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SOME CHANGES OF PROBABILITIES RELATED TO A GEOMETRIC BROWNIAN MOTION VERSION OF PITMAN'S $2M - X$ THEOREM

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Abstract

Rogers-Pitman have shown that the sum of the absolute value of $B^{(\mu)}$, Brownian motion with constant drift μ, and its local time $L^{(\mu)}$ is a diffusion $R^{(\mu)}$. We exploit the intertwining relation between $B^{(\mu)}$ and $R^{(\mu)}$ to show that the same addition operation performed on a one-parameter family of diffusions $\{X^{(\alpha,\mu)}\}_{\alpha\in\mathbf{R}_+}$ yields the same diffusion $R^{(\mu)}$. Recently we obtained an exponential analogue of the Rogers-Pitman result. Here we exploit again the corresponding intertwining relationship to yield a one-parameter family extension of our result.

1 Introduction

In our recent paper [9], we have obtained some interesting examples of a diffusion process $X = \{X_t, t \geqq 0\}$ on $\mathbf{R}$ and an additive functional $\{A_t, t \geqq 0\}$ of X such that there exists a particular function $\theta : \mathbf{R} \times \mathbf{R}_+ \to \mathbf{R}_+$ for which $\Theta_t = \theta(X_t, A_t)$ gives another diffusion process. The difficulty (or the interest) of the situation is that $\Theta = \{\Theta_t, t \geqq 0\}$ enjoys the Markov property with respect to its natural filtration $\{\mathcal{Z}_t, t \geqq 0\}$ and not with respect to the larger filtration, say $\mathcal{X} = \{\mathcal{X}_t, t \geqq 0\}$, of the original diffusion X.
In fact, to get more precisely into our framework, there exists a Markov kernel $\mathbb{K}$ such that

$$E[f(X_t)|\mathcal{Z}_t] = (\mathbb{K}f)(\Theta_t)$$

holds for every bounded Borel function $f : \mathbf{R} \to \mathbf{R}_+$ and the Markov property of Θ is inherited from that of X, via $\mathbb{K}$. Such situations have been described and studied by Rogers-Pitman [13]; see Kurtz [7] for a more recent discussion.
The purpose of this article is to examine, through concrete examples, how these properties are transformed after a change of probabilities of the form

$$dQ|_{\mathcal{X}_t} = D_t \cdot dP|_{\mathcal{X}_t} \tag{1.1}$$

for some functional $\{D_t\}$, which we shall assume to be of the form $D_t = \phi(X_t, A_t, t)$.
In Section 2 we show that a number of different diffusion processes $\{X_t, t \geqq 0\}$ have the property that $\{|X_t| + L_t(X), t \geqq 0\}$ is distributed as $\{|B_t^{(\mu)}| + L_t(B^{(\mu)}), t \geqq 0\}$, where $B^{(\mu)} = \{B_t^{(\mu)}, t \geqq 0\}$ denotes the Brownian motion with constant drift $\mu \in \mathbf{R}$ and $\{L_t(Y), t \geqq 0\}$ is the local time of a diffusion process $Y = \{Y_t, t \geqq 0\}$ at 0.
In Section 3 our choice for $\{D_t\}$ turns out to yield only "strict" local martingales, i.e., local martingales which are not martingales (see, e.g., Elworthy-Li-Yor [1], [2] for detailed study of such processes). Therefore the equation (1.1) has to be considered carefully and yields "explosive" real-valued diffusions (see Feller [4], McKean [10]).

2 Kennedy's Martingales

Let $B = \{B_t, t \geqq 0\}$ be a standard Brownian motion and $\{L_t, t \geqq 0\}$ be its local time at 0. Then it is known (cf. Kennedy [5], Revuz-Yor [12], Exercise (4.9), p.264) that

$$D_t^{\alpha,\mu} = (\cosh(\mu B_t) + \frac{\alpha}{\mu}\sinh(\mu|B_t|))\exp(-\alpha L_t - \frac{1}{2}\mu^2 t), \quad t \geqq 0,$$

defines a martingale for every $\alpha, \mu > 0$.
We set

$$R_t = |B_t| + L_t,$$

which is a three-dimensional Bessel process by virtue of Pitman's celebrated theorem. Then, since the conditional distribution of $|B_t|$ given $\mathcal{R}_t = \sigma\{R_s; s \leqq t\}$ is the uniform distribution on $[0, R_t]$, it is easy to obtain the following.

