tag{3}$$

(H_2) The functions $g_j : x \mapsto \dfrac{\mathrm{div}(a_j(t,x)p_t(x))}{p_t(x)}$ are Lipschitz.

The condition (H_1) will ensure us that the time reversed process $\overline{X}_t = X_{T-t}$ is again a diffusion process (see [6], Theorem 2.3). The condition (H_2) allows to calculate the backward derivative. Let us remark that this condition, which may seem a bit restrictive, is weaker than the hypothesis imposed in Proposition 4.1 of [12] for the computations of the Nelson derivatives. Finally, remark that the positivity assumption made on p_t is quite weak when X is of the type (2). It is for instance automatically verified when we can apply Girsanov theorem in (2), that is for instance when the Novikov condition is verified.

2.2 Stochastic derivatives of Nelson's type

Let us recall the following definition (*cf.* [2]):

Definition 1. *Let $Z = (Z_t)_{t\in[0,T]}$ be a process defined on $(\Omega, \mathscr{F}, \mathbb{P})$. We assume, for any $t \in [0,T]$, that $Z_t \in \mathrm{L}^1(\Omega)$. Fix $t \in [0,T]$. We say that $\mathscr{A}^t$ (resp. $\mathscr{B}^t$) is a* forward differentiating σ-field *(resp.* backward differentiating σ-field*) for Z at t if $E[\frac{Z_{t+h}-Z_t}{h} | \mathscr{A}^t]$ (resp.*

$E[\frac{Z_t - Z_{t-h}}{h}|\mathscr{B}^t])$ converges in probability when $h \downarrow 0$. In these cases, we define the so-called forward and backward derivatives

$$D_+^{\mathscr{A}^t} Z_t = \lim_{h\downarrow 0} E\left[\frac{Z_{t+h} - Z_t}{h}\,|\mathscr{A}^t\right], \tag{4}$$

$$D_-^{\mathscr{B}^t} Z_t = \lim_{h\downarrow 0} E\left[\frac{Z_t - Z_{t-h}}{h}\,|\mathscr{B}^t\right]. \tag{5}$$

As we already said it in the introduction, the present turns out to be a forward and backward differentiating σ-field for Brownian diffusions X of the form (1). Precisely, the σ-field $\mathscr{T}_t^X$ generated by X_t is both forward and backward differentiating for X at t. Equivalently, due to the Markov property of X (resp. of its time reversal $\overline{X}$), $\mathscr{P}_t^X$ (resp. $\mathscr{F}_t^X$) is forward (resp. backward) differentiating for X at t. For this reason, we call the derivatives defined by (4) and (5) *Nelson derivatives*. Indeed, in [7] Nelson introduced the processes which have stochastic derivatives in $L^2(\Omega)$ with respect to a fixed filtration $(\mathscr{P}_t)$ and a fixed decreasing filtration $(\mathscr{F}_t)$. Henceforth, we work with the stochastic derivatives of Nelson's type for Brownian diffusions and so we simply write, given a process X, $D_\pm$ instead of $D_\pm^{\mathscr{T}_t^X}$.

Now, we recall some well known results on time reversal for Brownian diffusions (see e.g. [3, 6, 8]) and their relation with stochastic derivatives of Nelson type (see e.g. [3]). Since we will need slight extensions of some of these results, we outline the proofs for the sake of completeness.

Föllmer [3] showed that the time reversals of Brownian semimartingales of the form $X_t = \int_0^t b_s ds + W_t$, under the finite energy condition $E\int_0^T b_s^2 ds < \infty$, remain Brownian semimartingales $\int_0^t \widehat{b}_s ds + \widehat{W}_t$, by relating the backward Nelson derivative with the time reversed drift $\widehat{b}$. More precisely, we have the following expression:

$$\widehat{b}_{T-t} = -b_t + \frac{\nabla p_t}{p_t}(X_t),$$

and moreover: $D_- X_t = -\widehat{b}_{T-t}$.

