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OSCILLATION AND NON–OSCILLATION IN SOLUTIONS OF NONLINEAR STOCHASTIC DELAY DIFFERENTIAL EQUATIONS

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Abstract
This paper studies the oscillation and nonoscillation of solutions of a nonlinear stochastic delay differential equation, where the noise perturbation depends on the current state, and the drift depends on a delayed argument. When the restoring force towards equilibrium is relatively strong, all solutions oscillate, almost surely. However, if the restoring force is superlinear, positive solutions exist with positive probability, and for suitably chosen initial conditions, the probability of positive solutions can be made arbitrarily close to unity.

1 Introduction

Among others, Ladas et al [7], Shreve [9] and Staikos and Stavroulakis [10] consider the oscillatory behaviour of the nonlinear delay differential equation

$$x'(t) = -f(x(t-\tau)). \tag{1}$$

The existence of an equilibrium solution $x(t) \equiv 0$ is ensured by requiring that $f(0) = 0$. The continuous forcing function f must act towards the equilibrium in order to generate an environment conducive to oscillatory behaviour, and therefore it is required that $xf(x) > 0$ for $x \neq 0$.
The crucial property which in effect allows solutions of (1) to oscillate is that f is linearisable at the equilibrium. That is to say, if there is $\infty > L > 0$ such that

$$\lim_{x\to 0} \frac{f(x)}{x} = L \tag{2}$$

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then oscillatory solutions exist. Oscillation can be guaranteed for every solution by ensuring that the delay term τ is large enough. More specifically, if $\tau L > \frac{1}{e}$ and (2) is true, then every solution of (1) oscillates.

If f does not obey (2), but the restoring force towards the zero equilibrium is weaker, in the sense that

$$\lim_{x\to 0} \frac{|f(x)|}{|x|^\gamma} = L \tag{3}$$

for some $\gamma > 1$ and $L > 0$, solutions of (1) do not have to oscillate. In fact, if ψ is any positive, continuous function on $[-\tau, 0]$, there exists $\alpha^* > 0$ such that the solution of (1) with $x(t) = \alpha\psi(t)$ for $t \in [-\tau, 0]$ and $0 < \alpha < \alpha^*$ is nonoscillatory.

The purpose of this paper is to consider how the solutions of (1) behave when perturbed stochastically. However the question of how to structure this perturbation is an important one. In order to avoid the need to significantly alter our definition of oscillation in the transition from a deterministic to a stochastic setting, it is crucial that the equilibrium solution be preserved.

The effects of such a perturbation on the oscillatory behaviour of a linear version of (1) were considered in [1]. It was shown that all solutions of

$$dX(t) = -bX(t-\tau)dt + \sigma X(t)dB(t) \tag{4}$$

are a.s. oscillatory when $b > 0$, $\sigma \neq 0$, $\tau > 0$. In the case $b < 0$ it was shown that a positive initial function resulted in a positive solution.

In this paper we prove a nonlinear analogue of this oscillation result, showing that all solutions of the nonlinear stochastic delay differential equation

$$dX(t) = -f(X(t-\tau))dt + \sigma h(X(t))dB(t) \tag{5}$$

are a.s. oscillatory when f obeys (2), and h has a linearisation at zero. In contrast with the deterministic case, we do not require any condition on the length of the delay. Therefore, once again, the multiplicative noise perturbation induces an oscillation about the zero equilibrium solution which need not be present in the deterministic case, where $\sigma = 0$. Although the noise has not completely replaced the delay as a cause of the oscillation, the delay is no longer the sole factor. Since the delay term in the drift is effectively sublinear, its equilibrium restoring effect will be stronger than that in the linear equation (4). This effect is complemented by the contribution of the noise. As we will show in sections 2 and 3, the oscillation of solutions of the nonlinear equation is intimately linked with the oscillation of solutions of a linear equation. Each path of X can be associated with the solution of a linear, nonautonomous delay differential equation

$$Z'(t) = -P(t)Z(t-\tau), \quad t > 0.$$

The nonlinearity in the original equation has been subsumed into the positive function P. More importantly however, P depends on increments of a Brownian motion. The large deviations of these increments ensure that P is large enough, often enough, to cause oscillation.

