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A MODIFIED KARDAR–PARISI–ZHANG MODEL

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Abstract

A one dimensional stochastic differential equation of the form

$$dX = AXdt + \tfrac{1}{2}(-A)^{-\alpha}\partial_\xi[((-A)^{-\alpha}X)^2]dt + \partial_\xi dW(t), \qquad X(0) = x$$

is considered, where $A = \frac{1}{2}\partial_\xi^2$. The equation is equipped with periodic boundary conditions. When $\alpha = 0$ this equation arises in the Kardar–Parisi–Zhang model. For $\alpha \neq 0$, this equation conserves two important properties of the Kardar–Parisi–Zhang model: it contains a quadratic nonlinear term and has an explicit invariant measure which is gaussian. However, it is not as singular and using renormalization and a fixed point result we prove existence and uniqueness of a strong solution provided $\alpha > \frac{1}{8}$.

1 Introduction

Let us consider the following Burgers equation on $(0, 2\pi)$ with periodic boundary conditions and perturbed by noise

$$\begin{cases} dX = \frac{1}{2}\left[\partial_\xi^2 X + \partial_\xi(X^2)\right]dt + \partial_\xi dW(t) \\ X(0,\xi) = x(\xi) \in L_0^2(0,2\pi), \quad X(t,0) = X(t,2\pi). \end{cases} \tag{1.1}$$

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where W is a cylindrical white noise of the form

$$W(t,\xi)=\sum_{k=1}^{\infty}e_k(\xi)\beta_k(t),$$

where

$$e_k(\xi)=\frac{1}{\sqrt{2\pi}}\,e^{\mathtt{i}k\xi},\quad k\in\mathbb{Z}_0,$$

$\mathbb{Z}_0=\mathbb{Z}\backslash\{0\}$ and $(\beta_k(t))_{k\in\mathbb{Z}_0}$ is a family of standard Brownian motions mutually independent in a filtered probability space $(\Omega,\mathscr{F},(\mathscr{F}_t)_{t\geq 0},\mathbb{P})$.
Equation (1.1) is known as the Kardar-Parisi-Zhang equation (KPZ equation) and was introduced in [15] as a model of the interface growing in the phase transitions theory. It can also seen as the limit equation of a suitable particle system, see [4].
As usual, we write equation (1.1) in an abstract form. It is no restriction to assume that the initial data has a zero average. Since this property is conserved by equation (1.1) we introduce the space $L^2_0(0,2\pi)$ of all square integrable functions in $[0,2\pi]$ with 0 mean valueÄ. We define

$$Ax=\tfrac12\,\partial_\xi^2 x,\quad x\in D(A):=\{y\in H^2(0,2\pi):\ y(0)=y(2\pi),\ y_\xi(0)=y_\xi(2\pi)\},$$

$$Bx=\partial_\xi x\quad x\in\{y\in H^1(0,2\pi):\ y(0)=y(2\pi)\}$$

and rewrite (1.1) as

$$\begin{cases} dX=(AX+\frac12\partial_\xi(X^2))dt+BdW(t),\\[2mm] X(0)=x.\end{cases}\tag{1.2}$$

Equation (1.2) can be written in mild form

$$X(t)=e^{tA}x+\tfrac12\int_0^t e^{(t-s)A}\partial_\xi(X^2(s))\,ds+W_A(t),$$

where $W_A(t)$ is the stochastic convolution (see [12])

$$W_A(t)=\int_0^t e^{(t-s)A}BdW(s)=\sum_{k\in\mathbb{Z}_0}\mathtt{i}k\,e_k(\xi)\int_0^t e^{-\frac12(t-s)\,k^2}d\beta_k(s).\tag{1.3}$$

