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A NON-BALLISTIC LAW OF LARGE NUMBERS FOR RANDOM WALKS IN I.I.D. RANDOM ENVIRONMENT

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Abstract
We prove that random walks in i.i.d. random environments which oscillate in a given direction have velocity zero with respect to that direction. This complements existing results thus giving a general law of large numbers under the only assumption of a certain zero-one law, which is known to hold if the dimension is two.

1. Notation, introduction, and results

An environment ω in $\mathbb{Z}^d$ $(d \geq 1)$ is an element of $\Omega := \mathcal{P}_+^{\mathbb{Z}^d}$ where $\mathcal{P}_+ \subset [0,1]^{2d}$ is the $(2d-1)$–dimensional simplex. The projections of ω on $\mathcal{P}_+$ are denoted as $(\omega(z,z+e))_{|e|=1,e\in\mathbb{Z}^d}$ $(z \in \mathbb{Z}^d)$ and thus fulfill $\omega(z,z+e) \geq 0$ and $\sum_e \omega(z,z+e) = 1$. Given such an environment ω and some $x \in \mathbb{Z}^d$, the so-called quenched probability measure $P_{x,\omega}$ on the path space $(\mathbb{Z}^d)^{\mathbb{N}}$ is characterized by

$$\begin{aligned} P_{x,\omega}[X_0 = x] &= 1 \quad \text{and} \\ P_{x,\omega}[X_{n+1} = z+e \mid X_n = z] &= \omega(z,z+e) \quad (z,e \in \mathbb{Z}^d, |e| = 1, n \geq 0) \end{aligned}$$

where $X_n : (\mathbb{Z}^d)^{\mathbb{N}} \to \mathbb{Z}^d$ is the n-th canonical projection.
Endowing Ω with its canonical product σ-algebra and some probability measure $\mathbb{P}$ (with corresponding expectation operator $\mathbb{E}$) turns ω into a random environment and $(X_n)_{n\geq 0}$ into a Random Walk in Random Environment (RWRE). The so-called annealed probability measure P_x, $x \in \mathbb{Z}^d$, is then defined as the semi-direct product $P_x := \mathbb{P} \times P_{x,\omega}$ on $\Omega \times (\mathbb{Z}^d)^{\mathbb{N}}$, that is $P_x[\cdot] := \mathbb{E}[P_{x,\omega}[\cdot]]$.
RWRE in $d \geq 2$ is currently an active research area, see [1] and [3] for recent results, surveys, and references. Using a certain renewal structure, Sznitman and Zerner (see [2] and [3, Theorem 3.2.2]) proved the following law of large numbers.

Theorem A. *Assume that*

$$(1) \qquad \begin{array}{l} (\omega(z,z+e))_{|e|=1}, \ z \in \mathbb{Z}^d, \quad \textit{are i.i.d. under } \mathbb{P} \textit{ with} \\ \omega(z,z+e) > 0 \quad \mathbb{P}\textit{-a.s. for all } z \in \mathbb{Z}^d, \ |e| = 1. \end{array}$$

Moreover, let $\ell \in \mathbb{R}^d \backslash \{0\}$ *and assume that* $P_0[A_\ell \cup A_{-\ell}] = 1$, *where*

$$A_\ell := \left\{ \lim_{n\to\infty} X_n \ell = \infty \right\}.$$

Then, there exist deterministic $v_\ell, v_{-\ell} \in \mathbb{R}^d$ *(possibly zero) such that*

$$\lim_{n\to\infty} \frac{X_n \ell}{n} = v_\ell \mathbf{1}_{A_\ell} + v_{-\ell}\mathbf{1}_{A_{-\ell}} \qquad P_0\text{-}a.s.$$

Remark: The version of this result which is quoted in [3, Theorem 3.2.2] assumes for some other purpose that the environment is uniformly elliptic, i.e. that $\mathbb{P}$-a.s. $\omega(z, z+e) > \varepsilon$ for some uniform "ellipticity constant" $\varepsilon > 0$. However, an inspection of the proof of this theorem shows that it actually does not use this condition.

