Elect. Comm. in Probab. **13** (2008), 325–336

ELECTRONIC COMMUNICATIONS in PROBABILITY

ON THE RELATIONSHIP BETWEEN SUBORDINATE KILLED AND KILLED SUBORDINATE PROCESSES

RENMING SONG
Department of Mathematics, University of Illinois, Urbana, IL 61801
email: `rsong@math.uiuc.edu`

ZORAN VONDRAČEK[1]
Department of Mathematics, University of Zagreb, Bijenička 30, 10000 Zagreb, Croatia
email: `vondra@math.hr`

Submitted 25 January 2008, *accepted in final form* 16 June 2008

AMS 2000 Subject classification: Primary 60J25; secondary 60J45, 60J75
Keywords: Markov process, subordination, killing, resurrection

Abstract

We study the precise relationship between the subordinate killed and killed subordinate processes in the case of an underlying Hunt process, and show that, under minimal conditions, the former is a subprocess of the latter obtained by killing at a terminal time. Moreover, we also show that the killed subordinate process can be obtained by resurrecting the subordinate killed one at most countably many times.

1 Introduction

Let X be a strong Markov process on a state space E. In this paper we will be interested in two types of probabilistic transformations of X. The first one is subordination of X via an independent subordinator T giving a Markov process $Y = (Y_t : t \ge 0)$ on E defined by $Y_t = X(T_t)$. The other transformation is killing X upon exiting an open subset D of E. The resulting process X^D is defined by $X^D_t = X_t$ for $t < \tau_D = \inf\{t > 0 : X_t \notin D\}$, and $X^D_t = \partial$ (the cemetery) otherwise. Now one can kill Y upon exiting D giving the process Y^D, and also subordinate X^D by the same subordinator T giving the process that we will denote by Z^D. Both processes are Markov with the same state space D. The process Y^D is called the killed subordinate process (first subordination, then killing), while Z^D is called the subordinate killed process (first killing, then subordination). It is an interesting problem to investigate the precise relationship between these two processes. This question can be traced back to [4] in the case when X is a Brownian motion and T a stable subordinator. In this context it was addressed in [10] where by use of pathwise approach it was shown that the semigroup of Z^D (subordinate killed) is subordinate to the semigroup of Y^D (killed subordinate). Recently, by

[1]THE RESEARCH OF THIS AUTHOR IS SUPPORTED IN PART BY THE MZOS GRANT 037-0372790-2801 OF THE REPUBLIC OF CROATIA.

use of Dirichlet form techniques, He and Ying gave in [5] an answer in a general setting of symmetric Borel right processes on a Lusin space E. Again, the semigroup of Z^D is subordinate to the semigroup of Y^D. The general theory then implies that Z^D can be obtained by killing Y^D via a multiplicative functional.

The goal of this paper is to give (in our opinion) the complete description of the relationship between Z^D and Y^D in the context of a Hunt process X (not necessarily symmetric) on a locally compact second countable Hausdorff space E. By defining X and the subordinator T on appropriate path spaces, and considering all relevant processes on the product of these path spaces, we show that Z^D is obtained by killing Y^D at an identifiable terminal time with respect to a filtration making Y a strong Markov process. Note that killing at a terminal time is a special case of killing by a multiplicative functional, but clearly more transparent. Moreover, we go a step further and show that the process Y^D can be recovered from Z^D by resurrecting the latter at most countably many times. This easily follows from our setting in which both Z^D and Y^D are described explicitly in terms of the underlying Hunt process X and the subordinator T. We also compute the resurrection kernel (given, implicitly, in [5]). Having the resurrection kernel, one can now start from any process with the same distribution as Z^D and use Meyer's resurrection procedure described in [7] to construct a process with the distribution of Y^D.

The paper is organized as follows: In the next section we precisely describe our setting. In Section 3 we give a description of the relationship between subordinate killed and killed subordinate processes. In Section 4 the resurrection kernel is computed. In the last section, as an application, we give sufficient conditions for Y to be not on the boundary ∂D at the exit time from D.

2 Setting and notation

Let E be a locally compact second countable Hausdorff space and let $\mathcal{E}$ be the corresponding Borel σ-algebra. Further, let Ω_1 be the set of all functions $\omega_1 : [0,\infty) \to E$ which are right continuous and have left limits. For each $t \geq 0$, let $X_t : \Omega_1 \to E$ be defined by $X_t(\omega_1) = \omega_1(t)$. The shift operator $\vartheta^1_t : \Omega_1 \to \Omega_1$ is defined by $\vartheta^1_t\omega_1(\cdot) = \omega_1(t+\cdot)$. Let $\mathbb{F}^0 = (\mathcal{F}^0_t : t \geq 0)$, $\mathcal{F}^0_t = \sigma(X_s : 0 \leq s \leq t)$, be the natural filtration generated by the process $X = (X_t : t \geq 0)$, and let $\mathcal{F}^0_{t+} = \cap_{s>t}\mathcal{F}^0_s$. Also, let $\mathcal{F} = \sigma(X_t : t \geq 0)$. We assume that $(\mathbb{P}^x_1 : x \in E)$ is a family of probability measures on $(\Omega_1, \mathcal{F})$ such that $(X_t, \mathbb{P}^x_1)$ is a strong Markov process. Let $\mathbb{F} = (\mathcal{F}_t : t \geq 0)$ be the usual augmentation of the natural filtration $\mathbb{F}^0$. From now on we assume that $X = (\Omega_1, \mathcal{F}, \mathcal{F}_t, X_t, \vartheta^1_t, \mathbb{P}^x_1)$ is a Hunt process with the state space $(E, \mathcal{E})$.