Note that $(e_k)_{k\in\mathbb{Z}_0}$ is a basis of eigenvectors of A.
$W_A(t)$ is a Gaussian random variable in $L^2_0(0,2\pi)$ and covariance operator

$$C(t)=I-e^{2tA}\quad t\geq 0.$$

An important characteristic of this problem is that (as it happens for the $2D$ periodic Navier-Stokes equation), though the problem is non linear, its invariant measure coincides with the Gaussian invariant measure of the corresponding free system, whose covariance operator reduces to the identity in our case. Consequently, the invariant measure does not live in $L^2_0(0,2\pi)$. It is not difficult to see that this measures lives in functional spaces of negative regularity, strictly less than 1/2.
A natural way to define the product in this context is to replace the nonlinear term $\partial_\xi(X^2)$ by $\partial_\xi(:X^2:)$, where $:X^2:$ represents the Wick product. In the case of periodic boundary conditions, the Wick product is the standard product renormalized by the subtraction of an

infinite constant. Thus the two products are in fact formally equal since the infinite constant disappears by differentiation. This method based on renormalization has been successfully used recently for some reaction-diffusion equations arising in field theory, see [2], [5], [9], [10], [11], [14], [16] and for $2D$-Navier-Stokes equations, see [1], [8], [13]. The case of the Navier-Stokes is very similar to the case considered here. Indeed, the Wick nonlinearity is formally equal to the original nonlinearity.

The KPZ equation is more difficult and it is not possible to define the Wick product in the classical way (see [10] for a discussion). A generalized Wick product has been introduced in [3], however it is very irregular and up to now it has not been possible to construct solutions of the KPZ equation with this generalized Wick product.

In this article, we adopt another strategy. As it has been done in the case of the stochastic quantization equation, we modify the equation in such a way that the nonlinear term has the same structure and that the equation has the same invariant measure as the KPZ equation. For this reason we shall introduce the following modified equation

$$dX = AXdt + \tfrac{1}{2}(-A)^{-\alpha}\partial_\xi[((-A)^{-\alpha}X)^2]dt + \partial_\xi dW(t), \tag{1.4}$$

trying to choose $\alpha > 0$ as small as possible. It is not difficult to see that indeed the Gaussian measure with covariance equal to the identity is formally invariant for (1.4).

It is convenient to introduce a new variable $X_\alpha(t) = (-A)^{-\alpha}X(t)$ and to replace the quadratic term X_α^2 with the renormalized power $:X_\alpha^2:$ - which just differs from X_α^2 by an infinite constant. So, equation (1.4) becomes

$$dX_\alpha = AX_\alpha dt + \tfrac{1}{2}(-A)^{-2\alpha}\partial_\xi[:X_\alpha^2:]dt + \partial_\xi(-A)^{-\alpha}dW(t).$$

With this transformation, the invariant measure has now the covariance given by $(-A)^{-2\alpha}$. For $\alpha > 1/4$, this measures lives in $L_0^2(0, 2\pi)$, it is not necessary to use the Wick product and this equation can be solved by standard arguments. We shall show that the Wick power is well defined provided

$$\sum_{k\in\mathbb{Z}_0} k^{-8\alpha} < \infty.$$

So, we shall choose $\alpha \in (\frac{1}{8}, \frac{1}{4}]$. Using the strategy introduced in [8, 9], we will construct strong solutions for this equation by a suitable fixed point. Using the fact that we know explicitly the invariant measures, we will also prove that the solutions are almost surely global in time. We think that this work is a step in the understanding of the KPZ model and hope that our techniques will generalize so that we can treat the original case $\alpha = 0$.