So far there is no general law of large numbers under the assumption of (1) only which would state the existence of a deterministic v towards which X_n/n converges P_0-a.s.. There are two problems to be solved in order to derive such a law from Theorem A:

(i) Show that $v_\ell = 0$ if $0 < P_0[A_\ell] < 1$. Or even show that $0 < P_0[A_\ell] < 1$ is impossible. The latter has been proven in [4] for $d = 2$ and is still unknown for $d \geq 3$.

(ii) Show that for the elements ℓ of some basis of $\mathbb{Z}^d$ the assumption $P_0[A_\ell \cup A_{-\ell}] = 1$ in Theorem A can be omitted. Since (1) implies $P_0[A_\ell \cup A_{-\ell}] \in \{0,1\}$ (see [4, Proposition 3] and also [2, Lemma 1.1], [3, Theorem 3.1.2]) this means that one needs to investigate the case $P_0[A_\ell \cup A_{-\ell}] = 0$ only.

The purpose of the present note is to settle problem (ii) by the following result.

Theorem 1. *Assume (1) and let* $e \in \mathbb{Z}^d$ *with* $|e| = 1$ *and* $P_0[A_e \cup A_{-e}] = 0$. *Then*

$$\lim_{n\to\infty} \frac{X_n e}{n} = 0 \qquad P_0\text{-}a.s. \tag{2}$$

Of course, the main statement here is that the limit in (2) P_0-a.s. exists. Once this has been established it follows from $P_0[A_e \cup A_{-e}] = 0$ that the value of this limit has to be 0. Theorems A and 1 immediately imply:

Corollary 2. *Assume (1) and* $P_0[A_e] \in \{0,1\}$ *for all* $e \in \mathbb{Z}^d$ *with* $|e| = 1$. *Then there is some deterministic* $v \in \mathbb{R}^d$ *such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \qquad P_0\text{-}a.s.$$

According to [4, Theorem 1] the assumption $P_0[A_e] \in \{0,1\}$ holds if $d = 2$ and (1) are fulfilled.

2. Proofs

We first introduce some more notation. Fix some standard basis vector $e \in \mathbb{Z}^d$, $|e| = 1$. The following quantities are functions of nearest neighbor paths $X_\cdot = (X_n)_n$. For any $0 \leq u \in \mathbb{R}$, denote by $T_u := \inf\{n \geq 0 \mid X_n e \geq u\}$ $(\leq \infty)$ the first time the e-coordinate of $X_\cdot$ reaches or exceeds the level u. The times spent by $X_\cdot$ inside the hyperplane at distance m from the origin before $X_\cdot$ enters the hyperplane at distance $m + L$ constitute the set

$$\mathcal{T}_{m,L} := \{n \geq 0 \mid T_m \leq n < T_{m+L},\ X_n e = m\} \quad (m, L \in \mathbb{N}).$$

Note that $T_m \in \mathcal{T}_{m,L}$ if $\mathcal{T}_{m,L} \neq \emptyset$. The diameter of $\mathcal{T}_{m,L}$ is denoted by $h_{m,L}$. Finally, some empirical cumulative distribution function related to $h_{m,L}$ is given by

$$F_{M,L}(c) := \frac{\#\{0 \leq m \leq M \mid h_{m,L} \leq c\}}{M+1} \in [0,1] \quad (M, L \in \mathbb{N};\ c \geq 0).$$

Notice that $\mathcal{T}_{m,L}$ and $h_{m,L}$ increase in L as sets and numbers, respectively, whereas $F_{M,L}(c)$ decreases in L and increases in c. The following lemma deals with properties of single paths.

Lemma 3. *Let* $(X_n)_n \in (\mathbb{Z}^d)^{\mathbb{N}}$ *be a fixed nearest neighbor path with* $X_0 = 0$ *and* $\limsup_{n\to\infty} X_n e/n > 0$. *Then*

$$\sup_{c\geq 0} \inf_{L\geq 1} \limsup_{M\to\infty} F_{M,L}(c) > 0. \tag{3}$$

Proof. By assumption there exist $\delta > 0$ and a strictly increasing sequence $(n_k)_k$ of positive integers such that

