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ON THE SECOND-ORDER CORRELATION FUNCTION OF THE CHARACTERISTIC POLYNOMIAL OF A REAL SYMMETRIC WIGNER MATRIX

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Abstract
We consider the asymptotic behaviour of the second-order correlation function of the characteristic polynomial of a real symmetric random matrix. Our main result is that the existing result for a random matrix from the Gaussian Orthogonal Ensemble, obtained by Brézin and Hikami [BH2], essentially continues to hold for a general real symmetric Wigner matrix. To obtain this result, we adapt the approach by Götze and Kösters [GK], who proved the analogous result for the Hermitian case.

1 Introduction

In recent years, the characteristic polynomials of random matrices have found considerable interest. This interest was sparked, at least in part, by the discovery by Keating and Snaith [KS] that the moments of a random matrix from the Circular Unitary Ensemble (CUE) seem to be related to the moments of the Riemann zeta-function along the critical line. Following this observation, several authors have investigated the moments and correlation functions of the characteristic polynomial also for other random matrix ensembles (see e.g. Brézin and Hikami [BH1, BH2], Mehta and Normand [MN], Strahov and Fyodorov [SF], Baik, Deift and Strahov [BDS], Borodin and Strahov [BS], Götze and Kösters [GK]).

One important observation is that the correlation functions of the characteristic polynomials of Hermitian random matrices are related to the "sine kernel" $\sin x/x$. More precisely, this holds both for the unitary-invariant ensembles (Strahov and Fyodorov [SF]) and – at least as far as the second-order correlation function is concerned – for the Hermitian Wigner ensembles (Götze and Kösters [GK]). Thus, the emergence of the sine kernel may be regarded as "universal" for Hermitian random matrices.

In contrast to that, the correlation functions of the characteristic polynomials of real symmetric random matrices lead to different results. This was first observed by Brézin and Hikami [BH2], who investigated the Gaussian Orthogonal Ensemble (GOE) (see e.g. Forrester [Fo] or Mehta [Me] for definitions) and came to the conclusion that the second-order correlation function of the characteristic polynomial is related to the function $\sin x/x^3 - \cos x/x^2$ in this case (see below for a more precise statement of this result). Moreover, Borodin and Strahov [BS] obtained similar results for arbitrary products and ratios of characteristic polynomials of the GOE.

The main purpose of this paper is to generalize the above-mentioned result by Brézin and Hikami [BH2] about the second-order correlation function of the characteristic polynomial of the GOE to arbitrary real symmetric Wigner matrices. Throughout this paper, we consider the following situation: let Q be a probability distribution on the real line such that

$$\int x \; Q(dx) = 0 \,, \quad a := \int x^2 \; Q(dx) = 1 \,, \quad b := \int x^4 \; Q(dx) < \infty \,, \tag{1.1}$$

and let $(X_{ii}/\sqrt{2})_{i\in\mathbb{N}}$ and $(X_{ij})_{i<j,\, i,j\in\mathbb{N}}$ be independent families of independent real random variables with distribution Q on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Also, let $X_{ji} := X_{ij}$ for $i < j$, $i, j \in \mathbb{N}$. Then, for any $N \in \mathbb{N}$, the real symmetric Wigner matrix of size $N \times N$ is given by $X_N = (X_{ij})_{1\le i,j\le N}$, and the second-order correlation function of the characteristic polynomial is given by

$$f(N; \mu, \nu) := \mathbb{E}(\det(X_N - \mu I_N) \cdot \det(X_N - \nu I_N)) \,, \tag{1.2}$$

where μ, ν are real numbers and I_N denotes the identity matrix of size $N \times N$.

In the special case where Q is given by the standard Gaussian distribution, the distribution of the random matrix X_N is the Gaussian Orthogonal Ensemble (GOE). (Note, however, that our scaling is slightly different from that mostly used in the literature (see e.g. Forrester [Fo] or Mehta [Me]), since the variance of the off-diagonal matrix entries is fixed to 1, and not to 1/2.) The result by Brézin and Hikami [BH2] then corresponds to the statement that

$$\begin{aligned} &\lim_{N\to\infty} \sqrt{\frac{2\pi}{N^3}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot f\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right) \\ &\qquad = e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot (2\pi\varrho(\xi))^3 \cdot \frac{1}{2}\left(\frac{\sin(\pi(\mu-\nu))}{(\pi(\mu-\nu))^3} - \frac{\cos(\pi(\mu-\nu))}{(\pi(\mu-\nu))^2}\right) \,, \end{aligned} \tag{1.3}$$

where $\xi \in (-2, +2)$, $\mu, \nu \in \mathbb{R}$, and $\varrho(\xi) := \frac{1}{2\pi}\sqrt{4 - \xi^2}$.

Our main result is the following generalization of (1.3):

Theorem 1.1. *Let Q be a probability distribution on the real line satisfying (1.1), let f be defined as in (1.2), let $\xi \in (-2, +2)$, and let $\mu, \nu \in \mathbb{R}$. Then we have*

$$\begin{aligned} &\lim_{N\to\infty} \sqrt{\frac{2\pi}{N^3}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot f\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right) \\ &\qquad = \exp\left(\tfrac{b-3}{2}\right) \cdot e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot (2\pi\varrho(\xi))^3 \cdot T(\pi(\mu-\nu)) \,, \end{aligned} \tag{1.4}$$

where $\varrho(\xi) := \frac{1}{2\pi}\sqrt{4-\xi^2}$ denotes the density of the semi-circle law,

$$T(x) := \frac{1}{2}\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right) \quad \text{for} \quad x \neq 0 \,, \tag{1.5}$$

and $T(x) := 1/6$ for $x = 0$, by continuous extension.