2 The main result

2.1 Notation

Let us consider the Hilbert space

$$H = \left\{ x \in L^2(0, 2\pi) : \int_0^{2\pi} x(\xi)d\xi = 0 \right\},$$

endowed with the scalar product

$$\langle x, y\rangle = \int_0^{2\pi} x(\xi)\overline{y(\xi)}d\xi, \quad x, y \in H,$$

and the associated norm denoted by $|\cdot|$.
A complete orthonormal system in H is given by $\{e_k\}_{k\in\mathbb{Z}_0}$, where

$$e_k(\xi) = \frac{1}{\sqrt{2\pi}}\, e^{\mathrm{i}k\xi}, \quad k \in \mathbb{Z}_0,$$

and $\mathbb{Z}_0 = \mathbb{Z}\backslash\{0\}$. It is well known that these are the eigenvectors of A:

$$Ae_k = -k^2 e_k, \quad k \in \mathbb{Z}_0.$$

We set $x_k = \langle x, e_k\rangle, \quad k \in \mathbb{Z}_0$. If x is real valued we have

$$x_{-k} = \overline{x_k}, \quad k \in \mathbb{Z}\backslash\{0\}.$$

In the following we shall identify H with $\ell^2(\mathbb{Z}_0)$ through the isomorphism:

$$x \in H \to \{x_k\}_{\mathbb{Z}_0} \in \ell^2(\mathbb{Z}_0).$$

We set $\mathscr{H} := \mathbb{R}^\infty$ so that H is identified as a subspace of $\mathscr{H}$. We denote by μ the product measure on $\mathscr{H}$

$$\mu = \mathop{\times}_{k\in\mathbb{Z}_0} \mathscr{N}\,(0,1).$$

We also use the L^2 based Sobolev spaces which in our case are easily characterized thanks to the eigenbasis of A. For $s \in \mathbb{R}$, we set

$$H^s = \{x = \{x_k\}_{\mathbb{Z}_0} \in \mathscr{H},\ \sum_{k\in\mathbb{Z}_0} |k|^{2s}|x_k|^2 < \infty\}$$

and

$$|x|_{H^s} = \left(\sum_{k\in\mathbb{Z}_0} |k|^{2s}|x_k|^2\right)^{1/2}.$$

Note that $H^s = D((-A)^{s/2})$. Setting $(-A)^{-\alpha}X = X_\alpha$, equation (1.4) reduces to

$$dX_\alpha = AX_\alpha dt + \tfrac{1}{2}(-A)^{-2\alpha}\partial_\xi[X_\alpha^2]dt + \partial_\xi(-A)^{-\alpha}dW(t). \tag{2.1}$$

Equations (1.4) and (2.1) are equivalent. We consider only (2.1) without mentioning the corresponding results for equation (1.4).
In order to lighten the notations, we omit the subscript α and below X denotes the unknown of equation (2.1). Since we work only with this equation, this should not yield any confusion.
We denote by μ_α the Gaussian measure (corresponding to the free system)

$$\mu_\alpha = N_{(-A)^{-2\alpha}}.$$

It lives in H if and only if Tr $[(-A)^{-2\alpha}] < +\infty$, that is if and only if $\alpha > \frac{1}{4}$. In this case, a local in time solution of equation (1.4) is easily obtained thanks to a classical fixed point argument. The same argument as in section 2.3 below can be used to prove global existence.
The case $\alpha \le \frac{1}{4}$ is more difficult, The measure μ_α lives in any space $H^{-\varepsilon} \subset \mathscr{H} := \mathbb{R}^\infty$ with $2\varepsilon > 1 - 4\alpha$.

Below, we shall use the following well known result *in finite dimensions*: given the following system of SDE (free system)

$$dZ_t = AZ_t\,dt + \sqrt{C}dW_t$$

whose invariant measure is $\mathcal{N}(0,Q)$ where $Q = \int_0^\infty e^{tA}Ce^{tA^*}dt$, the non linear system

$$dX_t = \big(AX_t + b(X_t)\big)\,dt + \sqrt{C}dW_t$$

has the same invariant measure $\mathcal{N}(0,Q)$ if and only if *i)* $\operatorname{div} b = 0$ and *ii)* $\langle b(x), Q^{-1}x\rangle = 0$ for any x.