However, when (3) holds, and h is kept unchanged, it can be shown that not all solutions of (5) need be oscillatory. Indeed, for any $\psi \in C([-\tau, 0]; \mathbb{R}^+)$, there exists $\alpha^* > 0$ such that for any $0 < \alpha < \alpha^*$ there is a positive probability of the solution with initial condition $X(t) = \alpha\psi(t)$, $t \in [-\tau, 0]$ being nonoscillatory. Moreover, if this probability is denoted P_α, we have that $P_\alpha \to 1$ as $\alpha \to 0$. Therefore by selecting an appropriate initial function, we can ensure that the probability of nonoscillation is as close to one as we like. Consequently, if the strength of the restoring force towards equilibrium is weak enough, the presence of noise may not be sufficient to induce an oscillation about that equilibrium. Under (3), the linearisation of (5) at zero is $d\overline{X}(t) = \sigma\overline{X}(t)dB(t)$, which has a nonoscillatory solution. Intuitively, this suggests that solutions of (5) should also be nonoscillatory.

Despite the effectiveness of such intuition in determining the nonoscillatory behaviour in this case, using the oscillatory behaviour of solutions of limiting equations as a guide for predicting the oscillatory behaviour of scalar stochastic delay differential equations is not always foolproof. For example in [3], the behaviour of a linear stochastic differential equation with an asymptotically vanishing delay term in the drift is shown to have different oscillatory behaviour to that of the corresponding limiting equation without a delay term in the drift. This result prompts the question of whether or not a stronger superlinear drift term than posited in (3) might give rise to a.s. oscillatory solutions of (5). We do not attempt to answer this here.

A second possible generalisation of (4), by Gushchin and Küchler [5], retains the linear multiplicative diffusion coefficient, but generalises the drift coefficient with an extra linear instantaneous feedback term, producing the equation

$$dX(t) = (aX(t) - f(X(t-\tau)))dt + \sigma X(t)dB(t), \tag{6}$$

defined with an initial condition that is nonnegative, not identically zero, but not necessarily strictly positive. In [5], the authors develop sufficient conditions on f for the a.s. oscillation of solutions of (6). They find that the relative intensities of the instantaneous feedback terms in the drift and in the noise play a crucial role in the stringency of these conditions, as the instantaneous feedback in the drift mitigates the effect of the nonlinear delay term. The mainly probabilistic methods employed contrast with the methods of this paper, which owe a debt to deterministic theory, and encompass both oscillation and nonoscillation. However, in the coincident special cases of (5) with $h(x) \equiv x$, and (6) with $a = 0$, the conditions which ensure a.s. oscillation given in this paper restrict f differently than those given in [5]. For example, herein f is not required to be nonincreasing. However examples given in [5] allow for discontinuity in f, whereas we impose a stronger continuity restriction here. We also require no sign condition on the initial function in order to prove results about oscillation. In fact it is this very distinction that causes our approach to oscillation in theorem 3.1 to depart from that of Gushchin and Küchler. We consider the finiteness of the last zero of the process, whereas Gushchin and Küchler can consider the occurrence of a series of stopping times defined by the arrival of successive zeros.

In section 2 we set up the problem and define exactly what we mean by almost sure oscillation and nonoscillation. In section 3 we establish a sufficient condition on the drift coefficient to ensure a.s. oscillation, and in section 4 we study a condition on the drift coefficient which allows for nonoscillatory behaviour to take place with non-zero probability. This result serves to distinguish the memory driven processes we study in this paper from the Markovian processes which obey classical zero-one laws.

2 Preliminaries

We study the oscillatory properties of the stochastic delay differential equation

$$\begin{aligned} dX(t) &= -f(X(t-\tau))dt + \sigma h(X(t))dB(t), && \text{(7a)} \\ X(t) &= \psi(t), \quad t \in [-\tau, 0] && \text{(7b)} \end{aligned}$$

where $\sigma \neq 0$, $\tau > 0$ are real constants. The initial data ψ is a continuous function on $[-\tau, 0]$. Suppose that $f \in C(\mathbb{R};\mathbb{R})$, and h is locally Lipschitz continuous on $\mathbb{R}$. It is shown in Appleby and Kelly [2] that (7) has a unique strong continuous solution on $[0,\infty)$, almost surely. The proof in [2] is a stochastic adaptation of the method of steps. We call the set on which the solution exists Ω_0^*, with $\mathbb{P}[\Omega_0^*] = 1$.

