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A GAUSSIAN OSCILLATOR

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Abstract.
We present a stochastic process with sawtooth paths whose distribution is given by a simple rule and whose stationary distribution is Gaussian. The process arose in a natural way in research on interaction of an inert particle with a Brownian particle.

The purpose of this short note is to present a very simple sawtooth-like process whose stationary distribution is Gaussian. The process arose in a natural way in the doctoral research [W] of the second author, devoted to the study of interaction between an inert particle with a Brownian particle. A model of such interaction was originally proposed by Knight [K].

We believe that the "Gaussian oscillator" is a sufficiently interesting abstract object on its own but we also believe that it points to the possibility of an alternative explanation for the occurrence of normal distributions in some real life data. The standard explanation for normally distributed data is that they are due to the Central Limit Theorem, acting in an explicit or implicit way. There is no CLT effect hidden in models discussed in this paper.

We have a whole family of sawtooth processes with Gaussian stationary distributions. First, we will present a particularly simple process in this family, and then we will describe the remaining ones. The informal presentation of the processes will be followed by a formal proof that the stationary distribution is normal.

The sawtooth processes we mentioned are not Markov but we can make them Markov by enlarging the state space as follows. Let K_t be a process with sawtooth paths (see Fig. 1) and following properties. The time derivative $V_t = (\partial/\partial t)K_t$ of K_t exists everywhere, except for a finite number of points on every finite interval, and takes values $\beta > 0$ and $-\beta < 0$. The process (K_t, V_t) is Markov. The process V_t switches its value from β to $-\beta$ on the interval Δt with

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probability $2K_t\Delta t$ if $K_t > 0$ (it cannot jump from β to $-\beta$ when $K_t < 0$). Similarly, V_t jumps from $-\beta$ to β on the interval Δt with probability $2|K_t|\Delta t$ if $K_t < 0$, and it does not jump from $-\beta$ to β when $K_t > 0$.

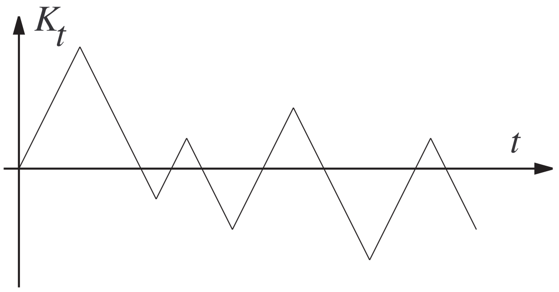

Figure 1. Sawtooth process K_t.

It is rather easy to see that K_t is not a Gaussian process, even in the stationary regime. For example, its two-dimensional distributions are not multidimensional normal. If $K_t = 0$ for some t then for small Δt, the distribution of $K_{t+\Delta t}$ is bimodal, and hence not normal, under the stationary distribution.

The process K_t described above is just one example from a larger family of processes. Suppose that we have non-negative continuous functions $d(k)$ and $b(k)$ such that $d(k) - b(k) = 2k$. The following definition of (K_t, V_t) contains the one given above as a special case. Recall that K_t is assumed to have sawtooth paths, the time derivative V_t of K_t exists everywhere, except for a finite number of points on every finite interval, V_t takes values $\beta > 0$ and $-\beta < 0$, and the process (K_t, V_t) is Markov. The process V_t switches its value from β to $-\beta$ on the interval Δt with probability $d(K_t)\Delta t$; V_t jumps from $-\beta$ to β on the interval Δt with probability $b(K_t)\Delta t$. Note that the assumption that the process K_t cannot change the direction when K_t and V_t have the opposite signs has been dropped.
The special case of (K_t, V_t) with $d(k) = ke^{ak}/\sinh(ak)$ and $b(k) = ke^{-ak}/\sinh(ak)$ arose in [W]. As usual, $\sinh(x) = (e^x - e^{-x})/2$. Note that as $a \to \infty$, one obtains the first example described in this paper.
We omit the formal construction of (K_t, V_t) as it presents no technical problems.

Theorem. *For any non-negative continuous $d(k)$ and $b(k)$ satisfying $d(k) - b(k) = 2k$, the process (K_t, V_t) has a stationary distribution and K_t is normal under this distribution, with mean zero and variance $\beta/2$.*

Note that the stationary distribution does not depend on $d(k)$ and $b(k)$.