Let us introduce Galerkin approximations of equation (2.1). For any $N \in \mathbb{N}$ we consider

$$\begin{cases} dX^N = \big(A_N X^N + \frac{1}{2}F_N(X^N)\big)\,dt + \partial_\xi(-A_N)^{-\alpha}dW(t), \\ X^N(0) = x_N, \end{cases} \tag{2.2}$$

where

$$P_N = \sum_{|k|\le N, k\neq 0} e_k \otimes e_k,\ A_N = P_N A,$$

and

$$F_N(x) = (-A_N)^{-2\alpha}\partial_\xi[(P_N x)^2].$$

Lemma 2.1. *The measure* $\mu_{\alpha,N} = \mathcal{N}\big(0,(-A_N)^{-2\alpha}\big)$ *is invariant for* (2.2).

Proof. It is clear that for any $x \in P_N H$

$$\langle F_N(x), (-A_N)^{2\alpha}x\rangle = 0. \tag{2.3}$$

Moreover, for $x \in P_N H$,

$$F_N(x) = \sqrt{2\pi}\,\mathtt{i} \sum_{0<|h|,|k|,|h+k|\le N} |h+k|^{-4\alpha}(h+k)\,x_h\,x_k\,e_{h+k}.$$

It follows

$$\langle F_N(x), e_j\rangle = \sqrt{2\pi}\,\mathtt{i} \sum_{0<|h|,|k|\le N, |h+k|=j} |j|^{2\alpha}\,j\,x_h\,x_k\,e_{h+k}.$$

and

$$D_{x_j}\langle F_N(x), e_j\rangle = 0$$

which yields

$$\operatorname{div} F_N(x) = \sum_{|j|\le N} D_{x_j}\langle F_N(x), e_j\rangle = 0.$$

Then this fact, together with (2.3), implies that $\mu_N = \mathcal{N}\big(0,(-A_N)^{-2\alpha}\big)$ is invariant for (2.2). $\square$

2.1.1 Definition of $:X^2:$

Let us recall the definition of Wick product $:X^2:$ in our specific case following the method of [10]. We denote by $\langle e_k, \cdot\rangle$ the k^{th} coordinate mapping defined on $\mathscr{H}$ and we set for $X \in \mathscr{H}$

$$X_N(\xi) = \sum_{1\le |k|\le N} \langle e_k, X\rangle e_k(\xi)$$

and

$$:X_N^2:(\xi) = [X_N(\xi)]^2 - \rho_N^2, \tag{2.4}$$

where

$$\rho_N^2 = \frac{1}{2\pi} \sum_{1\le |k|\le N} \frac{1}{|k|^{4\alpha}}. \tag{2.5}$$

Clearly, for any $X \in \mathscr{H}$, $:X_N^2:$ is an element of H and therefore of H^s for any $s \le 0$. The following result is proved in [10], section 7.

Theorem 2.2. *If $\frac{1}{8} < \alpha \le \frac{1}{4}$ then the sequence of functions $(:X_N^2:)$ has a limit in $L^2(H^{-\varepsilon}, \mu_\alpha; H^{-\varepsilon})$, for any $\varepsilon > \frac{1}{2}(1-4\alpha)$. We denote this limit by $:X^2:$.*

Unfortunately, the definition of the Wick product is much more complicated for $\alpha < \frac{1}{8}$. It is defined only in a space of generalized random variables (see [3]) and we are not able to handle it. Thus, we shall restrict ourselves from now on to the case $\alpha \in (\frac{1}{8}, \frac{1}{4}]$.
Note that for any $N \in \mathbb{N}$

$$F_N(X_N) = (-A_N)^{-2\alpha}\partial_\xi[X_N^2] = (-A_N)^{-2\alpha}\partial_\xi[:X_N^2:].$$

We deduce that the following result.