Proposition 2.1 *For every $\alpha, \mu > 0$ and $t > 0$, it holds that*

$$E[D_t^{\alpha,\mu}|\mathcal{R}_t] = \frac{\sinh(\mu R_t)}{\mu R_t}\exp(-\mu^2 t/2). \tag{2.1}$$

We may rewrite (2.1) in the following manner by using Girsanov's theorem.

Proposition 2.2 *Let $\{\gamma_t, t \geqq 0\}$ be a standard Brownian motion with $\gamma_0 = 0$ and $B^{\alpha,\mu} = \{B_t^{\alpha,\mu}, t \geqq 0\}$ be the solution of the stochastic differential equation*

$$dX_t = d\gamma_t + (\log\varphi_{\alpha,\mu})'(X_t)dt, \qquad X_0 = 0, \tag{2.2}$$

where $\varphi_{\alpha,\mu}(x) = \cosh(\mu x) + \mu^{-1}\alpha\sinh(\mu|x|)$. Then one has

$$\{|B_t^{\alpha,\mu}| + L_t^{\alpha,\mu}, t \geqq 0\} \overset{(\mathrm{law})}{=} \{\rho_t^{(\mu)}, t \geqq 0\}, \tag{2.3}$$

where $\{L_t^{\alpha,\mu}, t \geqq 0\}$ *is the local time of* $B^{\alpha,\mu}$ *at* 0 *and* $\{\rho_t^{(\mu)}, t \geqq 0\}$ *is the* $\mathbf{R}_+$*-valued diffusion process with infinitesimal generator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \mu \coth(\mu x)\frac{d}{dx}.$$

Remark 2.1 *The case* $\alpha = 0$ *is precisely the extension of Pitman's theorem by Rogers-Pitman [13]. Here we simply remark that* α *does not appear on the right hand side of* (2.3).

Proof. On one hand, using Girsanov's theorem, the law $Q^{\alpha,\mu}$ of $B^{\alpha,\mu}$, the solution of (2.2), satisfies

$$dQ^{\alpha,\mu}|_{\mathcal{X}_t} = D_t^{\alpha,\mu} \cdot dP|_{\mathcal{X}_t},$$

where $\mathcal{X}_t = \sigma\{X_s; s \leqq t\}, t > 0$, on the canonical space and P denotes the Wiener measure. On the other hand, if we set $R_t = |X_t| + L_t(X), t \geqq 0$, then, thanks to (2.1) and Girsanov's theorem again, $\{R_t, t \geqq 0\}$ satisfies, under $Q^{\alpha,\mu}$, the equation

$$dR_t = d\gamma_t + \mu \coth(\mu R_t)dt,$$

where $\{\gamma_t, t \geqq 0\}$ denotes a one-dimensional Brownian motion. $\square$

3 Local martingales Related to Geometric Brownian Motion

In this section we discuss some computations analogous to those in the previous section, but now we are concerned with geometric Brownian motion and related stochastic processes. Let $B^{(\mu)} = \{B_t + \mu t, t \geqq 0\}$ be a Brownian motion starting from 0 with constant drift $\mu > 0$, defined on a probability space $(\Omega, \mathcal{B}, P)$, and $\{\mathcal{B}_t^{(\mu)}, t \geqq 0\}$ be its natural filtration (which, obviously, does not depend on μ). We set

$$e_t^{(\mu)} = \exp(B_t^{(\mu)}) \qquad \text{and} \qquad A_t^{(\mu)} = \int_0^t (e_s^{(\mu)})^2 ds.$$