In particular, we find that the correlation function of the characteristic polynomial asymptotically factorizes into a universal factor involving the function $T(x)$, another universal factor involving the density $\varrho(\xi)$ of the semi-circle law, and a non-universal factor which depends on the underlying distribution Q only via its fourth moment b, or its fourth cumulant $b-3$.
It is interesting to compare the above result with the corresponding result for Hermitian Wigner matrices (Theorem 1.1 in GÖTZE and KÖSTERS [GK]), which states that under similar assumptions as in (1.1) and with similar notation as in (1.2), we have

$$\lim_{N\to\infty} \sqrt{\frac{2\pi}{N}} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot \hat{f}\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right)$$
$$= \exp\left(\hat{b} - \tfrac{3}{4}\right) \cdot e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot (2\pi\varrho(\xi)) \cdot \frac{\sin\pi(\mu-\nu)}{\pi(\mu-\nu)} . \qquad (1.6)$$

($\hat{f}$ and $\hat{b}$ denote the analogues of f and b, respectively.) Obviously, the structure of (1.6) and (1.4) is the same. The most notable difference is given by the fact that the "sine kernel" $\sin x/x$ in (1.6) is replaced with the function $T(x)$ in (1.4). It is noteworthy that both functions are closely related to Bessel functions, as already observed by BRÉZIN and HIKAMI [BH2]. Indeed, it is well-known (see e.g. p. 78 in ERDÉLYI [Er]) that

$$\frac{\sin x}{x} = \sqrt{\pi}\, J_{1/2}(x) \,/\, (2x)^{1/2} \quad \text{and} \quad \frac{1}{2}\left(\frac{\sin x}{x^3} - \frac{\cos x}{x^2}\right) = \sqrt{\pi}\, J_{3/2}(x) \,/\, (2x)^{3/2} \,,$$

where $J_p(x)$ denotes the Bessel function of order p. Thus, if one wishes, one may rewrite both (1.6) and (1.4) in the common form

$$\lim_{N\to\infty} \frac{\sqrt{2\pi}}{N^p} \cdot \frac{1}{N!} \cdot e^{-N\xi^2/2} \cdot f\left(N; \sqrt{N}\xi + \frac{\mu}{\sqrt{N}\varrho(\xi)}, \sqrt{N}\xi + \frac{\nu}{\sqrt{N}\varrho(\xi)}\right)$$
$$= \sqrt{\pi} \cdot \exp\left(b^*\right) \cdot e^{\xi(\mu+\nu)/2\varrho(\xi)} \cdot (2\pi\varrho(\xi))^{2p} \cdot \frac{J_p\big(\pi(\mu-\nu)\big)}{\big(2\pi(\mu-\nu)\big)^p}$$

with $p := \frac{1}{2}$, $\hat{b}^* := \hat{b} - \frac{3}{4}$ in the Hermitian case and $p := \frac{3}{2}$, $b^* := \frac{b-3}{2}$ in the real symmetric case.
Furthermore, in the special case that $\xi = \mu = \nu = 0$, Theorem 1.1 reduces to a result about determinants of random matrices, due to ŽURBENKO [Zu].
To prove Theorem 1.1, we show that the approach for Hermitian Wigner matrices adopted by GÖTZE and KÖSTERS [GK] can easily be adapted to real symmetric Wigner matrices. This stands in contrast to the "orthogonal polynomial approach" typically used in the analysis of the invariant ensembles, for which the transition from the unitary-invariant ensembles (such as the GUE) to the orthogonal-invariant ensembles (such as the GOE) is usually more complicated.
Similarly as in GÖTZE and KÖSTERS [GK], a crucial step in our analysis consists in deriving an explicit expression for the exponential-generating function of the second-order correlation function of the characteristic polynomial (see Lemma 2.3). After that, our main result can be deduced using standard techniques from asymptotic analysis.

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2 Generating Functions

In this section, we determine the exponential generating function of the correlation function of the characteristic polynomial of a real symmetric Wigner matrix. Our results generalize those by Žurbenko [Zu], who considered the special case of determinants.
We make the following conventions: the determinant of the "empty" (i.e., 0×0) matrix is taken to be 1. If A is an $n \times n$ matrix and z is a real or complex number, we set $A - z := A - zI_n$, where I_n denotes the $n \times n$ identity matrix. Also, if A is an $n \times n$ matrix and $i_1, \ldots, i_m$ and $j_1, \ldots, j_m$ are families of pairwise different indices from the set $\{1, \ldots, n\}$, we write $A^{[i_1,\ldots,i_m:j_1,\ldots,j_m]}$ for the $(n-m) \times (n-m)$-matrix obtained from A by removing the rows indexed by $i_1, \ldots, i_m$ and the columns indexed by $j_1, \ldots, j_m$. Thus, for any $n \times n$ matrix $A = (a_{ij})_{1 \le i,j \le n}$ $(n \ge 1)$, we have

$$\det(A) = \sum_{i,j=1}^{n-1} (-1)^{i+j-1} \, a_{i,n} \, a_{n,j} \, \det(A^{[n,i:n,j]}) + a_{n,n} \det(A^{[n:n]}) \,, \tag{2.1}$$

as follows by expanding the determinant about the last row and the last column. (For $n = 1$, note that the big sum vanishes.)
Recall that we write X_N for the real symmetric random matrix $(X_{ij})_{1 \le i,j \le N}$, where the X_{ij} are the random variables from the introduction. We will analyze the function

$$f(N;\mu,\nu) := \mathbb{E}\left(\det(X_N - \mu) \cdot \det(X_N - \nu)\right) \qquad (N \ge 0)\,.$$