$$\frac{X_{n_k} e}{n_k} > \delta \quad \text{for all } k. \tag{4}$$

Define $M_k := \lceil (1-\delta/2)\, n_k \delta/2 \rceil$. We are going to show that for all $L \in \mathbb{N}$,

$$\sum_{m=0}^{M_k} \# \mathcal{T}_{m,L} \;\leq\; \frac{3}{\delta}(M_k+1) \qquad \text{and} \tag{5}$$

$$\sum_{m=0}^{M_k} h_{m,L} \;\leq\; \frac{3L}{\delta}(M_k+1) \tag{6}$$

for all k large enough. To this end, fix L. Since $X.$ is a nearest neighbor path it follows from (4) that for all $\alpha \in [1-\delta/2, 1] n_k$

$$X_{\lceil \alpha \rceil} e \geq X_{n_k} e - n_k\delta/2 \geq n_k \delta/2 \geq \alpha\delta/2 \quad \text{and hence} \quad T_{\alpha\delta/2} \leq \lceil \alpha \rceil.$$

Applying this to $\alpha = (1-\delta/2)n_k + 2L/\delta$ shows that

$$T_{M_k+L} \leq \lceil (1-\delta/2)n_k + 2L/\delta \rceil \leq 2(M_k+L)/\delta + 1 \leq 3(M_k+1)/\delta \tag{7}$$

for k large enough.

For the proof of (5) observe that the sets $\mathcal{T}_{m,L}$ ($m = 0, \ldots, M_k$) are disjoint subsets of $\mathbb{N}$ and that all their elements are strictly less than T_{M_k+L}. Consequently, the left-hand side of (5) is at most T_{M_k+L} which along with (7) yields (5).

For the proof of (6) note that $h_{m,L} \leq T_{m+L} - T_m$ for all m. Hence the left-hand side of (6) is at most

$$\sum_{i=0}^{L-1} \sum_{\substack{m=0 \\ m \bmod L = i}}^{M_k} T_{m+L} - T_m \;\leq\; \sum_{i=0}^{L-1} T_{M_k+L} - T_i \;\leq\; L T_{M_k+L}$$

from which (6) follows again by (7).

Now assume that (3) is false. Then we define recursively a strictly increasing sequence $(L_i)_{i\geq 0}$ as follows. Set $L_0 := 0$, suppose that L_i has already been defined, and set $c_i := 9L_i/\delta$. By assumption we can choose $L_{i+1} > L_i$ such that

$$\limsup_{k\to\infty} F_{M_k, L_{i+1}}(c_i) < 1/3. \tag{8}$$

Then for any i,

$$1 \leq \frac{1}{M_k+1} \sum_{m=0}^{M_k} \mathbf{1}\{h_{m,L_i} \geq c_i\} + \mathbf{1}\{h_{m,L_{i+1}} \leq c_i\} + \mathbf{1}\{h_{m,L_i} < h_{m,L_{i+1}}\}.$$

We split the above sum canonically into three sums and get from (6) and the definition of c_i that the first sum is less than 1/3 for large k. Due to (8), the second sum is less than 1/3 for large k as well. Hence for any i,

$$\frac{1}{3} \le \frac{1}{M_k+1} \sum_{m=0}^{M_k} \mathbf{1}\{h_{m,L_i} < h_{m,L_{i+1}}\} \tag{9}$$

for large enough k. Now set $i_0 := \lceil 12/\delta \rceil$. Since $(L_i)_i$ is an increasing sequence, $(\mathcal{T}_{m,L_i})_i$ is an increasing sequence of sets for all m. Hence $\#\mathcal{T}_{m,L_i} < \#\mathcal{T}_{m,L_{i+1}}$ if $h_{m,L_i} < h_{m,L_{i+1}}$. Therefore, (5) with $L = L_{i_0}$ yields

$$\frac{3}{\delta} \ge \frac{1}{M_k+1} \sum_{m=0}^{M_k} \sum_{i=0}^{i_0-1} \mathbf{1}\{h_{m,L_i} < h_{m,L_{i+1}}\} \ge \frac{i_0}{3} \ge \frac{4}{\delta}$$