Let Ω_2 be the set of all functions $\omega_2 : [0,\infty) \to [0,\infty)$ which are right continuous and have left limits. For each $t \geq 0$, let $T_t : \Omega_2 \to [0,\infty)$ be defined by $T_t(\omega_2) = \omega_2(t)$. The shift operator $\vartheta^2_t : \Omega_2 \to \Omega_2$ is defined by $\vartheta^2_t\omega_2(\cdot) = \omega_2(t+\cdot)$. Let $\mathbb{G}^0 = (\mathcal{G}^0_t : t \geq 0)$, $\mathcal{G}^0_t = \sigma(T_s : 0 \leq s \leq t)$, be the natural filtration generated by the process $T = (T_t : t \geq 0)$, and let $\mathcal{G}^0_{t+} = \cap_{s>t}\mathcal{G}^0_s$. Also, let $\mathcal{G} = \sigma(T_t : t \geq 0)$. We assume that $(\mathbb{P}^y_2 : y \in [0,\infty))$ is a family of probability measures on $(\Omega_2, \mathcal{G})$ such that $(T_t, \mathbb{P}^y_2)$ is an increasing Lévy process. In particular, we assume that under $\mathbb{P}_2 := \mathbb{P}^0_2$, the law of T_t is given by

$$\mathbb{E}_2(\exp -\lambda T_t) = \exp(-t\phi(\lambda)),$$

where

$$\phi(\lambda) = b\lambda + \int_{(0,\infty)} (1 - e^{-\lambda x})\, \Pi(dx).$$

Here $b \geq 0$ is the drift, and Π the Lévy measure of the subordinator T. Further, let $U(dy)$ denote the potential measure of T under $\mathbb{P}_2$:

$$U(dy) = \mathbb{E}_2 \int_0^\infty 1_{(T_t \in \, dy)} \, dt \, .$$

For $y > 0$, let $\sigma_y = \inf\{t > 0 : T_t > y\}$ be the first passage time of T across the level y. Then σ_y is a $(\mathcal{G}^0_{t+})$-stopping time and the following identity holds true for all $t > 0$ and $y > 0$:

$$\{T_t < y\} = \{\sigma_y > t\} \, . \tag{2.1}$$

Let $\Omega = \Omega_1 \times \Omega_2$, and, for any $x \in E$ and $y \in [0, \infty)$, let $\mathbb{P}^{x,y} = \mathbb{P}^x_1 \times \mathbb{P}^y_2$ be the product probability measure on $\mathcal{H} = \mathcal{F} \times \mathcal{G}$. The probability $\mathbb{P}^{x,0}$ will be denoted as $\mathbb{P}^x$. The elements of Ω are denoted by $\omega = (\omega_1, \omega_2)$. For each $t \geq 0$ we define the shift operator $\theta_t : \Omega \to \Omega$ by

$$\theta_t(\omega)(\cdot) = \theta_t(\omega_1, \omega_2)(\cdot) := (\omega_1(\omega_2(t) + \cdot), \omega_2(t + \cdot) - \omega_2(t)) \, . \tag{2.2}$$

We will occasionally write $\theta_t(\omega) = (\theta^1_t(\omega), \theta^2_t(\omega))$. Note that for $s, t \geq 0$ we have

$$\begin{aligned}
\theta_s(\theta_t \omega) &= \theta_s(\omega_1(\omega_2(t) + \cdot), \omega_2(t + \cdot) - \omega_2(t)) \\
&= (\omega_1(\omega_2(t) + (\omega_2(t+s) - \omega_2(t)) + \cdot), (\omega_2(t + s + \cdot) - \omega_2(t)) - (\omega_2(t+s) - \omega_2(t))) \\
&= (\omega_1(\omega_2(t+s) + \cdot), \omega_2(t + s + \cdot) - \omega_2(t+s)) \\
&= \theta_{t+s}(\omega) \, .
\end{aligned}$$

The subordinate process $Y = (Y_t : t \geq 0)$ with the state space $(E, \mathcal{E})$ is defined on Ω by $Y_t(\omega) := X_{T_t(\omega_2)}(\omega_1) = \omega_1(\omega_2(t))$. Note that

$$\begin{aligned}
Y_s(\theta_t \omega) &= \omega_1(\omega_2(t) + (\omega_2(t+s) - \omega_2(t))) \\
&= \omega_1(\omega_2(t+s)) = Y_{s+t}(\omega) \, .
\end{aligned}$$

Following [3] we introduce the following filtration: For $t \geq 0$ let

$$\mathcal{S}_t = \{A^1 \times (A^2 \cap \{T_t \geq u\}) : \, A^1 \in \mathcal{F}^0_u, A^2 \in \mathcal{G}^0_t, u \geq 0\} \, ,$$

and let $\mathcal{H}_t = \sigma(\mathcal{S}_t)$. Then $\mathbb{H} = (\mathcal{H}_t : t \geq 0)$ is a filtration on Ω such that for all $t \geq 0$, $\sigma(Y_s : 0 \leq s \leq t) \subset \mathcal{H}_t$.

Remark 2.1. *Suppose that S is a function defined on, say, Ω_1. By abusing notation we will regard S as being defined on Ω by $S(\omega_1, \omega_2) = S(\omega_1)$. We use the same convention if S is a function defined on Ω_2.*

The following results are proved in [3]:

Proposition 2.2. *(i) ([3], p.65) If S is an $(\mathcal{F}^0_{t+})$-stopping time, then $\{S \leq T_t\} \in \mathcal{H}_{t+}$.*

(ii) ([3], p.66) If S is an $(\mathcal{H}_{t+})$-stopping time, then

- *(a) for each ω_1, $S(\omega_1, \cdot)$ is an $(\mathcal{G}^0_{t+})$-stopping time;*
- *(b) for each ω_2, $T_{S(\cdot,\, \omega_2)}(\omega_2)$ is an $(\mathcal{F}^0_{t+})$-stopping time.*

(iii) ([3], p. 67) The subordinate process $Y = (\Omega, \mathcal{H}, \mathcal{H}_{t+}, Y_t, \mathbb{P}^{x,y})$ is a strong Markov process.

In the next result we prove that the subordinate process Y is quasi-left-continuous.

Proposition 2.3. *Let $(S_n : n \geq 1)$ be an increasing sequence of $(\mathcal{H}_{t+})$-stopping times, and let $S = \lim_{n\to\infty} S_n$. Then $\lim_{n\to\infty} Y_{S_n} = Y_S$, $\mathbb{P}^x$-a.s. on $\{S < \infty\}$ for every $x \in E$.*

Proof. Without loss of generality we assume that $S < \infty$, $\mathbb{P}^x$-a.s., for every $x \in E$. Let $A = \{\omega = (\omega_1, \omega_2) : \lim_n T_{S_n(\omega_1,\omega_2)}(\omega_2) = T_{S(\omega_1,\omega_2)}(\omega_2)\}$, and let A_{ω_1} be the ω_1-section of A. For each fixed ω_1, it follows from Proposition 2.2(ii)(a) that $S_n(\omega_1, \cdot)$ is a $(\mathcal{G}^0_{t+})$-stopping time, hence by quasi-left-continuity of the subordinator T, we have that $\mathbb{P}_2(A_{\omega_1}) = 1$. Thus by Fubini's theorem we have that $\mathbb{P}^x(A) = 1$, for every $x \in E$.