Our main objects of study in [9] are the stochastic processes given by

$$Z_t^{(\mu)} = (e_t^{(\mu)})^{-1} A_t^{(\mu)} \qquad \text{and} \qquad \xi_t^{(\mu)} = (e_t^{(\mu)})^{-2} A_t^{(\mu)},$$

which turn out to be, in fact, diffusion processes [with respect to their own filtrations, respectively]. It should be remarked that $\sigma\{\xi_s^{(\mu)}; s \leqq t\}$ coincides with $\mathcal{B}_t^{(\mu)}$ and that $\mathcal{Z}_t^{(\mu)} \equiv \{Z_s^{(\mu)}; s \leqq t\}$ is strictly contained in $\mathcal{B}_t^{(\mu)}$. Here are the main facts, drawn from [9], about these diffusions.

Proposition 3.1 (i) $\{\xi_t^{(\mu)}, t \geqq 0\}$ *is a diffusion process with respect to its natural filtration* $\{\mathcal{B}_t^{(\mu)}, t \geqq 0\}$ *and it admits the infinitesimal generator*

$$2x^2\frac{d^2}{dx^2} + (2(1-\mu)x + 1)\frac{d}{dx}.$$

(ii) $\{Z_t^{(\mu)}, t \geqq 0\}$ *is a (transient) diffusion process with respect to its natural filtration* $\{\mathcal{Z}_t^{(\mu)}, t \geqq 0\}$ *with infinitesimal generator*

$$\frac{1}{2}z^2\frac{d^2}{dz^2} + \{(\frac{1}{2}-\mu)z + \left(\frac{K_{1+\mu}}{K_\mu}\right)\left(\frac{1}{z}\right)\}\frac{d}{dz}$$

and a scale function is $s_\mu(z) = -(I_\mu/K_\mu)(z^{-1})$.
(iii) *For any* $t>0$, $\mathcal{B}_t^{(\mu)} = \mathcal{Z}_t^{(\mu)} \vee \sigma(e_t^{(\mu)})$ *and the conditional law of* $e_t^{(\mu)}$ *given* $\mathcal{Z}_t^{(\mu)}$ *is expressed by*

$$P(e_t^{(\mu)} \in dx | \mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)} = z) = \frac{x^{\mu-1}}{2K_\mu(1/z)}\exp(-\frac{1}{2z}(x+\frac{1}{x}))dx, \tag{3.1}$$

where K_μ *is the usual Macdonald (modified Bessel) function.*

Now we set $\varphi_\mu(x) = x^{-\mu}I_\mu(x)$ for a modified Bessel function I_μ and consider the stochastic process

$$\Delta_t^{\mu,\delta} = \frac{\varphi_\mu(\delta e_t^{(\mu)})}{\varphi_\mu(\delta)}\exp(-\frac{\delta^2}{2}A_t^{(\mu)}).$$

Then, by using Itô's formula and the fact that I_μ solves the differential equation

$$u''(x) + \frac{1}{x}u'(x) - (1+\frac{\mu^2}{x^2})u(x) = 0,$$

it is easy to show that $\Delta^{\mu,\delta} = \{\Delta_t^{\mu,\delta}, t \geqq 0\}$ is a $(\mathcal{B}_t^{(\mu)})$-local martingale.
Another proof of the local martingale property of $\Delta^{\mu,\delta}$ consists in using Lamperti's representation (see, e.g., [12], Exercise (1.28), p.452)

$$e_t^{(\mu)} = R^{(\mu)}(A_t^{(\mu)}),$$

where $R^{(\mu)} = \{R^{(\mu)}(u), u \geqq 0\}$ denotes a Bessel process with index μ, and the well-known fact that the stochastic process $\{\varphi_\mu(\delta R_1^{(\mu)}(u))\exp(-\delta^2 u/2), u \geqq 0\}$ is a martingale with respect to the natural filtration of $R^{(\mu)}$ (see, e.g., Kent [6]); the corresponding result for $\Delta^{\mu,\delta}$ follows by time change.
However, the following proposition shows how different the situation is from that in the previous section.