In [6], Theorem 2.3, Millet, Nualart and Sanz extended this result to diffusions of the form $X_t = \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$ satisfying (H_1), using Malliavin calculus: $\overline{X}_t = X_{T-t}$ is again a diffusion process w.r.t. the increasing filtration $(\mathscr{F}_{T-t}^X)$ and the reversed drift may be expressed as

$$\widehat{b}(T-t, X_t) = -b(t, X_t) + \frac{\operatorname{div}(a(t, X_t)p_t(X_t))}{p_t(X_t)}.$$

This term can also be viewed as the byproduct of a "*grossissement de filtration*" (see, e.g., Pardoux [8]). Roughly speaking, if $\mathscr{G}_t$ denotes the σ-field generated by $W_u - W_r$ for $T-t \leqslant u < r \leqslant T$, then $\overline{W}_t - \overline{W}_0$ is a $\mathscr{G}_t$-Brownian motion and the question sums up to writing the Doob-Meyer decomposition of $\overline{W}_t - \overline{W}_0$ in the enlarged filtration $\mathscr{H}_t = \mathscr{G}_t \vee \overline{X}_t$. In particular, knowing this answer gives the decomposition of $\overline{X}$ with respect to its natural filtration.

As in [3] under the finite energy condition, we can relate the drift b and the time reversed drift $\widehat{b}$ to the forward and backward Nelson derivatives under conditions (H_1) and (H_2) using the following argument. Indeed, we have:

$$E\left[\frac{X_{t+h} - X_t}{h}\,|\mathscr{P}_t^X\right] = E\left[\frac{1}{h}\int_t^{t+h} b(s, X_s)ds\,|\mathscr{P}_t^X\right],$$

and

$$E\left|E\left[\frac{X_{t+h}-X_t}{h}\middle|\mathscr{P}_t^X\right]-b(t,X_t)\right| \leqslant \frac{1}{h}E\int_t^{t+h}|b(s,X_s)-b(t,X_t)|\,ds$$
$$= \frac{1}{h}\int_t^{t+h}E\,|b(s,X_s)-b(t,X_t)|\,ds.$$

Using the fact that b is Lipschitz and that $t\mapsto E|X_t|$ is locally integrable (see, *e.g.*, Theorem 2.9 in [4]), we can conclude by the differentiation Lebesgue theorem that for almost all $t\in(0,T)$:

$$\frac{1}{h}\int_t^{t+h}E\,|b(s,X_s)-b(t,X_t)|\,ds\to 0 \text{ a.s.},\quad \text{as } h\to 0.$$

Therefore D_+X_t exists and is equal to $b(t,X_t)$.
Moreover, assumption (H_1) implies that

$$t\mapsto E\left|\frac{\operatorname{div}(a_i(t,X_t)p_t(X_t))}{p_t(X_t)}\right|$$

is locally integrable. As above, using now (H_2), we obtain that D_-X_t exists and is equal to $-\overline{b}(T-t,\overline{X}_{T-t})$.

For the diffusions we are interested in, we may sum up these results:

Proposition 2. *If X given by (2) verifies assumption (H), we have for almost all $t\in(0,T)$:*

$$D_+X_t=b(t,X_t)\quad \textit{and}\quad D_-X_t=b(t,X_t)-\sigma^2\frac{\nabla p_t}{p_t}(X_t).$$

Finally, we will also need the following composition formula given by Nelson [7]. For the sake of completeness, we give all the details of its proof. It is based on the use of a Taylor expansion of f as in Nelson's work, plus suitable controls of some remainders:

Proposition 3. *Let $f\in C^{1,2}([0,T]\times\mathbb{R}^d)$ with bounded second order derivatives and let X be a diffusion of the form (2) satisfying (H). Then, for almost all $t\in(0,T)$:*

$$D_\pm f(t,X_t)=\left(\partial_t f+(\partial_x f)D_\pm X_t\pm\frac{\sigma^2}{2}\Delta f\right)(t,X_t). \tag{6}$$