Let $\widetilde{T}(\omega_1, \omega_2) = \lim_{n\to\infty} T_{S_n(\omega_1,\omega_2)}(\omega_2)$. Since for each fixed ω_2, $T_{S_n(\cdot,\omega_2)}(\omega_2)$ is an $(\mathcal{F}^0_{t+})$-stopping time by Proposition 2.2(ii)(b), it follows that $\widetilde{T}(\cdot, \omega_2)$ is also an $(\mathcal{F}^0_{t+})$-stopping time. The assumptions $S < \infty$ $\mathbb{P}^x$-a.s. implies that $\widetilde{T}(\cdot, \omega^2) < \infty$, $\mathbb{P}^x_1$-a.s., for every $x \in E$. By the quasi-left-continuity of the process X we obtain that

$$\lim_{n\to\infty} X_{T_{S_n(\omega_1,\omega_2)}}(\omega_1) = X_{\widetilde{T}(\omega_1,\omega_2)}(\omega_1) \quad \mathbb{P}^x_1\text{-a.e. } \omega_1 .$$

Let $B = \{\omega = (\omega_1, \omega_2) : \lim_{n\to\infty} X_{T_{S_n}}(\omega) = X_{\widetilde{T}}(\omega)\}$. Again by using Fubini's theorem, it follows that $\mathbb{P}^x(B) = 1$ for every $x \in E$. Therefore, $\mathbb{P}^x(A \cap B) = 1$, and for $\omega \in A \cap B$ it holds that $\lim_{n\to\infty} Y_{S_n}(\omega) = \lim_{n\to\infty} X_{T_{S_n}}(\omega) = X_{\widetilde{T}}(\omega) = X_{T_S}(\omega) = Y_S(\omega)$. □

Lemma 2.4. *Let S be an $(\mathcal{F}^0_{t+})$-stopping time. Then $\sigma_S = \inf\{t > 0 : T_t > S\}$ is an $(\mathcal{H}_{t+})$-stopping time.*

Proof. By (2.1) we have that $\{\sigma_S \leq t\} = \{T_t \geq S\}$. The claim now follows from Proposition 2.2(i). □

Let D be an open subset of E, and let $\tau_D = \inf\{t > 0 : X_t \notin D\}$ be the first exit time of X from D. We assume for simplicity that $\mathbb{P}^x_1(\tau_D < \infty) = 1$ for all $x \in E$. By the previous lemma it follows that σ_{τ_D} is an $(\mathcal{H}_{t+})$-stopping time. In the next lemma we prove that it is also a terminal time with respect to $(\mathcal{H}_{t+})$.

Lemma 2.5. *For every $t \geq 0$, $\sigma_{\tau_D} = t + \sigma_{\tau_D} \circ \theta_t$ on $\{t < \sigma_{\tau_D}\}$.*

Proof. First note that $\sigma_{\tau_D}(\omega) = \sigma_{\tau_D}(\omega_1, \omega_2) = \inf\{t > 0 : \omega_2(t) > \tau_D(\omega_1)\}$. Further, by (2.1), $\{t < \sigma_{\tau_D}\} = \{T_t < \tau_D\} = \{(\omega_1, \omega_2) : \omega_2(t) < \inf\{u > 0 : \omega_1(u) \notin D\}\}$. Therefore, on $\{t < \sigma_{\tau_D}\}$,

$$\begin{aligned} \sigma_{\tau_D}(\theta_t\omega) &= \inf\{s > 0 : \theta^2_t\omega_2(s) > \tau_D(\theta_t\omega)\} \\ &= \inf\{s > 0 : \omega_2(t+s) - \omega_2(t) > \inf\{u > 0 : (\theta^1_t\omega)(u) \notin D\}\} \\ &= \inf\{s > 0 : \omega_2(t+s) > \omega_2(t) + \inf\{u > 0 : \omega_1(\omega_2(t) + u) \notin D\}\} \\ &= \inf\{s > 0 : \omega_2(t+s) > \inf\{u > 0 : \omega_1(u) \notin D\}\}, \end{aligned}$$

where the last line follows from $\omega_2(t) < \inf\{u > 0 : \omega_1(u) \notin D\}$. Hence,

$$\begin{aligned} t + (\sigma_{\tau_D} \circ \theta_t)(\omega) &= t + \inf\{s > 0 : \omega_2(t+s) > \tau_D(\omega)\} \\ &= \inf\{s > 0 : \omega_2(s) > \tau_D(\omega)\} = \sigma_{\tau_D}(\omega). \end{aligned}$$

□

We also record the fact that for each fixed ω_2, $T_{\sigma_{\tau_D}(\cdot,\omega_2)}(\omega_2)$ is an $(\mathcal{F}^0_{t+})$-stopping time. This follows from Proposition 2.2(ii)(b). Further, for each fixed ω_2, $T_{\sigma_{\tau_D}(\cdot,\omega_2)}(\omega_2)$ is a function of τ_D, and hence $\mathcal{F}^0_{\tau_D}$-measurable.

3 Subordinate killed and killed subordinate processes

Let $\tau_D^Y = \inf\{t > 0 : Y_t \notin D\}$ be the first exit time of the subordinate process Y from D. Then τ_D^Y is an $(\mathcal{H}_{t+})$-stopping time. We note that even when $\mathbb{P}_1^x(\tau_D < \infty) = 1$ for all $x \in D$, it may happen that $\tau_D^Y = \infty$, $\mathbb{P}^x$-a.s. for every $x \in D$. Indeed, let X be a one-dimensional Brownian motion, $D = (-\infty, 0) \cup (0, \infty)$, and let T be an $\alpha/2$-stable subordinator with $0 < \alpha < 1$. The subordinate process Y is an α-stable process in $\mathbb{R}$. Since $0 < \alpha < 1$, points are polar for Y, and in particular, the hitting time to zero, which is precisely equal to τ_D^Y, is infinite. Clearly, $\mathbb{P}_1^x(\tau_D < \infty) = 1$.