Proposition 3.2 $\Delta^{\mu,\delta}$ *is a strict* $(\mathcal{B}_t^{(\mu)})$*-local martingale, that is, it is a local martingale, but not a martingale. More precisely, its "martingale default" may be computed from the formula*

$$E[\Delta_t^{\mu,\delta} | \mathcal{Z}_t^{(\mu)}] = P(L_{1/\delta}^{(\mu)} \geqq t | \mathcal{Z}_t^{(\mu)}), \tag{3.2}$$

where $L_y^{(\mu)} = \sup\{t \geqq 0; Z_t^{(\mu)} = y\}$. *Moreover, one has*

$$P(L_y^{(\mu)} \geqq t | \mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)} = z) = \min\{\frac{s_\mu(z)}{s_\mu(y)}, 1\}. \tag{3.3}$$

Remark 3.1 *A general study of "strict" local martingales and their martingale defaults has been undertaken by Elworthy-Li-Yor [1], [2]; see also Takaoka [14].*

Proof. Formula (3.3) is deduced from the conditional law (3.1); indeed, one has

$$\begin{aligned}
&E[\Delta_t^{\mu,\delta}|\mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)}=z] \\
&= \frac{1}{\varphi_\mu(\delta)} E[\varphi_\mu(\delta e_t^{(\mu)}) \exp(-\frac{\delta^2}{2} e_t^{(\mu)} Z_t^{(\mu)}) | \mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)}=z] \\
&= \frac{1}{\varphi_\mu(\delta)} \int_0^\infty \varphi_\mu(\delta x) \exp(-\frac{\delta^2}{2} xz) \frac{1}{2K_\mu(1/z)} x^{\mu-1} \exp(-\frac{1}{2z}(x+\frac{1}{x})) dx \\
&= \frac{1}{2I_\mu(\delta)K_\mu(1/z)} \int_0^\infty I_\mu(\frac{1}{z\delta} v) \exp(-\frac{1}{2}((1+\frac{1}{\delta^2 z^2})v + \frac{\delta^2}{v})) \frac{dv}{v}.
\end{aligned}$$

We now recall the integral representation of the product of the modified Bessel functions,

$$I_\mu(\delta a) K_\mu(\delta b) = \frac{1}{2} \int_0^\infty I_\mu(abv) \exp(-\frac{1}{2}((a^2+b^2)v + \frac{\delta^2}{v})) \frac{dv}{v}$$

for $0 \leqq a \leqq b$ (cf. [3], p.284 (56)). Then, noting that $\psi_\mu(x) = (I_\mu/K_\mu)(x)$ is an increasing function, we obtain

$$E[\Delta_t^{\mu,\delta}|\mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)}=z] = \min\{\frac{\psi_\mu(z^{-1})}{\psi_\mu(\delta)}, 1\}, \tag{3.4}$$

which implies that $\Delta^{\mu,\delta}$ is a strict local martingale.
Formula (3.2) is a particular case of the computation of the supermartingale $P(L_y^{(\mu)} \geqq t | \mathcal{Z}_t^{(\mu)})$ attached to the last passage time $L_y^{(\mu)}$ for a transient diffusion, here $\{Z_t^{(\mu)}\}$ (see, e.g., Pitman-Yor [11], Section 6 and also Revuz-Yor [12], Exercise (4.16), p.321). □

Despite Proposition 3.2, we wish to apply Girsanov's theorem with respect to P and the (strict) local martingale $\Delta^{\mu,\delta}$. This type of extension of Girsanov's theorem is dealt with in McKean [10], pp.63–64, who considers there explosive Itô stochastic differential equations, and our situation fits into his framework perfectly well. See also Yoeurp [15].