Corollary 2.3. *The sequence $F_N(X_N)$ converges in $L^2(H^{-\varepsilon}, \mu_\alpha; H^{-\varepsilon-1})$, for any $\varepsilon > \frac{1}{2}(1-4\alpha)$ to $(-A)^{-2\alpha}\partial_\xi[:X^2:]$.*

It is therefore natural to consider the equation

$$\begin{cases} dX = AXdt + \frac{1}{2}(-A)^{-2\alpha}\partial_\xi[:X^2:]dt + \partial_\xi(-A)^{-\alpha}dW(t), \\ \\ X(0) = x. \end{cases} \tag{2.6}$$

We are now able to define the nonlinear term for a random variable whose law is given by μ_α and, proceeding as in [16], this is sufficient to construct a weak stationary solution. We wish to go further and define the nonlinear term for a larger class of random variable. The following result proved by paraproduct techniques (see [6], [7]) is useful.

Lemma 2.4. *Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha + \beta > 0$, $\alpha, \beta < 1$, then for $x \in H^\alpha$, $y \in H^\beta$, we have $xy \in H^{\alpha+\beta-\frac{1}{2}}$ and*

$$|xy|_{H^{\alpha+\beta-\frac{1}{2}}} \le c(\alpha,\beta)|x|_{H^\alpha}|y|_{H^\beta}.$$

Consider now a random variable X with values in $\mathscr{H}$ which can be written as $X = Y + Z$ where Z has the law μ and $Y \in L^2(\Omega; H^\beta)$. We can write

$$:X_N^2:(\xi) = [X_N(\xi)]^2 - \rho_N^2 = [Y_N(\xi)]^2 + 2Y_N(\xi)Z_N(\xi) + :Z_N^2:(\xi).$$

Using Theorem 2.2 and Lemma 2.4, the three terms have a limit in $L^2(\Omega; H^\delta)$ provided $\beta > \frac{1}{2} - 2\alpha$ and $\delta < \beta - \frac{1}{2} + 2\alpha$. We are therefore able to define the nonlinear term for such random variables and we have the following natural formula :

$$:X^2:= Y^2 + 2YZ + \hat{\mathrm{E}}\ :Z^2: . \tag{2.7}$$

Finally, if we know only that $Y \in H^\beta$ almost surely, the above discussion still holds but the limit has to be understood in probability.
Again, $:X^2:$ is defined through the subtraction of an infinite constant and $\partial_\xi[:X^2:]$ is a natural definition for the nonlinear term.

2.2 Local existence

We write equation (2.6) in the mild form

$$X(t) = e^{tA}x + \tfrac{1}{2}\int_0^t e^{(t-s)A}(-A)^{-2\alpha}\partial_\xi[:X(s)^2:]ds + Z(t) - e^{At}Z(0), \tag{2.8}$$

where

$$Z(t) = \int_{-\infty}^t e^{(t-s)A}\partial_\xi(-A)^{-\alpha}dW(s). \tag{2.9}$$

It follows from the factorization method (see [12]) that

$$Z \in C([0,T]; H^{-\varepsilon}), \text{ for any } \varepsilon > \frac{1}{2} - 2\alpha. \tag{2.10}$$

The following Lemma is proved as in [8], [9].

Lemma 2.5. *We have*

$$:Z^2: \in L^p(0,T; H^{-\varepsilon}), \quad \forall\, p \geq 1,\ \varepsilon > \frac{1}{2} - 2\alpha. \tag{2.11}$$

Set

$$Y(t) = X(t) - Z(t), \quad t \geq 0.$$

We will see that Y is regular and thanks to (2.7), (2.8) becomes

$$\begin{aligned} Y(t) &= e^{tA}(x - Z(0)) + \int_0^t e^{(t-s)A}(-A)^{-2\alpha}\partial_\xi[Y(s)Z(s)]ds \\ &+\tfrac{1}{2}\int_0^t e^{(t-s)A}(-A)^{-2\alpha}\partial_\xi[Y^2(s)]ds \\ &+\tfrac{1}{2}\int_0^t e^{(t-s)A}(-A)^{-2\alpha}\partial_\xi[:Z^2(s):]ds \\ &=: T_0(x - Z(0))(t) + 2T_1(Y,Z)(t) + T_1(Y,Y)(t) + T_2(:Z^2:)(t), \quad t \geq 0. \end{aligned} \tag{2.12}$$

We are going to solve equation (2.12) by a fixed point argument in the space

$$\mathscr{X}_T := C([0,T]; H^{-\gamma}) \cap L^r(0,T; H^\beta),$$

where $\gamma > 0,\ \beta > 0$ and $r \geq 1$ will be chosen later and T is sufficiently small. We need the following lemma.