Proof. Let $h>0$.
1) *The forward case.* The Taylor formula yields:

$$f(t+h,X_{t+h})-f(t,X_t) = \partial_t f(t,X_t)h+\partial_x f(t,X_t)(X_{t+h}-X_t) \tag{7}$$
$$+\frac{1}{2}\sum_{i,j=1}^n (X_{t+h}^i-X_t^i)(X_{t+h}^j-X_t^j)\partial_{ij}^2 f(t,X_t)+R(t,h)$$

where the remainder $R(t,h)$ is given by

$$R(t,h) = \frac{1}{2}\sum_{i,j=1}^n (X_{t+h}^i-X_t^i)(X_{t+h}^j-X_t^j)\big(\partial_{ij}^2 f(u_{t,h})-\partial_{ij}^2 f(t,X_t)\big)$$
$$+h\sum_{j=1}^n (X_{t+h}^j-X_t^j)\partial_t\partial_j f(u_{t,h})$$

with $u_{t,h} = (t + \theta h, (1-\theta)X_t + \theta X_{t+h})$ and $\theta \in (0,1)$ depending on t and h.

We first treat the third term of the r.h.s of (7). For instance for the term $\frac{1}{h}E[(X^i_{t+h} - X^i_t)^2|X_t]$:

$$(X^i_{t+h}-X^i_t)^2 = \left(\int_t^{t+h} b(s,X_s)ds\right)^2 + \sigma^2(W^i_{t+h}-W^i_t)^2 + 2\sigma(W^i_{t+h}-W^i_t)\int_t^{t+h} b(s,X_s)ds. \quad (8)$$

We have by Schwarz inequality:

$$\left(\int_t^{t+h} b(s,X_s)ds\right)^2 \leqslant h\int_t^{t+h} b^2(s,X_s)ds.$$

Thus

$$\frac{1}{h}E\left(\int_t^{t+h} b(s,X_s)ds\right)^2 \leqslant \int_t^{t+h} E[b^2(s,X_s)]ds \longrightarrow 0,$$

since $t \to E|X_t|^2$ is locally integrable (see, *e.g.*, Theorem 2.9 in [4]). Again by Schwarz inequality, we deduce that $h^{-1}\big(W^i_{t+h} - W^i_t\big)\int_t^{t+h} b(s,X_s)ds$ tends to 0 in $L^1(\Omega)$. Moreover:

$$\frac{1}{h}E[(W^i_{t+h} - W^i_t)^2|X_t] = \frac{1}{h}E[(W^i_{t+h} - W^i_t)^2] = 1.$$

We now treat the remainder of (7). The fact that $\partial^2 f$ is bounded allows to show as above that $h^{-1}\left(\int_t^{t+h} b(s,X_s)ds\right)^2 (\partial^2_{ij}f(u_{t,h}) - \partial^2_{ij}f(t,X_t))$ and

$$\frac{W^i_{t+h} - W^i_t}{h}\int_t^{t+h} b(s,X_s)ds(\partial^2_{ij}f(u_{t,h}) - \partial^2_{ij}f(t,X_t))$$

converges to 0 in $L^1(\Omega)$. Moreover

$$\begin{aligned} E&\left[\tfrac{(W^i_{t+h}-W^i_t)^2}{h}(\partial^2_{ij}f(u_{t,h}) - \partial^2_{ij}f(t,X_t))\right] \\ &\le \tfrac{\sqrt{E|W^i_{t+h}-W^i_t|^4}}{h}\sqrt{E|\partial^2_{ij}f(u_{t,h}) - \partial^2_{ij}f(t,X_t)|^2} \\ &\le C\sqrt{E|\partial^2_{ij}f(u_{t,h}) - \partial^2_{ij}f(t,X_t)|^2}. \end{aligned}$$

Since $\partial^2 f$ is bounded and $u_{t,h}$ tends to (t,X_t) as $h \to 0$, we can apply the bounded convergence theorem and conclude.