Theorem 3.3 *Let $\mu \geqq 0, \delta > 0$, and $\{\beta_t, t \geqq 0\}$ be a standard Brownian motion with $\beta_0 = 0$.*
(i) *The solution of the equation*

$$X_t = \beta_t + \mu t + \int_0^t \left(\frac{\varphi'_\mu}{\varphi_\mu}\right)(\delta \exp(X_s)) \delta \exp(X_s) ds \tag{3.5}$$

is explosive a.s., that is, one can construct a process $\{X_t, t < \mathbf{e}\}$ which solves (3.5); moreover one has $P(\mathbf{e} < \infty) = 1$.
(ii) *Let $E^{\mu,\delta}$ denote the expectation with respect to the law $W^{\mu,\delta}$ of $\{X_t, t < \mathbf{e}\}$. Then one has*

$$E^{\mu,\delta}[F(X_s, s \leqq t) 1_{\{t<\mathbf{e}\}}] = E[F(B_s^{(\mu)}, s \leqq t) \Delta_t^{\mu,\delta}] \tag{3.6}$$

for every positive Borel functional F defined on $C([0,t]; \mathbf{R})$, where E on the right hand side of (3.6) denotes the expectation with respect to the Wiener measure P.
(iii) *Define the stochastic process $\{\eta_t, t < \mathbf{e}\}$ by $\eta_t = \exp(X_t)$. Then it satisfies the equation*

$$\eta_t = 1 + \int_0^t \eta_s d\beta_s + \int_0^t (\frac{1}{2} + \mu) \eta_s ds + \int_0^t \left(\frac{\varphi'_\mu}{\varphi_\mu}\right)(\delta \eta_s) \delta (\eta_s)^2 ds, \qquad t < \mathbf{e}. \tag{3.7}$$

Moreover, $\{\eta_t, t < \mathbf{e}\}$ *is defined implicitly in terms of an upward Bessel process* $\{R_1^{(\mu,\delta\uparrow)}(u), u \geqq 0\}$ *starting from* 1 *as follows :*

$$\eta_t = R_1^{(\mu,\delta\uparrow)}\Big(\int_0^t (\eta_s)^2 ds\Big), \qquad t < \mathbf{e} \equiv \int_0^\infty \frac{ds}{(R_1^{(\mu,\delta\uparrow)}(s))^2}. \tag{3.8}$$

Remark 3.2 *For generalized (upward and downward) Bessel processes, we refer to Pitman-Yor [11] and Watanabe [17].*

Proof. The assertions of (i) and (ii) follows from the general discussion of explosive diffusions on $\mathbf{R}$ (see, e.g., McKean [10], pp.66–67, and Feller [4]). In particular, we can check the Feller test for explosion as follows. Letting $b(x)$ be the drift coefficient given by

$$b(x) = \mu + \left(\frac{\varphi_\mu'}{\varphi_\mu}\right)(\delta \exp(x)) \cdot \delta \exp(x),$$

we see, after some elementary computations, that a scale function $s(x)$ is given by

$$s(x) = \int_0^\infty \exp(-2\int_0^\xi b(\eta)d\eta)d\xi = I_\mu(\delta)^2 \int_0^x I_\mu(\delta e^\xi)^{-2} d\xi.$$

The speed measure is then given by

$$m(dx) = 2I_\mu(\delta)^{-2} I_\mu(\delta \exp(x))^2 dx.$$

Noting that

$$I_\mu(z) = \frac{1}{\sqrt{2\pi z}} e^z \cdot (1 + o(1)) \qquad \text{as} \qquad z \to \infty \tag{3.9}$$

and

$$I_\mu(z) = \frac{z^\mu}{2^\mu \Gamma(1+\mu)}(1 + o(1)) \qquad \text{as} \qquad z \to 0$$

(cf. Lebedev [8], p.136), it is easy to show that $s(\infty) < \infty$ and $s(-\infty) = -\infty$. Moreover we have

$$\begin{aligned} v(x) \equiv \int_0^x (s(x) - s(y))m(dy) &= \int_\delta^{\delta\exp(x)} \frac{d\xi}{\xi I_\mu(\xi)^2} \int_\delta^\xi I_\mu(\eta)^2 \frac{d\eta}{\eta} \\ &= \int_\delta^{\delta\exp(x)} I_\mu(\eta)^2 \frac{d\eta}{\eta} \int_\eta^{\delta\exp(x)} \frac{d\xi}{\xi I_\mu(\xi)^2}. \end{aligned}$$