for large k due to (9), which is a contradiction. □

Proof of Theorem 1. The proof is by contradiction. Assume that

$$P_0\left[\limsup_{n\to\infty} \frac{X_n e}{n} > 0\right] > 0. \tag{10}$$

Since $\limsup_{M\to\infty} F_{M,L}(c)$ is measurable, increasing in c, and decreasing in L it follows from (10) and Lemma 3 that

$$P_0\left[\lim_{L\to\infty} \limsup_{M\to\infty} F_{M,L}(c) > 0\right] > 0 \tag{11}$$

for some finite c, which will be kept fixed for the rest of the proof. Now we need some more notation. Let $H_1(x) := \inf\{n \ge 0 \mid X_n = x\}$ be the first-passage time of the walk time through x ($x \in \mathbb{Z}^d$) and $H_r(x) := \inf\{n > H_{r-1}(x) \mid X_n = x\}$ be the time of the r-th visit to x ($r \ge 2$). Consider the last point visited by the walker in the hyperplane at distance m before the walker reaches the hyperplane at distance $m + L$. On the event $\{h_{m,L} \le c\}$, this point lies within $|\cdot|_1$-distance c from X_{T_m} and has been visited at most c times before T_{m+L}. This means on $\{h_{m,L} \le c\}$ there are $z \in \mathbb{Z}^d$ with $|z|_1 \le c$, $ze = 0$ and $r \in \mathbb{N}$ with $1 \le r \le c$ such that the event

$$B^1_{m,L}(z,r) := \left\{\sigma_m(z,r) < T_{m+L}, D^* \circ \theta_{\sigma_m(z,r)} + \sigma_m(z,r) \ge T_{m+L}\right\}$$

occurs, where

$$\begin{aligned} \sigma_m(z,r) &:= H_r(X_{T_m} + z), \\ D^* &:= \inf\{n \ge 1 \mid X_n e \le X_0 e\}, \end{aligned}$$

and $(\theta_n)_{n\ge 0}$ is the canonical shift on the path space $(\mathbb{Z}^d)^{\mathbb{N}}$. Thus the event $B^1_{m,L}(z,r)$ means that the last visit to the hyperplane at distance m before T_{m+L} is also the r-th visit to the point $X_{T_m} + z$. Consequently,

$$\lim_{L\to\infty} \limsup_{M\to\infty} F_{M,L}(c) \le \sum_{\substack{|z|_1 \le c \\ ze=0}} \sum_{r=1}^{c} \limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)}.$$

Since there are only finitely many such z and r it follows from (11) that for some of them

$$P_0\left[\limsup_{L\to\infty} \limsup_{M\to\infty} \frac{1}{M+1} \sum_{m=0}^{M} \mathbf{1}_{B^1_{m,L}(z,r)} > 0\right] > 0. \tag{12}$$