To this purpose, we will also need the auxiliary functions

$$\begin{aligned}
f_{11}^A(N;\mu,\nu) &:= \mathbb{E}(\det((X_N-\mu)^{[1:1]}) \cdot \det((X_N-\nu)^{[2:2]})) && (N \ge 2)\,,\\
f_{11}^B(N;\mu,\nu) &:= \mathbb{E}(\det((X_N-\mu)^{[1:2]}) \cdot \det((X_N-\nu)^{[1:2]})) && (N \ge 2)\,,\\
f_{11}^C(N;\mu,\nu) &:= \mathbb{E}(\det((X_N-\mu)^{[1:2]}) \cdot \det((X_N-\nu)^{[2:1]})) && (N \ge 2)\,,\\
f_{10}(N;\mu,\nu) &:= \mathbb{E}(\det(X_{N-1}-\mu) \cdot \det(X_N-\nu)) && (N \ge 1)\,,\\
f_{01}(N;\mu,\nu) &:= \mathbb{E}(\det(X_N-\mu) \cdot \det(X_{N-1}-\nu)) && (N \ge 1)\,.
\end{aligned}$$

Note that the functions f_{11}^B and f_{11}^C actually coincide, but we will not need this. Since μ and ν can be regarded as constants for the purposes of this section, we will only write $f(N)$ instead of $f(N;\mu,\nu)$, etc.
We have the following recursive equations:

Lemma 2.1.

$$\begin{aligned}
f(0) &= 1\,,\\
f(N) &= (2+\mu\nu)\, f(N-1) + b(N-1)\, f(N-2)\\
&\quad + (N-1)(N-2)\, f_{11}^A(N-1)\\
&\quad + (N-1)(N-2)\, f_{11}^B(N-1)\\
&\quad + (N-1)(N-2)\, f_{11}^C(N-1)\\
&\quad + \nu(N-1)\, f_{10}(N-1)\\
&\quad + \mu(N-1)\, f_{01}(N-1) \qquad (N \ge 1)\,,
\end{aligned} \tag{2.2}$$

$$\begin{aligned}
f_{11}^A(N) &= \mu\nu\, f(N-2) + (N-2)\, f(N-3) \\
&\quad + (N-2)(N-3)\, f_{11}^A(N-2) \\
&\quad + \nu(N-2)\, f_{10}(N-2) \\
&\quad + \mu(N-2)\, f_{01}(N-2) && (N \geq 2)\,, \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
f_{11}^B(N) &= f(N-2) + (N-2)\, f(N-3) \\
&\quad + (N-2)(N-3)\, f_{11}^B(N-2) && (N \geq 2)\,, \qquad (2.4)
\end{aligned}$$

$$\begin{aligned}
f_{11}^C(N) &= f(N-2) + (N-2) f(N-3) \\
&\quad + (N-2)(N-3)\, f_{11}^C(N-2) && (N \geq 2)\,, \qquad (2.5)
\end{aligned}$$

$$f_{10}(N) = -(N-1)\, f_{01}(N-1) - \nu\, f(N-1) \qquad (N \geq 1)\,, \qquad (2.6)$$

$$f_{01}(N) = -(N-1)\, f_{10}(N-1) - \mu\, f(N-1) \qquad (N \geq 1)\,. \qquad (2.7)$$

Proof. We give the proof for the recursive equation for $f(N)$ only, the proofs for the remaining recursive equations being very similar.

The result for $f(0)$ is clear. For $N \geq 1$, we expand the determinants of the matrices $(X_N - \mu)$ and $(X_N - \nu)$ as in (2.1) and use the independence of the random variables $X_{ij} = X_{ji}$ $(i \leq j)$, thereby obtaining

$$\begin{aligned}
& f(N) \\
&= \sum_{i,j=1}^{N-1} \sum_{k,l=1}^{N-1} (-1)^{i+j+k+l}\, \mathbb{E}\left(X_{i,N} X_{N,j} X_{k,N} X_{N,l}\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i,j=1}^{N-1} (-1)^{i+j+1}\, \mathbb{E}\left(X_{i,N} X_{N,j}\right) \cdot \mathbb{E}\left(X_{N,N} - \nu\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)\right) \\
&\quad + \sum_{k,l=1}^{N-1} (-1)^{k+l+1}\, \mathbb{E}\left(X_{k,N} X_{N,l}\right) \cdot \mathbb{E}\left(X_{N,N} - \mu\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \mathbb{E}\left((X_{N,N}-\mu)(X_{N,N}-\nu)\right) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)\right)\,.
\end{aligned}$$