2) *The backward case.* We calculate the Taylor expansion of $-(f(t-h,X_{t-h}) - f(t,X_t))$ and we write $(X^i_{t-h} - X^i_t)^2 = (\overline{X}^i_{T-t+h} - \overline{X}^i_{T-t})^2$. We then write the decomposition (8) for $\overline{X}$ with its time reversed drift $\overline{b}$ and its time reversed driving Brownian motion $\widehat{W}$. So the computations are identical to those of the first point. □

3 Dynamical study of gradient diffusions

3.1 First order derivatives

Gradient diffusions, also known as Langevin diffusions, are largely studied in the literature. For instance, a result of Kolmogorov [5] states that b is a gradient if and only if the law of X

given by (9) is reversible, i.e. $(X_t)_{t\in[0,T]}$ and $(X_{T-t})_{t\in[0,T]}$ have the same law. In this short section, we point out a characterization of the sub-class of stationary Langevin diffusions by means of first order Nelson derivatives. Actually, this fact is substantially contained in several works, see e.g. Föllmer's work [3] or a remark by Zheng and Meyer in [13] p.230.

We only consider Brownian diffusions of type (2) with a homogeneous drift, *i.e.* we work with X verifying

$$X_t = X_0 + \int_0^t b(X_s)ds + \sigma W_t, \quad t \in [0,T]. \tag{9}$$

For instance, knowing that b is a gradient allows to construct an invariant law for X. More precisely, when b equals ∇U with $U : \mathbb{R}^d \to \mathbb{R}$ regular enough and satisfying suitable integrability conditions, the probability law μ defined by

$$d\mu = c^{-1} \mathrm{e}^{\frac{2U(x)}{\sigma^2}} dx \quad \text{with} \quad c = \int_{\mathbb{R}^d} \mathrm{e}^{\frac{2U(x)}{\sigma^2}} dx < \infty$$

is invariant for X. We refer to Lemma 2.2.23 in [11] for this result and to Sections 2.2.2 and 2.2.3 in [11] for more details about Langevin diffusions.

Finally, thanks to formulas of Proposition 2, one can state the following:

Proposition 4. *Let X be the Brownian diffusion defined by (9). We moreover assume that X verifies assumption (H).*

1. *If $D_+X_t = -D_-X_t$ for any $t \in (0,T)$ then $b = \nabla U$ with $U : \mathbb{R}^d \to \mathbb{R}$ given by $U = \frac{\sigma^2}{2}\log p_t$. In particular, X is a stationary diffusion with initial law μ given by $d\mu = \mathrm{e}^{\frac{2U(x)}{\sigma^2}} dx$.*
2. *Conversely, if $b = \nabla U$ with $U : \mathbb{R}^d \to \mathbb{R}$ such that $c := \int_{\mathbb{R}^d} e^{\frac{2U(x)}{\sigma^2}} dx < \infty$ and if the law of X_0 is $d\mu = c^{-1}\mathrm{e}^{\frac{2U(x)}{\sigma^2}} dx$, then the probability law μ is invariant for X and, for any $t \in (0,T)$, we have $D_+X_t = -D_-X_t$.*

3.2 Main result: Characterization of gradient diffusions via second order derivatives

In [10] Theorem 5.4, Roelly and Thieullen give a very nice generalization of Kolmogorov's result [5] based on an integration by part formula from Malliavin calculus. Precisely, this time the drift is not assumed to be time homogeneous, nor the diffusion stationary. Their characterization requires that there exists one reversible law in the reciprocal class of the diffusion. In our case, we are also able to characterize a larger class of Brownian diffusions. However this further needs to use second order stochastic derivatives. The main result of our paper is the following theorem:

Theorem 5. *Let X be given by (9), verifying assumption (H), such that $b \in C^2(\mathbb{R}^d)$ with bounded derivatives, and such that for all $t \in (0,T)$ the second order derivatives of $\nabla \log p_t$ are bounded. Then, we have the following equivalence:*

$$D_+^2 X_t = D_-^2 X_t \ \text{ for almost all } t \in (0,T) \quad \Longleftrightarrow \quad b \text{ is a gradient.} \tag{10}$$

Remark 6. 1. Saying that b is a gradient means that we can write $b = \nabla U$ for a certain potential $U : \mathbb{R}^d \to \mathbb{R}$. It is equivalent, by Poincaré lemma, to verify that $G = \partial_x b - (\partial_x b)^*$ is identically zero.