2.1 Properties of the coefficients of (7).

We impose the following hypotheses on the continuous function h. Let $h(0) = 0$, and suppose there exists $0 < \underline{h} \leq 1 \leq \bar{h}$ such that

$$\underline{h}|x|^2 \leq xh(x) \leq \bar{h}|x|^2, \tag{8}$$

and

$$\lim_{x\to 0} \frac{h(x)}{x} = 1. \tag{9}$$

In addition, the continuous function f has the properties

$$f(0) = 0, \quad xf(x) > 0, \quad x \neq 0. \tag{10}$$

Notice now that if $\psi(t) = 0$ for all $t \in [-\tau, 0]$ that the unique solution of (7) is $X(t) = 0$ for all $t \geq 0$, a.s. It is the oscillation, or absence of oscillation, about this equilibrium solution that we intend to study.

2.2 Oscillation of stochastic processes.

The notion of oscillation of stochastic processes was introduced in [1]. We reprise those definitions here.

We say that a (non-trivial) continuous function $y : [t_0, \infty) \to \mathbb{R}$ is *oscillatory* if the set

$$Z_y = \{t \geq t_0 \ : \ y(t) = 0\}$$

satisfies $\sup Z_y = \infty$. A function which is not oscillatory is called *nonoscillatory*. We extend these notions to stochastic processes in the following intuitive manner: a stochastic process $(X(t,\omega))_{t\geq t_0}$ defined on a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$ with continuous sample paths is said to be *almost surely oscillatory* (a.s. oscillatory hereafter) if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is oscillatory. A stochastic process is *a.s. nonoscillatory* if there exists $\Omega^* \subseteq \Omega$ with $\mathbb{P}[\Omega^*] = 1$ such that for all $\omega \in \Omega^*$, the path $X(\cdot, \omega)$ is nonoscillatory. In the deterministic case ($\sigma = 0$), the second part of hypothesis (10) is material in producing an oscillation about the zero equilibrium, as it forces the solution towards the equilibrium (with a delay) whether it is above or below the equilibrium level.

2.3 The decomposition of solutions of (7).

Our proofs of oscillation and nonoscillation rely upon representing the solution of (7) as the product of a nowhere differentiable, but positive process, whose asymptotic behaviour is readily understood, and a process with continuously differentiable sample paths, which obeys a scalar random delay differential equation. To this end, we introduce the continuous function $\tilde{h}$

$$\tilde{h}(x) = \begin{cases} \frac{h(x)}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}, \tag{11}$$

so that $\underline{h} \leq \tilde{h}(x) \leq \bar{h}$, $x \in \mathbb{R}$. We may then define the process $(\varphi(t))_{t\geq -\tau}$ by $\varphi(t) = 1,\ t \in [-\tau, 0]$ and for $t \geq 0$ by

$$\varphi(t) = e^{\sigma \int_0^t \tilde{h}(X(s))dB(s) - \frac{1}{2}\sigma^2 \int_0^t \tilde{h}(X(s))^2 ds}. \tag{12}$$

The process is uniquely defined on $[0, \infty)$, as X is a well-defined process. We call the almost sure set on which φ exists $\Omega_1^* \subseteq \Omega_0^*$, with $\mathbb{P}[\Omega_1^*] = 1$. Observe further that φ satisfies

$$d\varphi(t) = \sigma \tilde{h}(X(t))\varphi(t)dB(t). \tag{13}$$

Since φ is positive, we may define

$$Z(t) = X(t)\varphi(t)^{-1}, \quad t \geq -\tau. \tag{14}$$

Then $Z(t) = \psi(t)$ for $t \in [-\tau, 0]$, and using stochastic integration by parts, (7) and (13) imply that

$$Z(t) = \psi(0) + \int_0^t -f(X(s-\tau))\varphi(s)^{-1} ds, \quad t \geq 0.$$

The continuity of the integrand implies that Z is continuously differentiable, and satisfies

$$Z'(t) = -\varphi(t)^{-1} f(X(t-\tau)),\ \ t > 0. \tag{15}$$

The following lemma places upper and lower bounds on the rate of decay of the process φ.