Therefore, using (3.9) again, we obtain $v(\infty) < \infty$ and, consequently, $P(e < \infty) = 1$.
Equation (3.7) follows from (3.5), using the Itô formula. The implicit representation (3.8) follows from (3.7) by performing the time change

$$t \mapsto A_t \equiv \int_0^t (\eta_s)^2 ds, \qquad t < \mathbf{e}.$$

To develop the passage formulae from η to $R_1^{(\mu,\delta\uparrow)}$, we note that, if $\{\sigma_u, u \geqq 0\}$ is the inverse of $A_t, t < \mathbf{e}$, then $(\eta_{\sigma_u})^2 d\sigma_u = du$. This, combined with (3.8), yields

$$\sigma_u = \int_0^u \frac{ds}{(R_1^{(\mu,\delta\uparrow)}(s))^2}$$

and one can define A_t so long as

$$t < \sigma_\infty \equiv \int_0^\infty (R_1^{(\mu,\delta\uparrow)}(s))^{-2} ds,$$

which is the explosion time $\mathbf{e}$. $\square$

We now come back to our original task, which is to extend further our exponential version of Pitman's theorem [9], that is, precisely to study the law of $\{Z_t^{(\mu)}, t < \mathbf{e}\}$ under $W^{\mu,\delta}$.

Theorem 3.4 *Keeping the notations in Theorem 3.3, we define*

$$Z_t = \exp(-X_t) \int_0^t \exp(2X_s) ds \equiv \frac{1}{\eta_t} \int_0^t (\eta_s)^2 ds, \qquad t < \mathbf{e}.$$

Then one has the equality in law

$$\{(Z_t, t < \mathbf{e}), W^{\mu,\delta}\} \overset{\text{(law)}}{=} \{(Z_t^{(\mu)}, t < L_{1/\delta}^{(\mu)}), P\}.$$

Consequently, in the filtration $\{\mathcal{Z}_t^{(\mu)}\}$, enlarged so that $L_{1/\delta}^{(\mu)}$ becomes a stopping time, the stochastic process $\{Z_t^{(\mu)}, t \leqq L_{1/\delta}^{(\mu)}\}$ satisfies the equation

$$\begin{aligned} Z_t = &\int_0^t Z_s d\gamma_s + (\frac{1}{2} - \mu) \int_0^t Z_s ds + \int_0^t \left(\frac{K_{1+\mu}}{K_\mu}\right)\left(\frac{1}{Z_s}\right) ds \\ &- \int_0^t \frac{1}{(I_\mu K_\mu)(1/Z_s)} 1_{\{Z_s < 1/\delta\}} ds. \end{aligned}$$

Proof. We provide two proofs. As a first proof, we project $\Delta_t^{\mu,\delta}$ on $\mathcal{Z}_t^{(\mu)}$ under P and remark that the right hand side of (3.6), where $F(B_s^{(\mu)}, s \leqq t)$ has been replaced by $F(Z_s^{(\mu)}, s \leqq t)$, thus obtained coincides with

$$E[F(Z_s^{(\mu)}, s \leqq t) 1_{\{L_{1/\delta}^{(\mu)} > t\}}].$$

Then, applying the progressive enlargement formula (see, e.g., Yor [16], Chapter 12), we obtain the assertion of the theorem.

Our second proof is deduced from (3.8), using time inversion. Indeed, there exists an upward Bessel process $\hat{R}_\delta^{(\mu,1\uparrow)}(t)$ starting from δ such that

$$t R_1^{(\mu,\delta\uparrow)}(1/t) = \hat{R}_\delta^{(\mu,1\uparrow)}(t).$$

Therefore, setting

$$u = \int_0^t (\eta_s)^2 ds,$$

one has

$$\frac{1}{Z_t} = u^{-1} R_1^{(\mu,\delta\uparrow)}(u) = \hat{R}_\delta^{(\mu,1\uparrow)}(\int_t^{\mathbf{e}} (Z_s)^{-2} ds), \qquad t < \mathbf{e} \equiv L_{1/\delta}^{(\mu)}.$$