2. When $d = 1$, that is when X is a *one-dimensional* Brownian diffusion, the equality $D_-^2 X - D_+^2 X = 0$ is always verified, see Lemma 7.

3. The proof we propose here is entirely based on probabilistic arguments. A more "classical" strategy for proving that $G \equiv 0$ when $D_-^2 X = D_+^2 X$ would use the fact that we then have $\mathrm{div}(p_t G_i) = 0$ for any index i and any time $t \in (0,T)$ (see Lemma 7). For instance, when $d = 2$, this system of equalities reduces to $(\partial_1 b_2 - \partial_2 b_1) p_t = c$ on $\mathbb{R}^2$, c denoting a constant. It is then not difficult to deduce that $\partial_1 b_2 = \partial_2 b_1$. In particular, b is a gradient. On the other hand this method seems hard to adapt in higher dimensions. In particular, it seems already difficult to integrate $\mathrm{div}(p_t G) = 0$ when $d = 3$.

First of all, we need the following technical lemma, which gives a remarkable identity for $D_+^2 X - D_-^2 X$:

Lemma 7. *Let X be given by (2), verifying assumption (H), such that $b \in C^{1,2}([0,T] \times \mathbb{R}^d)$ with bounded derivatives, and such that for all $t \in (0,T)$ the second order derivatives of $\nabla \log p_t$ are bounded. Therefore for any $i = 1, \ldots, n$:*

$$(D_-^2 X_t - D_+^2 X_t)^i = \frac{\mathrm{div}(p_t G_i)}{p_t}(t, X_t). \tag{11}$$

Recall that $G = (\partial_x b) - (\partial_x b)^*$, i.e. $G_i^j = \partial_i b^j - \partial_j b^i$.

Let us stress that the expression we obtain in (11) is the key point of our proof of Theorem 5. Moreover, it is valid for diffusions of the type (2) and not only for those of the type (9).

Proof. We have, by Proposition 3:

$$D_+^2 X_t = D_+ b(t, X_t) = \left(\partial_t b + (\partial_x b) b + \frac{\sigma^2}{2} \Delta b\right)(t, X_t), \tag{12}$$

and

$$\begin{aligned} D_-^2 X_t &= D_- \left(b - \sigma^2 \frac{\nabla p_t}{p_t}\right)(t, X_t) \\ &= \Bigg[\partial_t b + (\partial_x b) b - \frac{\sigma^2}{2} \Delta b - \sigma^2 \partial_t \frac{\nabla p_t}{p_t} - \sigma^2 (\partial_x b) \frac{\nabla p_t}{p_t} \\ &\quad -\sigma^2 \left(\partial_x \frac{\nabla p_t}{p_t}\right) b + \sigma^4 \left(\partial_x \frac{\nabla p_t}{p_t}\right) \frac{\nabla p_t}{p_t} + \frac{\sigma^4}{2} \Delta \frac{\nabla p_t}{p_t}\Bigg](t, X_t). \end{aligned}$$

With the Fokker-Planck equation $\partial_t p_t = -\mathrm{div}(p_t b) + \frac{\sigma^2}{2} \Delta p_t$ in mind, we can write:

$$\partial_t \frac{\nabla p_t}{p_t} = \nabla \frac{\partial_t p_t}{p_t} = \nabla \left(-\mathrm{div} b + \frac{\langle -b, \nabla p_t \rangle + \frac{\sigma^2}{2} \Delta p_t}{p_t}\right). \tag{13}$$

Therefore:

$$D_-^2 X_t - D_+^2 X_t = (\sigma^2 A + \sigma^4 B)(t, X_t)$$

with

$$\begin{aligned} A &= -\Delta b + \nabla \mathrm{div} b - (\partial_x b) \frac{\nabla p_t}{p_t} + \nabla \frac{\langle b, \nabla p_t \rangle}{p_t} - \left(\partial_x \frac{\nabla p_t}{p_t}\right) b, \\ B &= \left(\partial_x \frac{\nabla p_t}{p_t}\right) \frac{\nabla p_t}{p_t} + \frac{1}{2} \Delta \frac{\nabla p_t}{p_t} - \frac{1}{2} \nabla \frac{\Delta p_t}{p_t}. \end{aligned}$$