The process Y killed upon exiting D is defined by

$$Y_t^D := \begin{cases} Y_t, & t < \tau_D^Y \\ \partial, & t \geq \tau_D^Y \end{cases} = \begin{cases} X_{T_t}, & t < \tau_D^Y \\ \partial, & t \geq \tau_D^Y \end{cases}$$

where ∂ is a cemetery point . We call Y^D the killed subordinate process. Note that Y^D is a strong Markov process with respect to the filtration $(\mathcal{H}_{t+})$.

The other process that we are going to consider is obtained by killing Y at the terminal time σ_{τ_D}. Define

$$Z_t^D := \begin{cases} Y_t, & t < \sigma_{\tau_D} \\ \partial, & t \geq \sigma_{\tau_D} \end{cases} = \begin{cases} X_{T_t}, & T_t < \tau_D \\ \partial, & T_t \geq \tau_D \end{cases}$$

where the equality is a consequence of (2.1). Since σ_{τ_D} is a terminal time, it follows (similarly as in the proof of Theorem 12.23(i) of [9], p. 71) that Z^D is also a strong Markov process with respect to the filtration $(\mathcal{H}_{t+})$.

We can also introduce the process X^D as the process X killed upon exiting D. Clearly, if $T_t < \tau_D$, then $X_{T_t}^D = X_{T_t}$. This shows that Z^D is in fact obtained by first killing X as it exits D, and then by subordinating the killed process with the subordinator T. Therefore we call Z^D the subordinate killed process.

Note that if $t < \sigma_{\tau_D}$, then $T_t < \tau_D$, and therefore $Y_t = X_{T_t} \in D$. This shows that $\sigma_{\tau_D} \leq \tau_D^Y$. As a consequence, we see that Z^D can be obtained by killing Y^D at the terminal time σ_{τ_D}:

$$Z_t^D := \begin{cases} Y_t^D, & t < \sigma_{\tau_D} \\ \partial, & t \geq \sigma_{\tau_D}. \end{cases} \tag{3.1}$$

For any nonnegative Borel function f on D let $Q_t^D f(x) := \mathbb{E}^x[f(Y_t^D)]$ be the semigroup of Y^D, and $R_t^D f(x) := \mathbb{E}^x[f(Z_t^D)] = \mathbb{E}^x[f(Y_t^D), t < \sigma_{\tau_D}]$ be the semigroup of Z^D. The following result is now obvious.

Proposition 3.1. *The semigroup $(R_t^D : t \geq 0)$ is subordinate to the semigroup $(Q_t^D : t \geq 0)$ in the sense that for every nonnegative Borel function f on D it holds that $R_t^D f \leq Q_t^D f$.*

Let $S_1 = \sigma_{\tau_D}$ and define inductively $S_{n+1} = S_n + S_1 \circ \theta_{S_n}$ with the convention that if $S_n = \tau_D^Y$, then for all $k > n$, $S_k = \tau_D^Y$. Then $(S_n : n \geq 1)$ is an increasing sequence of $(\mathcal{H}_{t+})$-stopping times. The limit $S = \lim_{n\to\infty} S_n$ is an $(\mathcal{H}_{t+})$-stopping time. Clearly, $S \leq \tau_D^Y$. The next proposition shows that these stopping times are in fact equal $\mathbb{P}^x$-a.s. for every $x \in D$.

Proposition 3.2. *It holds that $\tau_D^Y = S$, $\mathbb{P}^x$-a.s. for every $x \in D$.*

Proof. Let $A = \{\omega = (\omega_1, \omega_2) : \lim_n T_{S_n(\omega_1,\omega_2)}(\omega_2) = T_{S(\omega_1,\omega_2)}(\omega_2)\}$. It was shown in Proposition 2.3 that $\mathbb{P}^x(A) = 1$ for all $x \in E$. Therefore, there exists $\widetilde{\Omega}_2 \subset \Omega_2$ with $\mathbb{P}_2(\widetilde{\Omega}_2) = 1$ and such that $\mathbb{P}_1^x(A_{\omega_2}) = 1$ for every $\omega_2 \subset \widetilde{\Omega}_2$. Let us fix $\omega_2 \in \widetilde{\Omega}_2$. If for some $n \geq 1$ it holds that $S_n(\cdot,\omega_2)=\tau_D^Y(\cdot,\omega_2)$ there is nothing to prove. Therefore we assume that $S_1(\cdot,\omega_2) < S_2(\cdot,\omega_2) < \cdots < S(\cdot,\omega_2)$. By Proposition 2.2, $T_{S(\cdot,\omega_2)}(\omega_2)$ and $T_{S_n(\cdot,\omega_2)}(\omega_2)$, $n\in\mathbb{N}$, are $(\mathcal{F}^0_{t+})$-stopping times. For $n\ge 1$ define

$$\tau_{n+1}=\inf\{t>T_{S_n(\cdot,\omega_2)}(\omega_2):\, X_t\notin D\}\,.$$

Then τ_{n+1} is an $(\mathcal{F}^0_{t+})$-stopping time. Moreover, we have that

$$T_{S_1(\cdot,\omega_2)}(\omega_2)\le\tau_2\le T_{S_s(\cdot,\omega_2)}(\omega_2)\le\tau_3\le\cdots\le T_{S(\cdot,\omega_2)}(\omega_2)\,.$$

Since $\mathbb{P}_1^x(A_{\omega_2})=1$, we have that $T_{S(\cdot,\omega_2)}(\omega_2)=\uparrow\lim_{n\to\infty}\tau_n$, $\mathbb{P}_1^x$-a.s., and by the quasi-left-continuity of X, we conclude that $X_{T_{S(\cdot,\omega_2)}(\omega_2)}=\lim_{n\to\infty}X_{\tau_n}$, $\mathbb{P}_1^x$-a.s. on $\{T_{S(\cdot,\omega_2)}(\omega_2)<\infty\}=\{S(\cdot,\omega_2)<\infty\}$ for every $x\in D$. Since $X_{\tau_n}\notin D$, we conclude that $\mathbb{P}_1^x(X_{T_{S(\cdot,\omega_2)}(\omega_2)}\notin D, S(\cdot,\omega_2)<\infty))=\mathbb{P}_1^x(S(\cdot,\omega_2)<\infty)$ for every $x\in D$. Since $\omega_2\in\widetilde{\Omega}_2$, integrating the last equality with respect to $\mathbb{P}_2$, gives that for every $x\in D$, $\mathbb{P}^x(Y_S\notin D, S<\infty)=\mathbb{P}^x(X_{T_S}\notin D, S<\infty)=\mathbb{P}^x(S<\infty)$. But this means that $S\ge\tau_D^Y$, $\mathbb{P}^x$-a.s. on $\{S<\infty\}$. Clearly, $S\ge\tau_D^Y$ on $\{S=\infty\}$. □