Lemma 2.1. *Let φ be defined by (12), where $\tilde{h}$ is given by (11). Then*

$$\limsup_{t\to\infty} \frac{1}{t} \log \varphi(t) \leq -\frac{1}{2}\sigma^2 \underline{h}^2, \quad a.s. \tag{16}$$

$$\liminf_{t\to\infty} \frac{1}{t} \log \varphi(t) \geq -\frac{1}{2}\sigma^2 \bar{h}^2, \quad a.s. \tag{17}$$

Proof. Define

$$M(t) = \int_0^t \tilde{h}(X(s))dB(s), \quad t \geq 0, \tag{18}$$

and the square variation process

$$\langle M \rangle(t) = \int_0^t \tilde{h}(X(s))^2\, ds. \tag{19}$$

Then φ may be rewritten as $\varphi(t) = e^{\sigma M(t) - \frac{1}{2}\sigma^2 \langle M\rangle(t)}$. By (8) and (11),

$$\underline{h}^2 t \leq \langle M \rangle(t) \leq \bar{h}^2 t.$$

To prove (16) and (17), note that $\lim_{t\to\infty} \langle M\rangle(t) = \infty$, a.s., so by the law of large numbers for martingales $M(t)/\langle M\rangle(t) \to 0$ as $t \to \infty$, a.s., and therefore, as

$$\left|\frac{M(t)}{t}\right| = \frac{\langle M\rangle(t)}{t} \cdot \left|\frac{M(t)}{\langle M\rangle(t)}\right| \leq \bar{h}^2 \left|\frac{M(t)}{\langle M\rangle(t)}\right|,$$

we get $M(t)/t \to 0$ as $t \to \infty$, a.s. Since $\underline{h}^2 t \leq \langle M\rangle(t) \leq \bar{h}^2 t$, the estimates (16) and (17) follow. ☐

We now state a fundamental lemma on the oscillation of all solutions of linear deterministic delay equations. It is a special case of Theorem 2 in [10].

Lemma 2.2. *Let p be a continuous and non-negative function, and x be a nontrivial solution of*

$$x'(t) = -p(t)x(t-\tau), \ \ t > 0.$$

If

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} p(s)\,ds > 1,$$

then x is an oscillatory solution of the equation.

We note that many similar results, for example covering equations with several delays, exist in the deterministic literature. These results could be used in conjunction with the methods here to develop more general results concerning the oscillation of stochastic delay differential equations.

3 Almost sure oscillation of (7)

Consider the stochastic differential delay equation (7), where in addition to the earlier hypotheses on f, h, we request that there exists $L > 0$ (possibly infinite) such that

$$\liminf_{x\to 0} \frac{f(x)}{x} = L. \tag{20}$$

Theorem 3.1. *Suppose that the continuous function f obeys (10) and (20), and the locally Lipschitz continuous function h obeys (8) and (9). If $\psi \in C([-\tau, 0], \mathbb{R})$, then all solutions of (7) are oscillatory, a.s.*

Proof. Note that if $\psi(t) \equiv 0$ on $[-\tau, 0)$, then $X(t) \equiv 0$, for all $t \geq 0$. So, in this case, the solution is oscillatory. Therefore we assume that $\psi(t) \not\equiv 0$ on $[-\tau, 0)$. If the solution exists on Ω^*, then $\Omega^* = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \emptyset$ such that the solution is a.s. oscillatory on Ω_1 and a.s. nonoscillatory on Ω_2. Suppose $\mathbb{P}[\Omega_2] > 0$, in contradiction to the result to be proved. Take $\omega \in \Omega^*$. Let φ be the process defined in (12) which obeys (13). Let Z be the process defined in (14) which satisfies the random delay differential equation (15). Now suppose $\omega \in \Omega_2$. Then there exists $\tau^*(\omega) < \infty$ such that, for all $t > \tau^*(\omega)$, $X(t, \omega) \neq 0$. Therefore, either $X(t, \omega) > 0$ for all $t > \tau^*(\omega)$, or $X(t, \omega) < 0$ for all $t > \tau^*(\omega)$. Suppose that $X(t, \omega) > 0$ for all $t > \tau^*(\omega)$. The proof in the case where $X(t, \omega) < 0$ for all $t > \tau^*(\omega)$, is analogous, and hence omitted. Then, for all $t > \tau^*(\omega) + \tau$, (10), (14) and (15) imply that

$$Z'(t, \omega) < 0, \quad Z(t, \omega) > 0.$$

Hence $0 < X(t) < \varphi(t)Z(\tau^* + \tau)$ for all $t > \tau^* + \tau$. Therefore, for all $t > \tau^* + \tau$

$$|X(t)| \leq \left|\frac{X(\tau^* + \tau)}{\varphi(\tau^* + \tau)}\right| \varphi(t).$$