Since the random variables $X_{ij} = X_{ji}$ $(i \leq j)$ are independent with $\mathbb{E}(X_{ij}) = 0$ $(i \leq j)$, several of the expectations vanish, and the sum reduces to

$$\begin{aligned}
f(N) &= (\mathbb{E}X_{N,N}^2 + \mu\nu) \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)\right) \\
&\quad + \sum_{i=j=k=l} \mathbb{E}X_{i,N}^4 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i=j\neq k=l} \mathbb{E}X_{i,N}^2 \cdot \mathbb{E}X_{k,N}^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i=k\neq j=l} \mathbb{E}X_{i,N}^2 \cdot \mathbb{E}X_{j,N}^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) \\
&\quad + \sum_{i=l\neq j=k} \mathbb{E}X_{i,N}^2 \cdot \mathbb{E}X_{k,N}^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)^{[k:l]}\right)
\end{aligned}$$

$$+\ \nu \sum_{i=j} \mathbb{E}X_{i,N}^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu)^{[i:j]} \cdot \det(X_{N-1}-\nu)\right)$$
$$+\ \mu \sum_{k=l} \mathbb{E}X_{k,N}^2 \cdot \mathbb{E}\left(\det(X_{N-1}-\mu) \cdot \det(X_{N-1}-\nu)^{[k:l]}\right) .$$

From this (2.2) follows by noting that $\mathbb{E}X_{N,N}^2 = 2$, $\mathbb{E}X_{i,N}^2 = 1$, $\mathbb{E}X_{i,N}^4 = b$, and by exploiting obvious symmetries. □

It turns out that the above recursions can be combined into a single recursion involving only the values $f(N)$. Using the abbreviations

$$c(N) := \frac{f(N)}{N!} \qquad (N \geq 0) \qquad \text{and} \qquad s(N) := \sum_{\substack{k=0,\ldots,N \\ k \text{ even}}} c(N-k) \qquad (N \geq 0)\,,$$

we have the following result:

Lemma 2.2. *The values $c(N)$ satisfy the recursive equation*

$$c(0) = 1\,, \tag{2.8}$$
$$\begin{aligned} Nc(N) = {} & 2 \cdot c(N-1) + (N+1) \cdot c(N-2) \\ & +\ \mu\nu \cdot \big(s(N-1) + s(N-3)\big) \\ & -\ (\mu^2+\nu^2) \cdot s(N-2) \\ & +\ (b-3) \cdot \big(c(N-2) - c(N-4)\big) \qquad (N \geq 1)\,, \end{aligned} \tag{2.9}$$

where all terms $c(\,\cdot\,)$ and $s(\,\cdot\,)$ with a negative argument are taken to be zero.

Proof. It follows from Lemma 2.1 that

$$f(N-2) = f_{11}^A(N-1) + f_{11}^B(N-1) + f_{11}^C(N-1) + (b-3)(N-3)\,f(N-4)$$

for all $N \geq 3$. Thus, we can substitute $f_{11}^A(N-1) + f_{11}^B(N-1) + f_{11}^C(N-1)$ on the right-hand side of (2.2) to obtain

$$\begin{aligned} f(N) &= (2+\mu\nu)\,f(N-1) + b(N-1)\,f(N-2) \\ &\quad +\ (N-1)(N-2)\left(f(N-2) - (b-3)(N-3)f(N-4)\right) \\ &\quad +\ \nu(N-1)\,f_{10}(N-1) \\ &\quad +\ \mu(N-1)\,f_{01}(N-1) \\ &= (2+\mu\nu)\,f(N-1) + (N+1)(N-1)\,f(N-2) \\ &\quad +\ \nu(N-1)\,f_{10}(N-1) \\ &\quad +\ \mu(N-1)\,f_{01}(N-1) \\ &\quad +\ (b-3) \cdot \Big((N-1)\,f(N-2) - (N-1)(N-2)(N-3)\,f(N-4)\Big) \end{aligned}$$

for all $N \geq 3$. Dividing by $(N-1)!$, it follows that

$$\begin{aligned} Nc(N) &= (2+\mu\nu) \cdot c(N-1) + (N+1)\,c(N-2) \\ &\quad +\ \nu \cdot f_{10}(N-1)\,/\,(N-2)! \\ &\quad +\ \mu \cdot f_{01}(N-1)\,/\,(N-2)! \\ &\quad +\ (b-3) \cdot \Big(c(N-2) - c(N-4)\Big) \end{aligned}$$

for all $N \geq 3$. (For $N = 3$, note that the second term in the large bracket vanishes.) Since

$$f_{10}(N-1) \,/\, (N-2)! = -\nu s(N-2) + \mu s(N-3)\,,$$
$$f_{01}(N-1) \,/\, (N-2)! = -\mu s(N-2) + \nu s(N-3)\,,$$

for all $N \geq 3$, as follows from (2.6) and (2.7) by a straightforward induction, the assertion for $N \geq 3$ is proved.
The assertion for $N < 3$ follows from Lemma 2.1 by direct calculation:

$$\begin{aligned}
c(0) &= f(0) = 1 \\
1c(1) &= f(1) = (2+\mu\nu)f(0) = (2+\mu\nu)c(0) = 2c(0) + \mu\nu s(0) \\
2c(2) &= f(2) = (2+\mu\nu)f(1) + bf(0) + \nu(-\nu f(0)) + \mu(-\mu f(0)) \\
&= (2+\mu\nu)f(1) + bf(0) - (\mu^2+\nu^2)f(0) \\
&= (2+\mu\nu)c(1) + bc(0) - (\mu^2+\nu^2)c(0) \\
&= 2c(1) + 3c(0) + \mu\nu c(1) - (\mu^2+\nu^2)c(0) + (b-3)c(0) \\
&= 2c(1) + 3c(0) + \mu\nu s(1) - (\mu^2+\nu^2)s(0) + (b-3)c(0)
\end{aligned}$$

□

Using Lemma 2.2, we can determine the exponential generating function of the sequence $(f(N))_{N\geq 0}$:

Lemma 2.3. *The exponential generating function $F(x) := \sum_{N=0}^{\infty} f(N)\, x^N \,/\, N!$ of the sequence $(f(N))_{N\geq 0}$ is given by*

$$F(x) = \frac{\exp\left(\mu\nu \cdot \frac{x}{1-x^2} - \frac{1}{2}(\mu^2+\nu^2)\cdot\frac{x^2}{1-x^2} + b^* x^2\right)}{(1-x)^{5/2}\cdot(1+x)^{1/2}}\,,$$

where $b^* := \frac{1}{2}(b-3)$.