Remark 3.3. *Assume that $S_n<S$ for every $n\ge1$. Note that in this case $Y_{S_n}\in D$. Since $S=\lim_{n\to\infty}S_n$, the quasi-left-continuity of Y implies that on $\{S<\infty\}$, $Y_S=\lim_{n\to\infty}Y_{S_n}\in\overline{D}$, the closure of D. From the proof of Proposition 3.2, we have that $Y_S\in D^c$ on $\{S<\infty\}$. Therefore, $Y_S=Y_{\tau_D^Y}\in\partial D$ on $\{S<\infty\}=\{\tau_D^Y<\infty\}$.*

4 Resurrection kernel

Proposition 3.2 clearly shows that the process Y^D can be obtained from Z^D by resurrecting the latter at most countably many times. Our next goal is to compute the resurrection kernel. For $t\ge0$ and $x\in E$, let $P_t(x,\,dy)$ denote the transition kernel of X. To be more precise, $P_t(x,\,dy)=\mathbb{P}_1^x(X_t\in\,dy)=\mathbb{P}^x(X_t\in\,dy)$. Similarly, for $x\in D$, let $P_t^D(x,\,dy)$ denotes the transition kernel of the killed process X^D. Throughout this paper we will assume the following
(**A1**) X admits a Lévy system of the form (J^X,dt).
Here $J^X(x,\,dy)$ is a kernel on $(E,\mathcal{E})$. The assumption (**A1**) is not very restrictive. For example, all Lévy processes satisfy this assumption. Under the assumption (**A1**), one can easily check that the killed process X^D has a Lévy system of the form (J^{X^D},dt), where $J^{X^D}(x,dy)$ is the restriction of J^X on $(D,\mathcal{B}(D))$. That is, for $x\in D$ and a Borel subset $B\subset D$, $J^{X^D}(x,B)=J^X(x,B)$. By slightly abusing the notation, we will denote J^{X^D} simply by J^X.
The subordinate process Y admits a Lévy system of the form (J^Y,dt), where

$$J^Y(x,\,dy)=bJ^X(x,\,dy)+\int_{(0,\infty)}P_t(x,\,dy)\,\Pi(dt)\,,\quad x\in E\,, \tag{4.1}$$

(see [8] for a proof, and also [3], p.74, for the case $b=0$).
Similarly, the subordinate killed process Z^D admits a Lévy system of the form (J^{Z^D},dt), where

$$J^{Z^D}(x,\,dy)=bJ^X(x,\,dy)+\int_{(0,\infty)}P_t^D(x,\,dy)\,\Pi(dt)\,,\quad x\in D\,. \tag{4.2}$$

It is also well known that the potential kernel $U^{Z^D}(x,\,dy)$ of Z^D is given by

$$U^{Z^D}(x,\,dy)=\int_0^\infty P_t^D(x,\,dy)\,U(dt)\,, \tag{4.3}$$

where $U(ds)$ is the potential measure of the subordinator T. Finally, we recall the following first-passage formulae for the subordinator (see, e.g., [1], p.76): For each fixed $x\geq 0$ and every $0\leq s\leq x<t$,

$$\mathbb{P}_2(T_{\sigma_x-}\in ds, T_{\sigma_x}\in dt)=U(ds)\Pi(dt-s)\,. \tag{4.4}$$

If $b=0$, then $\mathbb{P}_2(T_{\sigma_x}=x)=0$, while if $b>0$, then $U(dx)$ has a continuous density u and

$$\mathbb{P}_2(T_{\sigma_x}=x)=b\,u(x)\,, \tag{4.5}$$

(see, e.g., [1], pp.77-79). Moreover, $\mathbb{P}_2(T_{\sigma_x}=x)=\mathbb{P}_2(T_{\sigma_x-}=T_{\sigma_x}=x)$.

We are mainly interested in the case when the Lévy measure Π of T is infinite. So from now on, we assume

(**A2**) The Lévy measure Π of T is infinite.

In this case, the potential measure $U(dt)$ has no atoms (e.g. [6], Theorem 5.4).

Theorem 4.1. *Suppose that (**A1**) and (**A2**) are valid. Let $B\subset\overline{D}$ and C be Borel subsets of E. Then for every $x\in D$,*

$$\begin{aligned}
&\mathbb{P}^x(Y_{\sigma_{\tau_D}-}\in B, Y_{\sigma_{\tau_D}}\in C)\\
&=\int_{B\cap D}U^{Z^D}(x,\,dy)\int_{C\cap D^c}(J^Y(y,\,dz)-bJ^X(y,\,dz))\\
&\quad+\int_{B\cap D}U^{Z^D}(x,\,dy)\int_{C\cap D}(J^Y(y,\,dz)-J^{Z^D}(y,\,dz))\\
&\quad+b\,\mathbb{E}^x(u(\tau_D), X(\tau_D-)\in B, X(\tau_D)\in C\cap D^c)\,.
\end{aligned} \tag{4.6}$$

Remark: In case when $b=0$, the last line vanishes.

Proof. Note that

$$Y_{\sigma_{\tau_D}-}=\lim_{t\uparrow\sigma_{\tau_D}}Y_t=\lim_{t\uparrow\sigma_{\tau_D}}X(T_t)=X(T_{\sigma_{\tau_D}-}-)\,.$$

We first split the left-hand side in (4.6) depending on whether the subordinator jumps over, or hits, the level τ_D at the first passage over τ_D:

$$\begin{aligned}
\mathbb{P}^x(Y_{\sigma_{\tau_D}-}\in B, Y_{\sigma_{\tau_D}}\in C) &= \mathbb{P}^x(Y_{\sigma_{\tau_D}-}\in B, Y_{\sigma_{\tau_D}}\in C, T_{\sigma_{\tau_D}-}\leq\tau_D<T_{\sigma_{\tau_D}})\\
&\quad+\mathbb{P}^x(Y_{\sigma_{\tau_D}-}\in B, Y_{\sigma_{\tau_D}}\in C, T_{\sigma_{\tau_D}-}=\tau_D=T_{\sigma_{\tau_D}})\\
&=: A_1+A_2\,.
\end{aligned}$$