Lemma 2.6. *(i) For any $y \in H^{-\gamma}$, $T_0(y) \in \mathscr{X}_T$, provided*

$$r\,\frac{\beta+\gamma}{2} < 1. \tag{2.13}$$

Moreover

$$|T_0(y)|_{\mathscr{X}_T} \le c(r,\gamma,\beta)|y|_{H^{-\gamma}}.$$

(ii) For any $Y_1 \in L^r(0,T;H^\beta)$, $Y_2 \in C([0,T];H^{-\gamma})$, $T_1(Y_1,Y_2) \in \mathscr{X}_T$, provided

$$\frac{\gamma}{2} - 2\alpha < \frac{1}{4},\ \beta-\gamma>0,\ \text{and}\ -\frac{\beta}{2} - 2\alpha + \frac{1}{r} < \frac{1}{4}. \tag{2.14}$$

Moreover

$$|T_1(Y_1,Y_2)|_{\mathscr{X}_T} \le c\,T^\delta |Y_1|_{L^r(0,T;H^\beta)}|Y_2|_{C([0,T];H^{-\gamma})} \tag{2.15}$$

with $\delta = \min\{\frac{1}{4} - \frac{\gamma}{2} + 2\alpha; \frac{1}{4} - \frac{1}{r} + \frac{\beta}{2} + 2\alpha\}$.

(iii) For any $V \in L^p(0,T;H^{-\varepsilon})$, with $p \ge 1$, $\varepsilon > 0$, $T_2(V) \in \mathscr{X}_T$ provided

$$\frac{1}{2}\ (-\gamma+\varepsilon+1-4\alpha) + \frac{1}{p} < 1 \tag{2.16}$$

and

$$\frac{1}{2}\ (\beta+\varepsilon+1-4\alpha) + \frac{1}{p} < 1 + \frac{1}{r}. \tag{2.17}$$

Proof. (i) We first notice that, since $y \in H^{-\gamma}$, we have

$$e^{tA}y \in C([0,T];H^{-\gamma}).$$

Moreover, since for all $\beta,\gamma \in \mathbb{R}$

$$|e^{tA}y|_{H^\beta} \le c\,t^{-\frac{\beta+\gamma}{2}}|y|_{H^{-\gamma}},$$

we see that $e^{tA}y \in L^r(0,T;H^\beta)$ provided condition (2.13) is fulfilled.
(ii) By Lemma 2.4, if $\beta - \gamma > 0$,

$$|Y_1Y_2|_{H^{\beta-\gamma-\frac{1}{2}}} \le c\,|Y_1|_{H^\beta}\ |Y_2|_{H^{-\gamma}}.$$

Therefore

$$|(-A)^{-2\alpha}\partial_\xi[Y_1Y_2]|_{H^{\beta-\gamma-\frac{3}{2}+4\alpha}} \le c\,|Y_1|_{H^\beta}\ |Y_2|_{H^{-\gamma}}$$

and, by classical properties of the heat semigroup,

$$|e^{A(t-s)}(-A)^{-2\alpha}\partial_\xi[Y_1(s)Y_2(s)]|_{H^\beta} \le c|t-s|^{-\frac{1}{2}(\gamma+\frac{3}{2}-4\alpha)}|Y_1(s)|_{H^\beta}\ |Y_2(s)|_{H^{-\gamma}}.$$