Proof. Starting from Lemma 2.2, it is easy to see that for any μ, ν and any $\delta > 0$, there exists a constant $K(b,\mu,\nu,\delta) > 0$ (depending only on b,μ,ν,δ) such that

$$\left|f(N;\mu,\nu)\right| \leq K(b,\mu,\nu,\delta)\, N!\, (1+\delta)^N$$

for all $N \geq 0$. In particular, it follows that the exponential generating function converges for all $x \in \mathbb{C}$ with $|x| < 1$. (On the other hand, it turns out that there are singularities at $x = \pm 1$.) Thus, multiplying (2.9) by x^{N-1}, summing over N and recalling our convention concerning negative arguments, we have

$$\begin{aligned}
\sum_{N=1}^{\infty} Nc(N)x^{N-1} = \sum_{N=1}^{\infty} 2c(N-1)x^{N-1} &+ \sum_{N=2}^{\infty}(N+1)c(N-2)x^{N-1} \\
&+ \mu\nu\left(\sum_{N=1}^{\infty} s(N-1)x^{N-1} + \sum_{N=3}^{\infty} s(N-3)x^{N-1}\right) \\
&- (\mu^2+\nu^2)\sum_{N=2}^{\infty} s(N-2)x^{N-1} \\
&+ 2b^*\left(\sum_{N=2}^{\infty} c(N-2)x^{N-1} - \sum_{N=4}^{\infty} c(N-4)x^{N-1}\right)\,,
\end{aligned}$$

whence

$$F'(x) = 2F(x) + (3xF(x) + x^2F'(x)) \\ + \mu\nu\frac{1+x^2}{1-x^2}F(x) - (\mu^2+\nu^2)\frac{x}{1-x^2}F(x) + 2b^*\left(xF(x) - x^3F(x)\right) .$$

This leads to the differential equation

$$F'(x) = \left(\frac{2+3x}{1-x^2} + \mu\nu\frac{1+x^2}{(1-x^2)^2} - (\mu^2+\nu^2)\frac{x}{(1-x^2)^2} + 2b^*x\right)F(x)\,,$$

which has the solution

$$F(x) = \frac{F_0}{(1-x)^{5/2}\cdot(1+x)^{1/2}}\exp\left(\mu\nu\frac{x}{1-x^2} - \tfrac{1}{2}(\mu^2+\nu^2)\frac{1}{1-x^2} + b^*x^2\right) .$$

Here, F_0 is a multiplicative constant which must be chosen as $\exp(\frac{1}{2}(\mu^2+\nu^2))$ in order to satisfy (2.8). This completes the proof. □

3 The Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. In doing so, we will proceed similarly to the proof of Theorem 1.1 in Götze and Kösters [GK]. Throughout this section, $T(x)$ will denote the function defined in Theorem 1.1.

We will first establish the following slightly more general result:

Proposition 3.1. *Let Q be a probability distribution on the real line satisfying (1.1), let f be defined as in (1.2), let $(\xi_N)_{N\in\mathbb{N}}$ be a sequence of real numbers such that $\lim_{N\to\infty}\xi_N/\sqrt{N} = \xi$ for some $\xi \in (-2,+2)$, and let $\eta \in \mathbb{C}$. Then we have*

$$\lim_{N\to\infty}\sqrt{\frac{2\pi}{N^3}}\cdot\frac{1}{N!}\cdot\exp(-\xi_N^2/2)\cdot f\left(N;\xi_N+\frac{\eta}{\sqrt{N}},\xi_N-\frac{\eta}{\sqrt{N}}\right) \\ = \exp\left(\tfrac{b-3}{2}\right)\cdot(4-\xi^2)^{3/2}\cdot T\left(\sqrt{4-\xi^2}\cdot\eta\right).$$

It is easy to see that Proposition 3.1 implies Theorem 1.1:

Proof of Theorem 1.1. Taking

$$\xi_N := \sqrt{N}\xi + \frac{\pi(\mu+\nu)}{\sqrt{N}\cdot\sqrt{4-\xi^2}} \qquad \text{and} \qquad \eta := \frac{\pi(\mu-\nu)}{\sqrt{4-\xi^2}}$$

in Proposition 3.1, we have

$$\lim_{N\to\infty}\sqrt{\frac{2\pi}{N^3}}\cdot\frac{1}{N!}\cdot\exp\left(-N\xi^2/2 - 2\pi\xi(\mu+\nu)/2\sqrt{4-\xi^2}\right) \\ \cdot f\left(N;\xi_N+\frac{2\pi\mu}{\sqrt{N}\sqrt{4-\xi^2}},\xi_N+\frac{2\pi\nu}{\sqrt{N}\sqrt{4-\xi^2}}\right) \\ = \exp\left(\tfrac{b-3}{2}\right)\cdot(4-\xi^2)^{3/2}\cdot T\left(\pi(\mu-\nu)\right),$$

from which Theorem 1.1 follows by a simple rearrangement. □

Proof of Proposition 3.1. From the exponential generating function obtained in Lemma 2.3, we have the integral representation