Next, A_1 can be written as

$$\begin{aligned} A_1 &= \mathbb{P}^x\Big(X(T_{\sigma_{\tau_D}-}-)(\omega_1,\omega_2)\in B, X(T_{\sigma_{\tau_D}})(\omega_1,\omega_2)\in C,\\ &\qquad T_{\sigma_{\tau_D(\omega_1)}-}(\omega_2)\le \tau_D(\omega_1) < T_{\sigma_{\tau_D(\omega_1)}}(\omega_2)\Big)\\ &= (\mathbb{P}_1^x\times\mathbb{P}_2)\Big(X(T_{\sigma_{\tau_D(\omega_1)}-}(\omega_2)-)(\omega_1)\in B, X(T_{\sigma_{\tau_D(\omega_1)}}(\omega_2))(\omega_1)\in C,\\ &\qquad T_{\sigma_{\tau_D(\omega_1)}-}(\omega_2)\le \tau_D(\omega_1) < T_{\sigma_{\tau_D(\omega_1)}}(\omega_2))\Big)\\ &= \mathbb{E}_1^x\left[\int_0^{\tau_D(\omega_1)}\int_{\tau_D(\omega_1)}^\infty \mathbb{P}_2(T_{\sigma_{\tau_D(\omega_1)}-}\in ds, T_{\sigma_{\tau_D(\omega_1)}}\in dt)\,1_{(X_{s-}(\omega_1)\in B)}1_{(X_t(\omega_1)\in C)}\right]\\ &= \mathbb{E}_1^x\int_0^{\tau_D}U(ds)\int_{\tau_D}^\infty \Pi(dt-s)1_{(X_{s-}\in B)}1_{(X_t\in C)}\\ &= \int_0^\infty U(ds)\,\mathbb{E}_1^x\left[1_{(s<\tau_D)}1_{(X_s\in B)}\int_{\tau_D}^\infty \Pi(dt-s)1_{(X_t\in C)}\right], \end{aligned} \tag{4.7}$$

where the next to last line follows from (4.4), and the last line from the fact that U has no atoms, and $X_{s-}=X_s$, $\mathbb{P}_1^x$-a.e. for every fixed $s\ge 0$. Next,

$$\begin{aligned} &\mathbb{E}_1^x\left[1_{(s<\tau_D)}1_{(X_s\in B)}\int_{\tau_D}^\infty \Pi(dt-s)1_{(X_t\in C)}\right]\\ &\quad= \mathbb{E}_1^x\left[1_{(s<\tau_D)}1_{(X_s\in B\cap D)}\,\mathbb{E}_1^x\left[\int_{\tau_D}^\infty \Pi(dt-s)1_{(X_t\in C)}\,\middle|\mathcal{F}_{s+}^0\right]\right]\\ &\quad= \mathbb{E}_1^x\left[1_{(s<\tau_D)}1_{(X_s\in B\cap D)}\,\mathbb{E}_1^x\left[\int_{s+\tau_D\circ\vartheta_s^1}^\infty \Pi(dt-s)1_{(X_t\in C)}\,\middle|\mathcal{F}_{s+}^0\right]\right]\\ &\quad= \mathbb{E}_1^x\left[1_{(s<\tau_D)}1_{(X_s\in B\cap D)}\,\mathbb{E}_1^x\left[\int_{\tau_D\circ\vartheta_s^1}^\infty \Pi(dw)1_{(X_w\circ\vartheta_s^1\in C)}\,\middle|\mathcal{F}_{s+}^0\right]\right]\\ &\quad= \mathbb{E}_1^x\left[1_{(s<\tau_D)}1_{(X_s\in B\cap D)}\,\mathbb{E}_1^{X_s}\int_{\tau_D}^\infty \Pi(dw)1_{(X_w\in C)}\right]\\ &\quad= \int_{B\cap D}P_s^D(x,\,dy)f(y)\,, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} f(y) &= \mathbb{E}_1^y\left[\int_{\tau_D}^\infty \Pi(dw)1_{(X_w\in C)}\right]\\ &= \int_0^\infty \Pi(dw)\,\mathbb{P}_1^y(X_w\in C, w\ge\tau_D)\,. \end{aligned}$$

Since

$$\begin{aligned} \mathbb{P}_1^y(X_w\in C, w\ge\tau_D) &= \mathbb{P}_1^y(X_w\in C)-\mathbb{P}_1^y(X_w\in C, w<\tau_D)\\ &= \int_C P_w(y,\,dz)-\int_{C\cap D}P_w^D(y,\,dz)\,, \end{aligned}$$

it follows from (4.1) and (4.2) that

$$\begin{aligned} f(y) &= \int_C \int_0^\infty \Pi(dw) P_w(y,\, dz) - \int_{C\cap D} \int_0^\infty \Pi(dw) P_w^D(y,\, dz) \\ &= \int_C (J^Y(y,\, dz) - bJ^X(y,\, dz)) - \int_{C\cap D} (J^{Z^D}(y,\, dz) - bJ^X(y,\, dz)) \\ &= \int_{C\cap D^c} (J^Y(y,\, dz) - bJ^X(y,\, dz)) + \int_{C\cap D} (J^Y(y,\, dz) - J^{Z^D}(y,\, dz)) . \end{aligned} \tag{4.9}$$

Equalities (4.7), (4.8) and (4.9) yield that A_1 is equal to the first two lines on the right-hand side of (4.6).

In order to compute A_2 we proceed as follows:

$$\begin{aligned} A_2 &= \mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in B, Y_{\sigma_{\tau_D}} \in C, T_{\sigma_{\tau_D}-} = \tau_D = T_{\sigma_{\tau_D}}) \\ &= \mathbb{P}^x(X(\tau_D-) \in B, X(\tau_D) \in C, T_{\sigma_{\tau_D}-} = \tau_D = T_{\sigma_{\tau_D}}) \\ &= \mathbb{E}_1^x[X(\tau_D-)(\omega_1) \in B, X(\tau_D)(\omega_1) \in C, \mathbb{P}_2(T_{\sigma_{\tau_D(\omega_1)}} = \tau_D(\omega_1))] \\ &= b\,\mathbb{E}^x[u(\tau_D), X(\tau_D-) \in B, X(\tau_D) \in C \cap D^c] , \end{aligned}$$

where the last line follows from (4.5) and the fact that $X(\tau_D) \notin D$. □

For $y \in D$, let $q(y,dz) := J^Y(y,dz) - J^{Z^D}(y,dz)$. We call q the *resurrection kernel.* Note that for Borel sets $B \subset D$ and $C \subset D$, the formula (4.6) can be written as

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in B, Y_{\sigma_{\tau_D}} \in C) = \int_B U^{Z^D}(x,\, dy)\, q(y,C) .$$

By use of (4.1) and (4.2) one can write the resurrection kernel as

$$q(y,dz) = \int_{(0,\infty)} (P_t(y,dz) - P_t^D(y,dz))\, \Pi(dt) .$$

This is the form that the resurrection kernel appears in [5].