Since $\tau^* < \infty$ a.s. on Ω_2 and $t \mapsto X(t)$, $t \mapsto \varphi(t)$ are continuous, and hence bounded, on $[0, \tau^* + \tau]$, the quantity

$$C(\omega) := \left|\frac{X(\tau^* + \tau, \omega)}{\varphi(\tau^* + \tau, \omega)}\right|,$$

is positive and finite for all $\omega \in \Omega_2$, and

$$|X(t,\omega)| \leq C(\omega)\varphi(t,\omega), \quad t > \tau^*(\omega) + \tau.$$

By (16), $\varphi(t) \to 0$ as $t \to \infty$. Thus $X(t) \to 0$ as $t \to \infty$, on Ω_2. For $t \geq \tau^* + 2\tau$, $\tilde{f}(t)$, given by

$$\tilde{f}(t) = \frac{f(X(t-\tau))}{X(t-\tau)},$$

is well defined. Then $\tilde{f}(t) > 0$ for $t \geq \tau^* + 2\tau$ and, as $X(t) \to 0$ as $t \to \infty$, (20) implies

$$\liminf_{t\to\infty} \tilde{f}(t,\omega) = L > 0, \quad \omega \in \Omega_2. \tag{21}$$

Letting $P(t) = \varphi(t)^{-1}\varphi(t-\tau)\tilde{f}(t)$, $t \geq \tau^* + 2\tau$, we see that $P(t,\omega) > 0$ for all $\omega \in \Omega_2$, $t \geq \tau^*(\omega) + 2\tau$, and (15) can be rewritten as

$$Z'(t) = -P(t)Z(t-\tau), \quad t > \tau^* + 2\tau. \tag{22}$$

Therefore, if we can show

$$\limsup_{t\to\infty} \int_{t-\tau}^{t} P(s,\omega)\,ds = \infty, \tag{23}$$

for almost all $\omega \in \Omega_2$, then $t \mapsto Z(t,\omega)$ is oscillatory for a.a. $\omega \in \Omega_2$, by applying lemma 2.2 for each $\omega \in \Omega_2$. But as the zeros of $X(t,\omega)$ and $Z(t,\omega)$ coincide, this implies that $t \mapsto X(t,\omega)$ is oscillatory for a.a. $\omega \in \Omega_2$. This contradicts the construction of Ω_2, and so the result follows from (23). By (21), we see that (23) is true if

$$\limsup_{t\to\infty} \left(\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds \right)(\omega) = \infty, \quad \text{a.a. } \omega \in \Omega_2. \tag{24}$$

We now turn to proving this claim. For $t > \tau$, we have

$$\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds = \int_{t-\tau}^{t} e^{\frac{1}{2}\sigma^2(\langle M\rangle(s)-\langle M\rangle(s-\tau))} e^{\sigma(M(s-\tau)-M(s))}\,ds.$$

But $\langle M\rangle(s) - \langle M\rangle(s-\tau) \geq \underline{h}^2\tau$, so

$$\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds \geq e^{\frac{1}{2}\sigma^2\underline{h}^2\tau} \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s)-\widetilde{B}(\langle M\rangle(s-\tau))}\,ds.$$

Note that $t \mapsto \langle M\rangle(t)$ is a strictly increasing and C^1 function, with $\underline{h}^2 \leq \langle M\rangle'(t) \leq \overline{h}^2$. Therefore, $\langle M\rangle(s) > \overline{h}^2\tau$ for $s > \overline{h}^2\tau/\underline{h}^2$, and moreover,

$$\widetilde{B}(\langle M\rangle(s-\tau)) \leq \max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]} \widetilde{B}(\langle M\rangle(s)-u), \quad s > \overline{h}^2\tau/\underline{h}^2. \tag{25}$$

Next we suppose, without loss of generality, that $\sigma < 0$. It then follows from (25) for $t > \tau + \overline{h}^2\tau/\underline{h}^2$ that

$$\begin{aligned}
\overline{h}^2 \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s)-\widetilde{B}(\langle M\rangle(s-\tau)))}\,ds &\geq \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s)-\widetilde{B}(\langle M\rangle(s-\tau)))}\langle M\rangle'(s)\,ds \\
&\geq \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}\widetilde{B}(\langle M\rangle(s)-u)}\langle M\rangle'(s)\,ds \\
&\geq \int_{\langle M\rangle(t-\tau)}^{\langle M\rangle(t)} e^{-\sigma(\widetilde{B}(v)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}\widetilde{B}(v-u))}\,dv.
\end{aligned}$$