$$\frac{f(N;\mu,\nu)}{N!} = \frac{1}{2\pi i}\int_\gamma \frac{\exp\left(\mu\nu\cdot\frac{z}{1-z^2} - \frac{1}{2}(\mu^2+\nu^2)\cdot\frac{z^2}{1-z^2} + b^* z^2\right)}{(1-z)^{5/2}\cdot(1+z)^{1/2}}\ \frac{dz}{z^{N+1}}\,, \tag{3.1}$$

where $\gamma \equiv \gamma_N$ denotes the counterclockwise circle of radius $R \equiv R_N = 1-1/N$ around the origin. (We will assume that $N \geq 2$ throughout the proof.)
Setting $\mu = \xi_N + \eta/\sqrt{N}$ and $\nu = \xi_N - \eta/\sqrt{N}$ and doing a simple calculation, we obtain

$$\begin{aligned}
&\exp\left(\mu\nu\cdot\frac{z}{1-z^2} - \tfrac{1}{2}(\mu^2+\nu^2)\cdot\frac{z^2}{1-z^2} + b^* z^2\right)\\
&= \exp\left((\xi_N^2-\eta^2/N)\cdot\frac{z}{1-z^2} - (\xi_N^2+\eta^2/N)\cdot\frac{z^2}{1-z^2} + b^* z^2\right)\\
&= \exp\left(\tfrac{1}{2}\xi_N^2+\eta^2/N\right)\cdot\exp\left(-\tfrac{1}{2}\xi_N^2\cdot\frac{1-z}{1+z} - (\eta^2/N)\cdot\frac{1}{1-z} + b^* z^2\right).
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{1}{N!}\cdot f\left(N;\xi_N+\frac{\eta}{\sqrt{N}},\xi_N-\frac{\eta}{\sqrt{N}}\right)\\
&= \exp\left(\tfrac{1}{2}\xi_N^2+\eta^2/N\right)\cdot\frac{1}{2\pi i}\int_\gamma \frac{\exp\left(-\frac{1}{2}\xi_N^2\cdot\frac{1-z}{1+z} - (\eta^2/N)\cdot\frac{1}{1-z} + b^* z^2\right)}{(1-z)^{5/2}\cdot(1+z)^{1/2}}\ \frac{dz}{z^{N+1}}\,.
\end{aligned} \tag{3.2}$$

Putting

$$h(z) := \frac{\exp\left(-\frac{1}{2}\xi_N^2\cdot\frac{1-z}{1+z} - (\eta^2/N)\cdot\frac{1}{1-z} + b^* z^2\right)}{(1-z)^{5/2}\cdot(1+z)^{1/2}}$$

and

$$h_0(z) := \frac{\exp(b^*)}{\sqrt{2}}\cdot\frac{\exp\left(-\frac{1}{4}\xi_N^2\cdot(1-z) - (\eta^2/N)\cdot\frac{1}{1-z}\right)}{(1-z)^{5/2}}\,,$$

we can rewrite the integral in (3.2) as

$$\frac{1}{2\pi i}\int_\gamma h(z)\ \frac{dz}{z^{N+1}} = I_1 + I_2\,,$$

where

$$I_1 := \frac{1}{2\pi i}\int_\gamma h_0(z)\ \frac{dz}{z^{N+1}} \qquad\text{and}\qquad I_2 := \frac{1}{2\pi i}\int_\gamma (h(z)-h_0(z))\ \frac{dz}{z^{N+1}}\,.$$

Note that I_1 and I_2 implicitly depend on N. Clearly, to complete the proof, it suffices to show that

$$\lim_{N\to\infty}\frac{I_1}{N^{3/2}} = \frac{\exp(b^*)}{\sqrt{2\pi}}\cdot(4-\xi^2)^{3/2}\cdot T\big(\sqrt{4-\xi^2}\cdot\eta\big) \tag{3.3}$$

and

$$\lim_{N\to\infty}\frac{I_2}{N^{3/2}} = 0\,. \tag{3.4}$$

We first prove (3.3). To begin with, since $\xi_N \in \mathbb{R}$, we have

$$\Big|\exp\Big(- \tfrac{1}{4}\xi_N^2 \cdot (1-z)\Big)\Big| = \exp\big(- \tfrac{1}{4}\xi_N^2 \cdot \mathrm{Re}\,(1-z)\big) \le 1 \tag{3.5}$$

for any $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) \le 1$. Also, we have

$$\Big|\exp\Big(-(\eta^2/N) \cdot \tfrac{1}{1-z}\Big)\Big| \le \exp\Big((|\eta|^2/N) \cdot \tfrac{1}{|1-z|}\Big) \le \exp(|\eta|^2) \tag{3.6}$$

for any $z \in \mathbb{C}$ with $\mathrm{Re}\,(z) \le 1-(1/N)$. It follows that

$$\begin{aligned}\frac{1}{2\pi i}\int_\gamma \frac{\exp\Big(-\frac{1}{4}\xi_N^2 \cdot (1-z) - (\eta^2/N)\cdot \frac{1}{1-z}\Big)}{(1-z)^{5/2}}\,\frac{dz}{z^{N+1}} \\ = \frac{1}{2\pi i}\int_{R-\mathbf{i}\,\infty}^{R+\mathbf{i}\,\infty} \frac{\exp\Big(-\frac{1}{4}\xi_N^2 \cdot (1-z) - (\eta^2/N)\cdot \frac{1}{1-z}\Big)}{(1-z)^{5/2}}\,\frac{dz}{z^{N+1}}\,.\end{aligned} \tag{3.7}$$