Corollary 4.2. *Assume that (**A1**) and (**A2**) are valid.*

(i) If the subordinator has no drift, i.e., $b = 0$, then $\mathbb{P}^x(T_{\sigma_{\tau_D}} = \tau_D) = \mathbb{P}^x(T_{\sigma_{\tau_D}-} = \tau_D) = 0$ and $\mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in D) = 1$ for every $x \in D$.

(ii) If $b > 0$, then for every Borel set $C \subset E$,

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}} \in C, T_{\sigma_{\tau_D}} = \tau_D) = b\,\mathbb{E}^x[u(\tau_D), X(\tau_D) \in C \cap D^c)] .$$

Proof.

(i) That $\mathbb{P}^x(T_{\sigma_{\tau_D}} = \tau_D) = \mathbb{P}^x(T_{\sigma_{\tau_D}-} = \tau_D) = 0$ is an immediate consequence of the first passage formulae stated before Theorem 4.1 and the fact that the potential measure U has no atoms. In order to show that $\mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in D) = 1$ we use (4.7):

$$\begin{aligned} \mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in D) &= \int_0^\infty U(ds)\, \mathbb{E}_1^x\left[1_{(s<\tau_D)} 1_{(X_s \in D)} \int_{\tau_D}^\infty \Pi(dt-s) 1_{(X_t \in E)}\right] \\ &= \int_0^\infty U(ds)\, \mathbb{E}_1^x\left[1_{(s<\tau_D)} \int_{\tau_D}^\infty \Pi(dt-s)\right] \\ &= \mathbb{E}_1^x\left[\int_0^{\tau_D} U(ds) \int_{\tau_D}^\infty \Pi(dt-s)\right] = 1 , \end{aligned}$$

because the quantity in the last bracket is $\mathbb{P}_1^x$-a.s. equal to 1.

(ii) This follows immediately from the expression for A_2 in the proof of Theorem 4.1 by taking $B = \overline{D}$.

□

Corollary 4.3. *Suppose that (**A1**) and (**A2**) are valid. Assume that $J^X(x, \partial D) = 0$ and $P_t(x, \partial D) = 0$, for all $x \in D$ and all $t > 0$. Then for every Borel subset $C \subset \partial D$,*

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}} \in C) = b\,\mathbb{E}^x[u(\tau_D), X(\tau_D) \in C]\,, \quad x \in D\,. \tag{4.10}$$

In particular, if $b = 0$, $J^X(x, \partial D) = 0$ and $P_t(x, \partial D) = 0$, for all $x \in D$ and all $t > 0$, then

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}} \in \partial D) = 0\,. \tag{4.11}$$

Proof. It follows from (4.1) that $J^Y(x, \partial D) = 0$, $x \in D$. Now (4.10) follows from Theorem 4.1 by taking $B = \overline{D}$ and $C = \partial D$, having (4.11) as an immediate consequence. □

Corollary 4.4. *Suppose that (**A2**) is valid. Assume that X has continuous paths and $P_t(x, \partial D) = 0$ for all $x \in D$ and all $t > 0$. Then for every Borel set $C \subset \partial D$ it holds that*

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in C) = \mathbb{P}^x(Y_{\sigma_{\tau_D}} \in C) = b\,\mathbb{E}^x[u(\tau_D), X(\tau_D) \in C]\,.$$

In particular,

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}-} \in \partial D) = \mathbb{P}^x(Y_{\sigma_{\tau_D}} \in \partial D) = b\,\mathbb{E}^x[u(\tau_D)]\,.$$

Proof. The second equality follows from Corollary 4.3. For the first, notice that by continuity of X, $Y_{\sigma_{\tau_D}-} = X(T_{\sigma_{\tau_D}}-) = X(T_{\sigma_{\tau_D}})$. If $Y_{\sigma_{\tau_D}-} \in C \subset \partial D$, then $T_{\sigma_{\tau_D}}- = \tau_D = T_{\sigma_{\tau_D}}$ (see the proof of Corollary 4.2). Therefore, $Y_{\sigma_{\tau_D}-} = Y_{\sigma_{\tau_D}}$. □

5 An application

It is of some interest to find sufficient conditions for $Y_{\tau_D^Y} \notin \partial D$. In [11], Sztonyk gave a sufficient condition for a rotationally invariant Lévy process with infinite Lévy measure and no Gaussian component not to hit the boundary ∂D upon exiting a Lipschitz domain D. The given condition is satisfied for a rotationally invariant processes. In this section we give two sufficient conditions in our setting.

For $t \geq 0$ and $x \in E$ define $N_t(x, f) = \mathbb{E}_1^x[f(X_t)]$, where f is a nonnegative Borel function on E. Note that $T_{\sigma_{\tau_D}} = \tau_D + (T_{\sigma_{\tau_D}} - \tau_D)$, and for each fixed ω_2, $T_{\sigma_{\tau_D}(\cdot,\omega_2)}(\omega_2) - \tau_D(\cdot)$ is $\mathcal{F}^0_{\tau_D}$-measurable. Therefore, by an extended version of the strong Markov property (see [2], pp.43-44), for each fixed ω_2 (with ω_2 suppressed in notation),

$$\mathbb{E}_1^x[1_D(Y_{\sigma_{\tau_D}}) \,|\, \mathcal{F}^0_{\tau_D+}] = \mathbb{E}_1^x[1_D(X_{\tau_D + (T_{\sigma_{\tau_D}} - \tau_D)}) \,|\, \mathcal{F}^0_{\tau_D+}] = N_{T_{\sigma_{\tau_D}} - \tau_D}(X_{\tau_D}, 1_D)\,, \quad \mathbb{P}_1^x - a.s. \tag{5.1}$$

Proposition 5.1. *Suppose that (**A2**) is valid. Assume that X has continuous paths, $P_t(x,\partial D)=0$ for all $x\in D$ and all $t>0$, and that there exists a constant $c\in(0,1)$ such that*

$$\mathbb{P}_1^x(X_t\in D)\le c \quad \text{for every } x\in\partial D \text{ and every } t>0\,. \tag{5.2}$$

Assume further that the subordinator T has no drift. Then $\mathbb{P}^x(Y_{\tau_D^Y}\in\partial D)=0$ for every $x\in D$.