We deduce

$$|T_1(Y_1,Y_2)(t)|_{H^\beta} \le c\int_0^t |t-s|^{-\frac{1}{2}(\gamma+\frac{3}{2}-4\alpha)}|Y_1(s)|_{H^\beta}\ |Y_2(s)|_{H^{-\gamma}}ds,$$

provided

$$\frac{\gamma}{2} - 2\alpha < \frac{1}{4}.$$

Then by Hausdorff-Young inequality

$$|T_1(Y_1,Y_2)|_{L^r(0,T;H^\beta)} \le cT^{1-\frac{1}{2}(\gamma+\frac{3}{2}-4\alpha)}|Y_1|_{L^r(0,T;H^\beta)}|Y_2|_{C([0,T];H^{-\gamma})}$$

Similarly

$$|T_1(Y_1,Y_2)(t)|_{H^{-\gamma}} \le c\int_0^t |t-s|^{-\frac{1}{2}(-\beta+\frac{3}{2}-4\alpha)}|Y_1(s)|_{H^\beta}\ |Y_2(s)|_{H^{-\gamma}}ds,$$

and

$$|T_1(Y_1,Y_2)|_{C([0,T];H^{-\gamma})} \le cT^{1-\frac{1}{r}+\frac{\beta}{2}-\frac{3}{4}+2\alpha}|Y_1|_{L^r(0,T;H^\beta)}|Y_2|_{C([0,T];H^{-\gamma})}$$

provided

$$-\frac{\beta}{2}-2\alpha+\frac{1}{r}<\frac{1}{4}.$$

The claim follows.
The proof of (iii) is easier and left to the reader. □
The following lemma states that the conditions of Lemma 2.6 are compatible.

Lemma 2.7. *There exist $\beta>0$, $\gamma>\frac{1}{2}-2\alpha$, $\varepsilon>\frac{1}{2}-2\alpha$, p and r such that all conditions of Lemma 2.6 are verified.*

Proof. Taking ε sufficiently close to $\frac{1}{2}-2\alpha$ and p sufficiently large, (2.16) and (2.17) are satisfied provided

$$-\gamma<6\alpha+\frac{1}{2},\ \beta<6\alpha+\frac{1}{2}+\frac{2}{r}.$$

The first condition is clearly satisfied for $\gamma>0$. Hence, we can summarize (2.14), (2.16) and (2.17) as

$$0<\gamma<\frac{1}{2}+4\alpha,\ -\frac{1}{2}-4\alpha+\frac{2}{r}<\beta<\frac{1}{2}+6\alpha+\frac{2}{r}$$

which have to supplemented by

$$\beta+\gamma<\frac{2}{r},\ \beta>\gamma>\frac{1}{2}-2\alpha.$$

We take $\epsilon_1,\ \epsilon_2,\ \epsilon_3>0$ and $\gamma=\frac{1}{2}-2\alpha+\epsilon_1,\ \beta=\frac{1}{2}-2\alpha+\epsilon_2,\ \frac{2}{r}=1-4\alpha+\epsilon_3$, r exists provided

$$\epsilon_3<1+4\alpha.$$

It is easy to check that all conditions are satisfied for

$$\epsilon_1<\epsilon_2,\ \epsilon_1+\epsilon_2<\epsilon_3<\epsilon_2+6\alpha.$$

□

Using Lemma 2.6, it is now easy to prove the following result. Note that we have chosen $\gamma>\frac{1}{2}-2\alpha$ so that we know the Z has paths in $C([0,T];H^{-\gamma})$.

Proposition 2.8. *Let $\beta>0$, $\gamma>\frac{1}{2}-2\alpha$, ε, p and r be as in Lemma 2.7. For any $x\in H^{-\gamma}$, there exists a unique solution of (2.6) in $\mathscr{X}_T$ with*

$$T=c\left(|x|_{H^{-\gamma}}+|:Z^2:|_{L^p(0,T;H^{-\varepsilon})}+|Z|_{C([0,T];H^{-\gamma})}\right)^{-1/\delta}.$$

2.3 Global existence

Recall that we have denoted by μ_α the Gaussian measure $\mathcal{N}\big(0,(-A)^{-2\alpha}\big)$ on $H^{-\gamma}$. We have

Theorem 2.9. *For μ_α almost every all initial data $x \in H^{-\gamma}$, $\gamma > \frac{1}{2} - 2\alpha$, there exists a unique global solution of* (2.6).