Indeed, for any $R' > 1$, we can replace the contour γ by the contour δ consisting of the line segment between the points $R - \mathbf{i}\sqrt{(R')^2 - R^2}$ and $R + \mathbf{i}\sqrt{(R')^2 - R^2}$, and the arc of radius R' around the origin to the left of this line segment. Using (3.5) and (3.6), it is easy to see then that the integral along the arc is bounded by

$$\frac{1}{2\pi} \cdot 2\pi R' \cdot \frac{\exp(|\eta|^2)}{(R'-1)^{5/2}} \cdot \frac{1}{(R')^{N+1}}\,,$$

which tends to zero as $R' \to \infty$. This proves (3.7).
Next, performing a change of variables, we obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{R-\mathbf{i}\,\infty}^{R+\mathbf{i}\,\infty} \frac{\exp\Big(-\frac{1}{4}\xi_N^2 \cdot (1-z) - (\eta^2/N)\cdot \frac{1}{1-z}\Big)}{(1-z)^{5/2}}\,\frac{dz}{z^{N+1}} \\ = N^{3/2} \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp\Big(-\frac{1}{4}(\xi_N^2/N) \cdot (1-iu) - \eta^2 \cdot \frac{1}{1-iu}\Big)}{(1-iu)^{5/2}}\,\frac{du}{(1-\frac{1-iu}{N})^{N+1}}\,.\end{aligned} \tag{3.8}$$

Since $\lim_{N\to\infty} \xi_N/\sqrt{N} = \xi$, an application of the dominated convergence theorem yields

$$\begin{aligned}\lim_{N\to\infty} \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp\Big(-\frac{1}{4}(\xi_N^2/N) \cdot (1-iu) - \eta^2 \cdot \frac{1}{1-iu}\Big)}{(1-iu)^{5/2}}\,\frac{du}{(1-\frac{1-iu}{N})^{N+1}} \\ = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp\Big((1-\frac{1}{4}\xi^2) \cdot (1-iu) - \eta^2 \cdot \frac{1}{1-iu}\Big)}{(1-iu)^{5/2}}\,du\,.\end{aligned} \tag{3.9}$$

Another application of the dominated convergence theorem shows that the right-hand side in (3.9) is equal to

$$\sum_{l=0}^{\infty} \frac{(-1)^l\,\eta^{2l}}{l!} \cdot \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\exp\big((1-\frac{1}{4}\xi^2)\cdot(1-iu)\big)}{(1-iu)^{l+5/2}}\,du\,,$$

which in turn can be rewritten as

$$\sum_{l=0}^{\infty} \frac{(-1)^l \, \eta^{2l}}{l!} \cdot \frac{(1-\frac{1}{4}\xi^2)^{l+3/2}}{\Gamma(l+5/2)} = \sum_{l=0}^{\infty} \frac{(-1)^l \, \eta^{2l}}{l!} \cdot \frac{1}{\sqrt{\pi}} \cdot \frac{(l+1)!}{(2l+3)!} \cdot (4-\xi^2)^{l+3/2} \,,$$

as follows from the Laplace inversion formula (see e.g. Chapter 24 in Doetsch [Do]) and the functional equation of the Gamma function. Hence, putting it all together, we obtain

$$\lim_{N\to\infty} \frac{I_1}{N^{3/2}} = \frac{\exp(b^*)}{\sqrt{2\pi}} \cdot (4-\xi^2)^{3/2} \cdot \sum_{l=0}^{\infty} \frac{(-1)^l (l+1) \big(\sqrt{4-\xi^2} \cdot \eta\big)^{2l}}{(2l+3)!} \,.$$

Since

$$\sum_{l=0}^{\infty} \frac{(-1)^l \, (l+1) \, z^{2l}}{(2l+3)!} = \frac{1}{2} \left(\frac{\sin z}{z^3} - \frac{\cos z}{z^2} \right) = T(z) \,,$$

this proves (3.3).

To prove (3.4), we will show that $\limsup_{N\to\infty} \left| I_2 / N^{3/2} \right| \le \delta$ for any $\delta > 0$. Thus, fix $\delta > 0$, and let $c = c(b^*, \eta, \delta) > 0$ denote a constant depending only on b^*, η, δ which will be chosen later. Then we have

$$\begin{aligned} |I_2| \le \frac{1}{2\pi} \int_{c/N}^{2\pi - c/N} \left| h_0(Re^{it}) \right| \, \frac{dt}{R^N} + \frac{1}{2\pi} \int_{c/N}^{2\pi - c/N} \left| h(Re^{it}) \right| \, \frac{dt}{R^N} \\ + \frac{1}{2\pi} \int_{-c/N}^{+c/N} \left| h(Re^{it}) - h_0(Re^{it}) \right| \, \frac{dt}{R^N} \,. \end{aligned}$$

Denote the summands on the right-hand side by I_{21}, I_{22} and I_{23}, respectively.