Proof. By the assumptions, $X_{\tau_D}\in\partial D$ and $N_t(x,1_D)=\mathbb{P}_1^x(X_t\in D)\le c$, for all $x\in\partial D$ and all $t\ge 0$. By (5.1), this implies that for each fixed ω_2, $\mathbb{P}_1^x(Y_{\sigma_{\tau_D}}\in D\,|\,\mathcal{F}^0_{\tau_D+})\le c$, $x\in D$, and therefore

$$\mathbb{P}^x(Y_{\sigma_{\tau_D}}\in D)\le c\,,\quad x\in D\,. \tag{5.3}$$

Recall the notations $S_1=\sigma_{\tau_D}$, and for $n\ge 1$, $S_{n+1}=S_n+S_1\circ\theta_{S_n}$. By the strong Markov property of Y, it follows from (5.3) that

$$\mathbb{P}^x(Y_{S_n}\in D)\le c^n\,,\quad n\ge 1\,.$$

Let $N:=\inf\{n\ge 1:\ S_n=\tau_D^Y\}$ with the usual convention $\inf\emptyset=\infty$. It follows from the last displayed formula that $\mathbb{P}^x(N=\infty)=0$ for every $x\in D$. Hence, there are only finitely many S_n which are less than τ_D^Y. From Corollary 4.4, $\mathbb{P}^x(Y_{\sigma_{\tau_D}}\in\partial D)=\mathbb{P}^x(Y_{S_1}\in\partial D)=0$, and therefore by iteration $\mathbb{P}^x(Y_{S_n}\in\partial D)=0$ for all $n\in\mathbb{N}$ and all $x\in D$. Since $\tau_Y^D=S_n$ for some $n\in\mathbb{N}$, the claim of the proposition follows. □

If X is a Brownian motion in $\mathbb{R}^d$, then it was shown in [10] that (5.2) holds true provided D is a bounded domain satisfying the exterior cone condition.
The next result should be compared to Lemma 1 from [12].

Proposition 5.2. *Suppose that (**A2**) is valid. Assume that X has continuous paths, $P_t(x,\partial D)=0$ for all $x\in D$ and all $t>0$, $b=0$, and $\sup_{x\in D}\mathbb{P}^x(Y_{\sigma_{\tau_D}}\in D)<1$. Then $\mathbb{P}^x(Y_{\tau_D^Y}\in\partial D)=0$ for every $x\in D$.*

Proof. Again note that by Corollary 4.4, $\mathbb{P}^x(Y_{\sigma_{\tau_D}}\in\partial D)=0$. Therefore, if $Y_{\tau_D^Y}\in\partial D$, then $\sigma_{\tau_D}<\tau_D^Y$, and hence $Y_{\sigma_{\tau_D}}\in D$. Let $\gamma:=\sup_{x\in D}\mathbb{P}^x(Y_{\tau_D^Y}\in\partial D)$. By the strong Markov property of Y at σ_{τ_D} and the assumptions we have

$$\begin{aligned}
\mathbb{P}^x(Y_{\tau_D^Y}\in\partial D) &= \mathbb{P}^x(Y_{\tau_D^Y}\in\partial D, Y_{\sigma_{\tau_D}}\in D)\\
&= \mathbb{P}^x(\mathbb{P}^{Y_{\sigma_{\tau_D}}}(Y_{\tau_D^Y}\in\partial D), Y_{\sigma_{\tau_D}}\in D)\\
&\le \gamma\,\mathbb{P}^x(Y_{\sigma_{\tau_D}}\in D).
\end{aligned}$$

By taking the supremum over $x\in D$, it follows that $\gamma\le\gamma\sup_{x\in D}\mathbb{P}^x(Y_{\sigma_{\tau_D}}\in D)$. Therefore, $\gamma=0$. □

Acknowledgment: The second named author gratefully acknowledges the hospitality of Department of Mathematics at the University of Illinois at Urbana-Champaign where the research for this paper was done.

References

[1] J. Bertoin, *Levy Processes*, Cambridge University Press, Cambridge, 1996. MR1406564

[2] R. M. Blumenthal, R. K. Getoor, *Markov Processes and Potential Theory*, Academic Press, New York, 1968. MR0264757

[3] N. Bouleau, Quelques résultats probabilistes sur la subordination au sens de Bochner, in *Seminar on potential theory*, Paris, No. 7, 54–81, Lecture Notes in Math., 1061, Springer, Berlin, 1984. MR0779295

[4] J. Glover, M. Rao, H. Šikić, R. Song, Γ-potentials, in *Classical and modern potential theory and applications* (Chateau de Bonas, 1993), 217–232, Kluwer Acad. Publ., Dordrecht, 1994. MR1321619

[5] P. He, J. Ying, Subordinating the killed process versus killing the subordinate process, *Proceedings Amer. Math. Soc.* **135**(2007), 523–529. MR2255299

[6] A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, Berlin, 2006. MR2250061

[7] P. A. Meyer, Renaisssance, recollements, mélanges, ralentissement de processus de Markov, *Ann. Inst. Fourier* **25**(1975), 465–497. MR0415784

[8] A. Miyake, The subordination of Lévy system for Markov processes, *Proc. Japan Acad.* **45**(1969), 601–604. MR0260019

[9] M. Sharpe, *General Theory of Markov Processes*, Academic Press, San Diego, 1988. MR0958914

[10] R. Song, Z. Vondraček, Potential theory of the killed Brownian motion in a domain, *Probab. Theory Rel. Fields* **125**(2003), 578–592. MR1974415

[11] P. Sztonyk, On harmonic measure for Lévy processes, *Probab. Math. Statist.*, **20** (2000), 383–390. MR1825650

[12] J.-M. Wu, Harmonic measures for symmetric stable processes, *Studia Math.* **149**(2002), 281–293. MR1893056