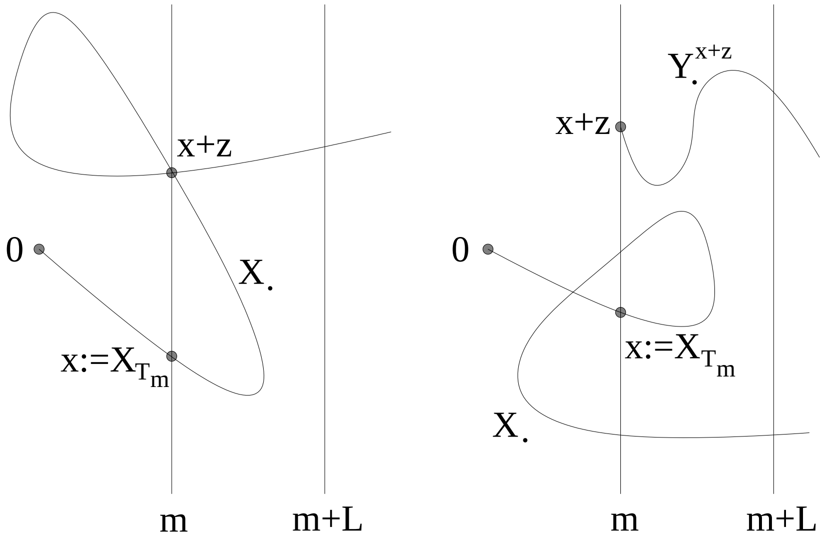

FIGURE 1. Left figure: The second visit in $x+z$ is the last visit in the hyperplane at distance m before reaching the hyperplane at distance $m+L$, that is $B^1_{m,L}(z, r=2)$ occurs. Right figure: The hyperplane at distance $m+L$ is reached before $x+z$ has been visited twice. Hence $B^1_{m,L}(z, r=2)$ does not occur. However, the auxiliary walk $Y^{x+z}_{\cdot}$ lets $B^2_{m,L}$ happen by entering the hyperplane at distance $m+L$ before visiting the hyperplane at distance m for a second time.

We keep a pair of z and r with property (12) fixed and drop z and r from the notation. For fixed L, the sequence $(\mathbf{1}_{B^1_{m,L}})_m$ seems to have a complicated dependence structure under P_0. However, one can dominate this sequence by an auxiliary sequence $(\mathbf{1}_{B_{m,L}})_m$, which consists of L i.i.d. sequences. To this end, we create for given ω in addition to the RWRE $X_{\cdot}$, for each starting point $y \in \mathbb{Z}^d$ an additional RWRE $Y^y_{\cdot}$ such that $X_{\cdot}$ and all $Y^y_{\cdot}$ ($y \in \mathbb{Z}^d$) are independent of each other, given ω. More precisely, we consider the probability measure

$$\widetilde{P}_{0,\omega} := P_{0,\omega} \otimes \bigotimes_{y \in \mathbb{Z}^d} P_{y,\omega} \quad \text{on} \quad (\mathbb{Z}^d)^{\mathbb{N}} \times \left((\mathbb{Z}^d)^{\mathbb{N}}\right)^{\mathbb{Z}^d} \tag{13}$$

endowed with its canonical σ-algebra and then realize $X_{\cdot}$ and the $Y^y_{\cdot}$'s as projections. Again, $\widetilde{P}_0 := \mathbb{E} \times \widetilde{P}_{0,\omega}$. We then define

$$\begin{aligned} B^2_{m,L} &:= \left\{\sigma_m \geq T_{m+L},\ D^*\left(Y^{X_{T_m}+z}_{\cdot}\right) \geq T_{m+L}\left(Y^{X_{T_m}+z}_{\cdot}\right)\right\} \qquad \text{and set} \\ B_{m,L} &:= B^1_{m,L} \cup B^2_{m,L}, \end{aligned}$$

see Figure 1. Here and in the following, if a stopping time is applied to a path other than $X_{\cdot}$, then this path is added in parentheses after the symbol for the stopping time.

If we interpret $B^1_{m,L}$ like $B^2_{m,L}$ as an event in the big sample space given in (13), we have $B^1_{m,L} \subseteq B_{m,L}$ and hence by (12),

$$\begin{aligned} 0 &< \widetilde{P}_0\left[\limsup_{L\to\infty}\limsup_{M\to\infty}\frac{1}{M+1}\sum_{m=0}^{M}\mathbf{1}_{B_{m,L}}>0\right] \\ &\le \widetilde{P}_0\left[\limsup_{L\to\infty}\frac{1}{L}\sum_{i=0}^{L-1}\limsup_{M\to\infty}\frac{L}{M+1}\sum_{\substack{m=0\\ m \bmod L=i}}^{M}\mathbf{1}_{B_{m,L}}>0\right]. \end{aligned} \tag{14}$$

We shall show at the end of the proof that for any $0 \le i < L$, the events

$$\begin{aligned} &B_{m,L}\ (m \bmod L = i) \text{ are independent under } \widetilde{P}_0 \text{ with} \\ &\widetilde{P}_0[B_{m,L}] = P_0\left[D^* \ge T_L\right]. \end{aligned} \tag{15}$$

Assuming this, we get from the ordinary strong law of large numbers and (14) that

$$0 < \limsup_{L\to\infty} P_0\left[D^* \ge T_L\right] \le \limsup_{L\to\infty} P_0\left[D^* \ge L\right] = P_0[D^* = \infty] \le P_0[A_e].$$

Here the last inequality follows from [4, Lemma 4], which implies that it is P_0-a.s. impossible for the walker to visit the strip $\{y \in \mathbb{Z}^d \mid 0 \le ye \le u\}$ $(u \ge 0)$ infinitely often without ever visiting the half space $\{y \mid ye < 0\}$ to the left of the strip. However, $P_0[A_e] > 0$ contradicts the assumption $P_0[A_e \cup A_{-e}] = 0$. Hence (10) is false. Repeating the argument with e in (10) replaced by $-e$ proves (2).