Proof. Our argument will be based on the analysis of the generator (see [EK]). It is clear from the informal description of the process that the generator A of (K_t, V_t) is given by

$$Af(k,\beta) = \beta\frac{\partial}{\partial k}f(k,\beta) + d(k)[f(k,-\beta) - f(k,\beta)],$$
$$Af(k,-\beta) = -\beta\frac{\partial}{\partial k}f(k,-\beta) + b(k)[f(k,\beta) - f(k,-\beta)].$$

We omit the domain of integration from the integrals. The variable k, representing K_t, always ranges from $-\infty$ to ∞, and dv is a measure representing V_t, giving weights 1 to β and $-\beta$.

The adjoint A^* of A is the operator satisfying, for all suitable f, g,

$$\int (Af)g\,dk\,dv = \int f(A^*g)\,dk\,dv.$$

Applying this to our particular A,

$$\begin{aligned}\int (Af)g\,dk\,dv = &\int \beta\big(\frac{\partial}{\partial k}f(k,\beta)\big)g(k,\beta)dk + \int d(k)[f(k,-\beta)g(k,\beta) - f(k,\beta)g(k,\beta)]dk\\ &+ \int (-\beta)\big(\frac{\partial}{\partial k}f(k,-\beta)\big)g(k,-\beta)dk + \int b(k)[f(k,\beta)g(k,-\beta) - f(k,-\beta)g(k,-\beta)]dk.\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}\int (Af)g\,dk\,dv = &\int \{-\beta f(k,\beta)\frac{\partial}{\partial k}g(k,\beta) + f(k,\beta)[b(k)g(k,-\beta) - d(k)g(k,\beta)]\}dk\\ &+ \int \{\beta f(k,-\beta)\frac{\partial}{\partial k}g(k,-\beta) + f(k,-\beta)[-b(k)g(k,-\beta) + d(k)g(k,\beta)]\}dk.\end{aligned}$$

From this we see that

$$\begin{aligned}A^*g(k,\beta) &= -\beta\frac{\partial}{\partial k}g(k,\beta) + b(k)g(k,-\beta) - d(k)g(k,\beta), \qquad (1)\\ A^*g(k,-\beta) &= \beta\frac{\partial}{\partial k}g(k,-\beta) - b(k)g(k,-\beta) + d(k)g(k,\beta).\end{aligned}$$

The stationary distribution μ of a process is characterized by

$$\int Af\ d\mu = 0.$$

When $d\mu$ is of the form $g(k,v)dk\,dv$, this is equivalent to

$$\int (Af)g\,dk\,dv = \int f(A^*g)\,dk\,dv = 0,$$

so that it is sufficient to find $g(k,v)$ satisfying $A^*g(k,v) \equiv 0$. By (1), we are looking for $g(k,v)$ such that

$$0 = A^*g(k,\beta) + A^*g(k,-\beta) = -\beta\big(\frac{\partial}{\partial k}g(k,\beta) - \frac{\partial}{\partial k}g(k,-\beta)\big),$$

so that $g(k,-\beta) = g(k,\beta) + c$. Since we have to have

$$1 = \int (g(k,\beta) + g(k,-\beta))dk = \int (g(k,\beta) + g(k,\beta))dk + \int c\,dk, \qquad (2)$$

the constant c must vanish so $g(k,\beta) = g(k,-\beta)$. Using this fact and (1), we get

$$\begin{aligned}A^*g(k,\beta) &= -\beta\frac{\partial}{\partial k}g(k,\beta) + (b(k) - d(k))g(k,\beta)\\ &= -\beta\frac{\partial}{\partial k}g(k,\beta) - 2kg(k,\beta).\end{aligned}$$

Setting this equal to zero and solving for $g(k,\beta)$ gives $g(k,\beta) = c_1 \exp(-k^2/\beta)$. The normalization (2) yields

$$g(k,\beta) = g(k,-\beta) = \frac{1}{2\sqrt{\pi\beta}} \exp(-k^2/\beta).$$

Now we can trace back our steps to see that the last formula represents the stationary distribution.
Q.E.D.

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