Since $\langle M\rangle(t-\tau) \le \langle M\rangle(t) - \underline{h}^2\tau$ for $t > \tau + \overline{h}^2\tau/\underline{h}^2$, we have

$$\int_{t-\tau}^{t} \varphi(s)^{-1}\varphi(s-\tau)\,ds \ge e^{\frac{1}{2}\sigma^2\underline{h}^2\tau} \int_{t-\tau}^{t} e^{-\sigma(\widetilde{B}(\langle M\rangle(s)-\widetilde{B}(\langle M\rangle(s-\tau))}\,ds$$
$$\ge e^{\frac{1}{2}\sigma^2\underline{h}^2\tau}\frac{1}{\overline{h}^2}\int_{\langle M\rangle(t)-\underline{h}^2\tau}^{\langle M\rangle(t)} e^{-\sigma(\widetilde{B}(v)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}\widetilde{B}(v-u))}\,dv.$$

Thus, as $\langle M\rangle(t)\to\infty$ as $t\to\infty$, a.s., and since it is clearly true that for $\sigma<0$ and any standard Brownian motion W

$$\limsup_{t\to\infty}\int_{t-\underline{h}^2\tau}^{t} e^{-\sigma(W(s)-\max_{u\in[\underline{h}^2\tau,\overline{h}^2\tau]}W(s-u)))}\,ds = \infty, \quad \text{a.s.,} \tag{26}$$

we have established (24), and completed the proof. A statement similar to (26) is proved in Lemma 5.1 of Appleby and Buckwar [1]. □

Remark 3.2. It is possible to comment upon the structure of the zero set $Z_X(\omega) = \{t \ge 0 : X(t,\omega) = 0\}$. As remarked earlier, this coincides with the set of times that $Z(\omega)$ is zero. But from (22), and the fact that $P(t,\omega) > 0$ for all $t > \tau^* + 2\tau$, it can readily be seen that the zeros of $Z(\omega)$, and hence of $X(\omega)$, must be isolated.

Remark 3.3. Theorem 1 in Gushchin and Küchler [5] guarantees the oscillation of solutions of (7) in the special case where $h(x)\equiv x$. The following condition is imposed, requiring a weaker regularity, but stronger monotonicity, condition on f:
If f is nondecreasing on $\mathbb{R}$, and there exist real δ and b with $\delta>0$, $b>0$, such that

$$\frac{f(x)}{x} \ge b \tag{27}$$

for all $x\ne 0$ satisfying $|x|<\delta$.
For example, $f(x)=\operatorname{sgn}(x)$ satisfies (27), and Gushchin and Küchler guarantee the existence of a unique strong solution of (7) for this choice of f, and $h(x)\equiv x$.
In Appleby and Kelly [2], the existence of unique solutions of (7) has been established with a continuity requirement on f, excluding functions with discontinuities, like $f(x)=\operatorname{sgn}(x)$, from consideration in this paper. But, by analogy with the deterministic theory, it should be possible to prove the existence and uniqueness of a strong solution of (7) with a weaker regularity requirement on f.

4 Nonoscillation of solutions of (7)

We now study non-oscillation of solutions of (7). Evidently, this requires f to satisfy different conditions than in Theorem 3.1. The hypotheses which we impose are the following: there exists $\gamma>1$ and $0<L\le\bar{L}$ such that

$$\lim_{x\to 0}\frac{|f(x)|}{|x|^\gamma} = L, \tag{28}$$

and

$$|f(x)| \le \bar{L}|x|^\gamma, \quad x\in\mathbb{R}. \tag{29}$$

As f obeys condition (28), it is said to be *superlinear* at zero. The significance of this property of f is emphasised in [9], where examples are given of deterministic equations with nonlinear coefficient satisfying (28) and (29) which do not oscillate.

Remark 4.1. If f satisfies a global linear bound of the form $|f(x)| \leq K(1+|x|)$ for all $x \in \mathbb{R}$ and some $K > 0$ then this along with (28) implies (29). Such a global linear bound, together with the local Lipschitz continuity of f, and with similar global linear bounds and local Lipschitz continuity on h, guarantees that (7) has a unique strong solution [8].