Let $\varepsilon^2 > 0$ denote a positive constant such that $\cos t \le 1 - \varepsilon^2 t^2$ for $-\pi \le t \le +\pi$. Then, for any $\alpha > 0$ and any $-\pi \le t_1 < t_2 \le +\pi$, we have the estimate

$$\begin{aligned} \int_{t_1}^{t_2} \frac{1}{|1 - Re^{it}|^\alpha} \, dt &= \int_{t_1}^{t_2} \frac{1}{(1 + R^2 - 2R\cos t)^{\alpha/2}} \, dt \le \int_{t_1}^{t_2} \frac{1}{((1-R)^2 + \varepsilon^2 t^2)^{\alpha/2}} \, dt \\ &= N^\alpha \int_{t_1}^{t_2} \frac{1}{(1 + N^2 \varepsilon^2 t^2)^{\alpha/2}} \, dt = K N^{\alpha - 1} \int_{N\varepsilon t_1}^{N\varepsilon t_2} \frac{1}{(1+u^2)^{\alpha/2}} \, du \,, \end{aligned} \tag{3.10}$$

where K denotes some absolute constant. Let us convene that this constant K may change from occurrence to occurrence in the subsequent calculations. Then, using the estimates (3.5), (3.6),

$$\left| \exp\left(b^* z^2 \right) \right| \le \exp\left(|b^*| |z|^2 \right) \le \exp(|b^*|) \tag{3.11}$$

(for $z \in \mathbb{C}$ with $|z| < 1$) as well as (3.10), we obtain

$$\begin{aligned} |I_{21}| &\le \frac{1}{2\pi} \int_{c/N}^{2\pi - c/N} \frac{\exp(|\eta|^2 + |b^*|)}{|1 - Re^{it}|^{5/2}} \, \frac{dt}{R^N} \\ &\le K \exp(|\eta|^2 + |b^*|) \cdot \int_{c/N}^{\pi} \frac{1}{|1 - Re^{it}|^{5/2}} \, dt \\ &\le K \exp(|\eta|^2 + |b^*|) \cdot N^{3/2} \int_{c\varepsilon}^{\infty} \frac{1}{u^{5/2}} \, du \\ &\le K \exp(|\eta|^2 + |b^*|) \cdot N^{3/2} \, c^{-3/2} \,. \end{aligned}$$

Similarly, using the estimate

$$\left|\exp\left(-\tfrac{1}{2}\xi_N^2\cdot\tfrac{1-z}{1+z}\right)\right| = \exp\left(-\tfrac{1}{2}\xi_N^2\cdot\mathrm{Re}\left(\tfrac{1-z}{1+z}\right)\right) = \exp\left(-\tfrac{1}{2}\xi_N^2\cdot\tfrac{1-|z|^2}{|1+z|^2}\right) \le 1 \tag{3.12}$$

(for $z\in\mathbb{C}$ with $|z|<1$) instead of (3.5), we obtain

$$\begin{aligned}
|I_{22}| &\le \frac{1}{2\pi}\int_{c/N}^{2\pi-c/N} \frac{\exp(|\eta|^2+|b^*|)}{|1-Re^{it}|^{5/2}\cdot|1+Re^{it}|^{1/2}}\ \frac{dt}{R^N}\\
&\le K\exp(|\eta|^2+|b^*|)\cdot\left(\int_{c/N}^{\pi/2}\frac{1}{|1-Re^{it}|^{5/2}}\ dt+\int_0^{\pi/2}\frac{1}{|1-Re^{it}|^{1/2}}\ dt\right)\\
&\le K\exp(|\eta|^2+|b^*|)\cdot\left(N^{3/2}\int_{c\varepsilon}^{\infty}\frac{1}{u^{5/2}}\ du+N^{-1/2}\left(1+\int_1^{N\varepsilon\pi/2}\frac{1}{u^{1/2}}\ du\right)\right)\\
&\le K\exp(|\eta|^2+|b^*|)\cdot\left(N^{3/2}\,c^{-3/2}+1\right).
\end{aligned}$$

Hence, if we pick $c=c(b^*,\eta,\delta)>0$ sufficiently large, we clearly have

$$\limsup_{N\to\infty}\left|I_{21}/N^{3/2}\right|\le\delta/2 \qquad\text{and}\qquad \limsup_{N\to\infty}\left|I_{22}/N^{3/2}\right|\le\delta/2\,.$$

For the remaining integral I_{23}, note that

$$\frac{I_{23}}{N^{3/2}}=\frac{1}{2\pi}\int_{-c}^{+c}\left|\frac{h(Re^{iu/N})-h_0(Re^{iu/N})}{N^{5/2}}\right|\ \frac{du}{R^N}\,.$$

Now, it is easy to check that for each $u\in[-c,+c]$, we have

$$1+Re^{iu/N}=2+\mathcal{O}_c(1/N) \qquad\text{and}\qquad N(1-Re^{iu/N})=1-iu+\mathcal{O}_c(1/N)\,,$$

the constants implicit in the $\mathcal{O}_c$-bounds depending only on c. Thus, since $\lim_{N\to\infty}\xi_N/\sqrt{N}=\xi$, we have

$$\lim_{N\to\infty}\frac{h(Re^{iu/N})-h_0(Re^{iu/N})}{N^{5/2}}=0$$

for any $u\in[-c,+c]$. Moreover, using (3.5), (3.6), (3.11) and (3.12), we also have

$$\left|\frac{h(Re^{iu/N})-h_0(Re^{iu/N})}{N^{5/2}}\right|\le 2\cdot\frac{\exp(|\eta|^2+|b^*|)}{\sqrt{2}\cdot|1-iu|^{5/2}}\cdot\big(1+\mathcal{O}_c(1/N)\big)$$

for any $u\in[-c,+c]$. It therefore follows from the dominated convergence theorem that $\lim_{N\to\infty}\left|I_{23}/N^{3/2}\right|=0$.

This completes the proof of (3.4), and thus of Proposition 3.1. $\square$

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