4 Concluding Remarks

In order to reinforce the parallel between the discussions in Sections 2 and 3, we now remark that, for $\delta, \gamma > 0$, the stochastic process

$$\begin{aligned}\Delta_t^{\mu,\delta,\gamma} &= \frac{\varphi_\nu(\delta e_t^{(\mu)})}{\varphi_\nu(\delta)}(e_t^{(\mu)})^k \exp(-\frac{\delta^2}{2}A_t^{(\mu)})\exp(-\frac{\gamma^2 t}{2}) \\ &= \frac{I_\nu(\delta e_t^{(\mu)})}{I_\nu(\delta)}(e_t^{(\mu)})^{k-\nu}\exp(-\frac{\delta^2}{2}A_t^{(\mu)})\exp(-\frac{\gamma^2}{2}t)\end{aligned}$$

is a $(\mathcal{B}_t^{(\mu)})$-local martingale, if we take $\nu = \sqrt{\gamma^2+\mu^2}$ and $k = \nu - \mu$. This may be deduced from the Cameron-Martin relationship between the processes $\{e_t^{(\mu)}\}$ and $\{e_t^{(\nu)}\}$ and the fact, which we already derived in Section 3, that $\{\Delta_t^{\nu,\delta}, t \geqq 0\}$ is a $(\mathcal{B}_t^{(\nu)})$-local martingale.
The local martingale property of $\{\Delta_t^{\mu,\delta,\gamma}\}$ can also be shown in the following way. Let us set

$$M_t = f(e_t^{(\mu)})\exp(-\frac{\delta^2}{2}A_t^{(\mu)})\exp(-\frac{\gamma^2}{2}t)$$

and look for a function f such that $\{M_t\}$ is a $(\mathcal{B}_t^{(\mu)})$-local martingale with $M_0 = 1$. Then it is easy to show from Itô's formula that f should satisfy

$$f''(x) + \frac{1+2\mu}{x}f'(x) - (\delta^2 + \frac{\gamma^2}{x^2})f(x) = 0 \qquad \text{and} \qquad f(1) = 1. \tag{4.1}$$

This is Bessel's equation. Hence, one obtains that f is a linear combination of $x^{-\mu}I_\nu(\delta x)$ and $x^{-\mu}K_\nu(\delta x)$, where $\nu = \sqrt{\mu^2+\gamma^2}$.
As an extension of Proposition 3.2, we obtain, using the same type of arguments as above,

$$E[\Delta_t^{\mu,\delta,\nu}|\mathcal{Z}_t^{(\mu)}, Z_t^{(\mu)} = z] = P(L_{1/\delta}^{(\nu)} \geqq t|\mathcal{Z}_t^{(\nu)}, Z_t^{(\nu)} = z)\ \frac{K_\nu(1/z)}{K_\mu(1/z)}\exp(-\frac{\gamma^2 t}{2}). \tag{4.2}$$

Thus, comparing (4.2) with formula (2.1), we see that (4.2) exhibits some dependency on δ, whereas there is no dependency on α in (2.1) ; this difference is due to the following facts.
(i) $\{D_t^{\alpha,0} \equiv (1+\alpha|B_t|)\exp(-\alpha L_t), t \geqq 0\}$ is a true martingale ; indeed, it projects on $\{\mathcal{R}_t\}$ as a true martingale, which must be of the form $h(R_t)$. However, the only such martingales (for the $BES(3)$ process or, more generally, transient diffusions) are constant, hence the projection of $\{D_t^{\alpha,0}\}$ is equal to 1.
(ii) $\{\Delta_t^{\mu,\delta}, t \geqq 0\}$ is already a strict local martingale, hence the previous arguments are not applicable. On the other hand, we may consider the true martingale

$$\Delta_t^{\mu,0,\gamma} \equiv (e_t^{(\mu)})^{\nu-\mu}\exp(-\frac{\gamma^2 t}{2}), \qquad t \geqq 0,$$

and its projection on $(\mathcal{Z}_t^{(\mu)})$ is

$$\frac{K_\nu(1/Z_t^{(\mu)})}{K_\mu(1/Z_t^{(\mu)})}\exp(-\frac{\gamma^2 t}{2}), \qquad \nu = \sqrt{\gamma^2+\mu^2}.$$

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