Remark 4.2. Suppose that f and h are locally Lipschitz continuous, and obey global linear bounds. Then it is automatically true [8] that the equation has a unique solution. Moreover, if it holds, we need only use conditions (10), (28), (8) and (9) in the sequel.

In advance of proving a result on the non-oscillation of solutions of (7), we require a technical result, the proof of which will require some further auxiliary processes. If M is the process defined in (18), we see that $\langle M\rangle(t) \to \infty$, as $t \to \infty$, a.s. because $\tilde{h}(X(t))^2 \geq \underline{h}^2 > 0$. Therefore, by the martingale time change theorem, there exists a standard Brownian motion $\widetilde{B}$ such that

$$M(t) = \widetilde{B}(\langle M\rangle(t)) \text{ for all } t \geq 0 \ a.s. \tag{30}$$

We also introduce the process $\widehat{B}$ given by

$$\widehat{B}(t) = \min_{t \leq w \leq t+\tau\overline{h}^2} \widetilde{B}(w). \tag{31}$$

Lemma 4.3. *Let $\widetilde{B}$ and $\widehat{B}$ be the processes defined in (30) and (31). If*

$$I = \int_0^\infty \varphi(s-\tau)^\gamma \varphi(s)^{-1}\, ds, \tag{32}$$

and

$$\bar{I} = \frac{1}{\underline{h}^2}\left\{\int_0^{\tau\bar{h}^2} e^{\frac{\sigma^2}{2}u-\sigma\widetilde{B}(u)}\, du + e^{\frac{1}{2}\sigma^2\bar{h}^2\tau}\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\hat{B}(u)}\, du\right\}, \tag{33}$$

then

$$I \leq \bar{I} < \infty. \tag{34}$$

Proof. We assume, without loss of generality, that $\sigma > 0$. First, we prove that

$$\int_0^\tau \varphi(s-\tau)^\gamma\varphi(s)^{-1}\, ds \leq \frac{1}{\underline{h}^2}\int_0^{\tau\bar{h}^2} e^{\frac{\sigma^2}{2}u-\sigma\widetilde{B}(u)}\, du, \quad \text{a.s.} \tag{35}$$

where $\widetilde{B}$ is defined via (18) and (30). Let M be the process defined by (18), with square variation given by (19). As $t \mapsto \langle M\rangle(t)$ is strictly increasing, and continuously differentiable, we may define $S(u) = \langle M\rangle^{-1}(u)$. Note also that $\langle M\rangle'(t) \geq \underline{h}^2$, a.s. Hence

$$\begin{aligned}\int_0^\tau \varphi(s-\tau)^\gamma\varphi(s)^{-1}ds &= \int_0^\tau \varphi(s)^{-1}ds = \int_0^\tau e^{\frac{\sigma^2}{2}\langle M\rangle(s)-\sigma M(s)}ds \\ &= \int_0^{\langle M\rangle(\tau)} e^{\frac{\sigma^2}{2}u-\sigma M(\langle M\rangle^{-1}(u))}\frac{1}{\langle M\rangle'(S(u))}du.\end{aligned}$$

Hence by (30), (35) is immediate. Now, we prove that

$$\int_0^\infty \varphi(s)^\gamma\varphi(s+\tau)^{-1}ds \leq \frac{1}{\underline{h}^2}e^{\frac{1}{2}\sigma^2\bar{h}^2\tau}\int_0^\infty e^{\{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\hat{B}(u)\}}du \tag{36}$$

where $\widehat{B}$ satisfies (31) above. Define $\tau(u) = \langle M\rangle(S(u)+\tau) - \langle M\rangle(S(u))$. Then $\tau(u) \le \tau\overline{h}^2$, a.s. Let $\widehat{B}$ be defined by (31). Observe that

$$\lim_{u\to\infty} \frac{\widehat{B}(u)}{u} = 0, \quad \text{a.s.} \tag{37}$$

and

$$\widehat{B}(u) \le \widetilde{B}(u+\tau(u)), \quad u \ge 0, \quad \text{a.s.}$$

Thus (37) and $\gamma > 1$ imply that

$$\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widehat{B}(u)}\,du < \infty, \quad \text{a.s.}$$