It remains to show (15). Let $k \ge 0$ and $0 \le m_0 < \ldots < m_k$ with $m_j \bmod L = i$ for all $j = 0, \ldots, k$. As above it follows from [4, Lemma 4] and $P_0[A_{-e}] = 0$ that T_{m_k} is P_0-a.s. finite because otherwise the walker could visit a strip of finite width infinitely often without visiting the half space to the right of the strip. Therefore, since $B^1_{m,L}$ and $B^2_{m,L}$ are disjoint,

$$\begin{aligned} &\widetilde{P}_0[B_{m_0,L} \cap \ldots \cap B_{m_k,L}] \\ &= \sum_{x:\ xe=m_k} \mathbb{E}\Big[\widetilde{P}_{0,\omega}\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L} \cap B^1_{m_k,L}, X_{T_{m_k}} = x\right] \\ &\qquad + \widetilde{P}_{0,\omega}\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L} \cap B^2_{m_k,L}, X_{T_{m_k}} = x\right]\Big] \\ &= \sum_{x:\ xe=m_k} \mathbb{E}\Big[\widetilde{P}_{0,\omega}\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L}, X_{T_{m_k}} = x, \sigma_{m_k} < T_{m_k+L}\right] \\ &\qquad \times P_{x+z,\omega}\left[D^* \ge T_{m_k+L}\right] \qquad\qquad (16) \\ &\qquad + \widetilde{P}_{0,\omega}\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L}, X_{T_{m_k}} = x, \sigma_{m_k} \ge T_{m_k+L}\right] \\ &\qquad \times \widetilde{P}_{0,\omega}\left[D^*(Y^{x+z}_\cdot) \ge T_{m_k+L}(Y^{x+z}_\cdot)\right]\Big]. \qquad\qquad (17) \end{aligned}$$

Here we used in (16) the strong Markov property with respect to σ_{m_k} and in (16) and (17) the fact that $\widetilde{P}_{0,\omega}$ is a product measure, see (13). However,

$$P_{x+z,\omega}\left[D^* \ge T_{m_k+L}\right] = \widetilde{P}_{0,\omega}\left[D^*(Y^{x+z}_\cdot) \ge T_{m_k+L}(Y^{x+z}_\cdot)\right].$$

Consequently, the whole sum in (16) and (17) can be rewritten as

$$\sum_x \mathbb{E}\left[\widetilde{P}_{0,\omega}\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L}, X_{T_{m_k}} = x\right] P_{x+z,\omega}\left[D^* \ge T_{m_k+L}\right]\right]. \tag{18}$$

Here the reason for introducing the auxiliary RWREs $Y^y_\cdot$ becomes clear: We got rid of the event $\{\sigma_{m_k} < T_{m_k+L}\}$, which links the environment to the left of m_k with the environment in the strip of width L to the right of m_k. Now the $\widetilde{P}_{0,\omega}$ term in (18) is $\sigma(\omega(y,\cdot) \mid ye < m_k)$-measurable since $m_j + L \le m_k$ for all $j < k$ whereas the $P_{x+z,\omega}$ term is $\sigma(\omega(y,\cdot) \mid ye \ge m_k)$-measurable. Hence by independence and translation invariance (18) equals

$$\begin{aligned}&\sum_x \widetilde{P}_0\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L}, X_{T_{m_k}} = x\right] P_{x+z}\left[D^* \ge T_{m_k+L}\right] \\ &\quad = \quad \widetilde{P}_0\left[B_{m_0,L} \cap \ldots \cap B_{m_{k-1},L}\right] P_0\left[D^* \ge T_L\right].\end{aligned}$$

From this (15) follows by induction over k. □

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