Proof. Using classical arguments (see the proof of Theorem 5.1 in [8] for details), it suffices to obtain a uniform a priori estimate on the solutions of the Galerkin approximations. We follow here the method in [9], [13]. In order to lighten the notations, we perform directly the computations below on the solutions of equation (2.8). A rigorous proof is easily obtained by translating these computations on the Galerkin solutions.
We have

$$X(t,x) = e^{tA}x + \tfrac{1}{2}\int_0^t e^{(t-s)A}(-A)^{-2\alpha}\partial_\xi[:X(s)^2:]\,ds + Z(t) - e^{tA}Z(0),$$

and so, for γ as in Lemma 2.7,

$$|X(t,x)|_{H^{-\gamma}} \le |x|_{H^{-\gamma}} + \tfrac{1}{2}\int_0^t |(-A)^{-2\alpha}\partial_\xi[:X(s)^2:]|_{H^{-\gamma}}ds + |Z(t)|_{H^{-\gamma}} + |Z(0)|_{H^{-\gamma}}.$$

Consequently

$$\sup_{t\in[0,T]}|X(t,x)|_{H^{-\gamma}} \le |x|_{H^{-\gamma}} + \tfrac{1}{2}\int_0^T |(-A)^{-2\alpha}\partial_\xi[:X(t)^2:]|_{H^{-\gamma}}dt + 2\sup_{t\in[0,T]}|Z(t)|_{H^{-\gamma}}.$$

Now it follows that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|_{H^{-\gamma}}\right) \le |x|_{H^{-\gamma}}$$

$$+\tfrac{1}{2}\int_0^T \mathbb{E}|(-A)^{-2\alpha}\partial_\xi[:X(t)^2:]|_{H^{-\gamma}}dt + 2\mathbb{E}\left(\sup_{t\in[0,T]}|Z(t)|_{H^{-\gamma}}\right),$$

and consequently

$$\int_{H^{-\varepsilon}}\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|_{H^{-\gamma}}\right)\mu_\alpha(dx) \le \int_{H^{-\varepsilon}}|x|_{H^{-\gamma}}\,\mu_\alpha(dx)$$

$$+\tfrac{1}{2}\int_0^T\int_{H^{-\varepsilon}}\mathbb{E}|(-A)^{-2\alpha}\partial_\xi[:X(t)^2:]|_{H^{-\gamma}}\,\mu_\alpha(dx)\,dt + 2\mathbb{E}\left(\sup_{t\in[0,T]}|Z(t)|_{H^{-\gamma}}\right).$$

From the inequality

$$\int_{H^{-\varepsilon}}|x|_{H^{-\gamma}}\,\mu_\alpha(dx) < +\infty.$$

and the fact that μ_α is invariant, we have

$$\int_0^T\int_{H^{-\varepsilon}}\mathbb{E}|(-A)^{-2\alpha}\partial_\xi[:X(t)^2:]|_{H^{-\gamma}}\,\mu_\alpha(dx)\,dt$$

$$= T\int_{H^{-\varepsilon}}|(-A)^{-2\alpha}\partial_\xi[:x^2:]|_{H^{-\gamma}}\,\mu_\alpha(dx) < +\infty.$$

Finally, it is not difficult to see, using the factorization method (see [12]), that

$$\mathbb{E}\left(\sup_{t\in[0,T]}|Z(t)|_{H^{-\gamma}}\right)<+\infty.$$

In conclusion

$$\mathbb{E}\left(\sup_{t\in[0,T]}|X(t,x)|_{H^{-\gamma}}\right)<+\infty,$$

for μ_α–almost all x and then the global existence for μ_α–almost all x follows. This ends the proof of Theorem 2.9. $\square$

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