Let us simplify A. By the Leibniz rule we have:

$$\nabla \frac{\langle b, \nabla p_t\rangle}{p_t} = (\partial_x b)^* \frac{\nabla p_t}{p_t} + \left(\partial_x \frac{\nabla p_t}{p_t}\right)^* b.$$

Since $p_t \in C^2$, the Schwarz lemma yields $\left(\partial_x \frac{\nabla p_t}{p_t}\right)^* = \left(\partial_x \frac{\nabla p_t}{p_t}\right)$. Thus

$$A = -\Delta b + \nabla \mathrm{div} b + G\ \frac{\nabla p_t}{p_t},$$

from which we deduce

$$A^i = \frac{\mathrm{div}(p_t G_i)}{p_t}.$$

Let us simplify B. We have:

$$2\left[\left(\partial_x \frac{\nabla p_t}{p_t}\right)\frac{\nabla p_t}{p_t}\right]^i = 2\sum_j \partial_i \left(\frac{\partial_j p_t}{p_t}\right)\frac{\partial_j p_t}{p_t} = \partial_i \sum_j \left(\frac{\partial_j p_t}{p_t}\right)^2.$$

But, again par the Schwarz lemma:

$$\left[\Delta \frac{\nabla p_t}{p_t}\right]^i = \sum_j \partial_j^2 \frac{\partial_i p_t}{p_t} = \partial_i \sum_j \partial_j \left(\frac{\partial_j p_t}{p_t}\right).$$

We then deduce that $B = 0$, which concludes the proof.

$\square$

Now, we go back to the proof of Theorem 5. In order to simplify the exposition, in the sequel we assume without loss of generality that $\sigma = 1$.

Proof. If b is a gradient then, for any $i \in \{1, \cdots, d\}$, we have $G_i = 0$. So Lemma 7 yields $D_-^2 X_t - D_+^2 X_t = 0$.

Conversely, assume that $D_-^2 X_t - D_+^2 X_t = 0$ for any $t \in (0, T)$. Fix $i \in \{1, \cdots, d\}$ and let $\widetilde{X}$ be the unique solution of

$$d\widetilde{X}_t = (b + G_i)(\widetilde{X}_t)dt + dW_t, \quad t \in [0, T], \quad \widetilde{X}_0 = X_0 \in \mathrm{L}^2(\Omega). \tag{14}$$

We denote by $\widetilde{\mathcal{L}}$ the infinitesimal generator of $\widetilde{X}$, considered as a $(L^2(\mathbb{R}^d), \langle\cdot,\cdot\rangle)$ operator. Also $\mathcal{L}$ will denote the generator of X. It is well-known that the adjoint $\widetilde{\mathcal{L}}^*$ of $\widetilde{\mathcal{L}}$ is given by

$$\widetilde{\mathcal{L}}^* = -\mathrm{div}[(b + G_i)\ \cdot\ ] + \frac{1}{2}\Delta\ . \tag{15}$$

Let $f \in C_0^\infty(\mathbb{R}^d)$. By the Dynkin formula for X, we have:

$$E[f(X_t)] - f(x) = E\left[\int_0^t \mathcal{L} f(X_s)ds\right]. \tag{16}$$

But

$$\begin{aligned} E\left[\int_0^t \mathcal{L} f(X_s)ds\right] &= \int_0^t \int_{\mathbb{R}^d} \mathcal{L} f(y) p_s(y) dy ds \\ &= \int_0^t \int_{\mathbb{R}^d} f(y) \mathcal{L}^* p_s(y) dy ds \\ &= \int_0^t E\left[f(X_s)\frac{\mathcal{L}^* p_s(X_s)}{p_s(X_s)}\right] ds. \end{aligned} \tag{17}$$

Since for all $s \in (0,T)$, $\frac{\mathrm{div}(p_s G_i)}{p_s}(X_s) = 0$ a.s., we deduce from (17) and (15) that:

$$E\left[\int_0^t \mathcal{L}f(X_s)ds\right] = \int_0^t E\left[f(X_s)\frac{\widetilde{\mathcal{L}}^* p_s(X_s)}{p_s(X_s)}\right]ds = E\left[\int_0^t \widetilde{\mathcal{L}}f(X_s)ds\right].$$

Therefore:

$$E[f(X_t)] - f(x) = E\left[\int_0^t \widetilde{\mathcal{L}}f(X_s)ds\right]. \tag{18}$$

So the process M defined by

$$M_t = f(X_t) - f(x) - \int_0^t \widetilde{\mathcal{L}}f(X_s)ds$$

is a $(\mathscr{P}^W, \mathbb{P})$-martingale (recall that we decided to note $\mathscr{P}_t^W$ the σ-field generated by W_s for $s \in [0,t]$, see section 2.1). Indeed, by the Markov property applied to X, we can write

$$E(M_t - M_s|\mathscr{P}_s^W) = E_{X_s}\left(f(X_{t-s}) - f(x) - \int_0^{t-s} \widetilde{\mathcal{L}}f(X_s)ds\right) = 0.$$

Thus the law of X solves the martingale problem associated with the Markov diffusion $\widetilde{X}$. But b has linear growth and since the second order derivatives of b are bounded it is also the case for G_i and so for $b + G_i$. This allows to apply the Stroock-Varadhan theorem (see *e.g.* [10, Th 24.1 p.170]) which establishes the existence and uniqueness of solutions for the martingale problem. Therefore X and $\widetilde{X}$ have the same law.

Set $d\mathbb{Q} = Zd\mathbb{P}$, where

$$\begin{aligned} Z &= \exp\left(-\int_0^T \langle G_i(\widetilde{X}_s), dW_s\rangle - \frac{1}{2}\int_0^T |G_i(\widetilde{X}_s)|^2 ds\right) \\ &= \exp\left(-\int_0^T \langle G_i(\widetilde{X}_s), d\widetilde{W}_s\rangle + \frac{1}{2}\int_0^T |G_i(\widetilde{X}_s)|^2 ds\right), \end{aligned}$$

where $\widetilde{W}_t = W_t + \int_0^t G_i(\widetilde{X}_s)ds$. By Girsanov theorem, $\widetilde{W}$ is a Brownian motion under $\mathbb{Q}$ (since G_i is bounded, the Novikov condition is obviously satisfied). By uniqueness in law of weak solution of SDE under linear growth and Lipschitz conditions, the law of $\widetilde{X}$ under $\mathbb{Q}$ is the same as the law of X under $\mathbb{P}$. Consequently, for every $n > 0$, $\phi \in C_b^\infty(\mathbb{R}^n)$ and $0 \le t_1 < \ldots < t_n \le T$:

$$E^{\mathbb{Q}}[\phi(\widetilde{X}_{t_1}, \cdots, \widetilde{X}_{t_n})] = E^{\mathbb{P}}[\phi(X_{t_1}, \cdots, X_{t_n})].$$

Since X and $\widetilde{X}$ have same law, we also have:

$$E^{\mathbb{P}}[\phi(\widetilde{X}_{t_1}, \cdots, \widetilde{X}_{t_n})] = E^{\mathbb{Q}}[\phi(\widetilde{X}_{t_1}, \cdots, \widetilde{X}_{t_n})].$$

But the cylindrical random variables $\phi(\widetilde{X}_{t_1}, \cdots, \widetilde{X}_{t_n})$ are dense in $L^2(\Omega, \mathscr{F}^W)$ (use, for instance, Girsanov theorem). Therefore, $Z = 1$ $\mathbb{P}$-a.s. This means that $G_i(\widetilde{X}) \equiv 0$. Since $\mathscr{L}(\widetilde{X}_t)$ has a positive density for any $t \in (0,T)$, we finally have $G_i \equiv 0$. This concludes the proof.

□

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