Now, as $\sigma > 0$

$$\begin{aligned}
&\int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widehat{B}(u)}\,du \\
&\quad\ge \int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)u+\sigma\gamma\widetilde{B}(u)-\sigma\widetilde{B}(u+\tau(u))}\,du \\
&\quad= \int_0^\infty e^{-\frac{1}{2}\sigma^2(\gamma-1)\langle M\rangle(s)+\sigma\gamma\widetilde{B}(\langle M\rangle(s))-\sigma\widetilde{B}(\langle M\rangle(s+\tau))}\langle M\rangle'(s)\,ds \\
&\quad= \int_0^\infty \varphi(s)^\gamma\varphi(s+\tau)^{-1}e^{\frac{1}{2}\sigma^2(\langle M\rangle(s)-\langle M\rangle(s+\tau))}\langle M\rangle'(s)\,ds \\
&\quad\ge \underline{h}^2 e^{-\frac{1}{2}\sigma^2\bar{h}^2\tau}\int_0^\infty \varphi(s)^\gamma\varphi(s+\tau)^{-1}\,ds
\end{aligned}$$

which is (36). Combining (35) and (36) yields (34). □

We now prove the main result in this section. To show that solutions of (7) do not oscillate with positive probability when f obeys (28) and (29), we show, for certain positive initial data that solutions can remain positive with non-zero probability. Suppose $\psi(t) > 0$ for all $t \in [-\tau, 0]$ and define the stopping time

$$\tau_\psi = \inf\{t > 0 : X(t,\psi) = 0\}, \tag{38}$$

where we set $\tau_\psi(\omega) = +\infty$ if $X(t,\omega) > 0$ for all $t \ge 0$. Suppose the solution of (7) is defined on Ω_0^*, with $\mathbb{P}[\Omega_0^*] = 1$, and define, as before $\Omega_1^* \subseteq \Omega_0^*$ the almost sure set on which φ exists, is strictly positive, and obeys conditions (16) and (17).

Theorem 4.4. *Let $(X(t))_{t\ge 0}$ be the unique continuous strong solution of (7). Suppose f satisfies (10), (28) and (29) and h satisfies (8) and (9). Suppose that $\psi(t) > 0$ for all $t \in [-\tau, 0]$ and τ_ψ is defined by (38). Then*

(i) There exists α^*, possibly infinite, such that for all $\alpha < \alpha^*$

$$\mathbb{P}[\tau_{\alpha\psi} < \infty] < 1.$$

(ii) Moreover

$$\lim_{\alpha\to 0} \mathbb{P}[\tau_{\alpha\psi} < \infty] = 0. \tag{39}$$

Proof. Define $\|\psi\| := \sup_{t\in[-\tau,0]} \psi(t) > 0$, $\Omega_\psi = \{\omega \in \Omega_1^* : \tau_\psi(\omega) < \infty\}$ and also

$$D_\psi = \left\{\omega \in \Omega_1^* : \bar{I}(\omega) < \frac{\psi(0)}{\|\psi\|^\gamma \bar{L}}\right\} \tag{40}$$

where $\bar{I}$ is given by (33), and $\bar{L}$ is given by (29). The main step in the analysis is to prove the following: if I is defined by (32), then

$$I(\omega) \geq \frac{1}{\bar{L}} \frac{\psi(0)}{\|\psi\|^\gamma}, \quad \omega \in \Omega_\psi. \tag{41}$$

If (41) is true, by (40), lemma 4.3, and (34), we have $\Omega_\psi \subseteq \overline{D_\psi}$.
Since $\bar{I} < \infty$, a.s., there exists some $C > 0$ such that

$$A := \{\omega \in \Omega_1^* : \bar{I}(\omega) > C\}$$

satisfies $\mathbb{P}[A] < 1$. Now, define α^* by $\frac{\alpha^*\psi(0)}{\bar{L}\alpha^{*\gamma}\|\psi\|^\gamma} = C$. Hence, for $\alpha < \alpha^*$, $\omega \in \overline{D_{\alpha\psi}}$ implies

$$\bar{I}(\omega) \geq \frac{1}{\bar{L}} \frac{\alpha\psi(0)}{\alpha^\gamma \|\psi\|^\gamma} > \frac{\alpha^*\psi(0)}{\bar{L}\alpha^{*\gamma}\|\psi\|^\gamma} = C$$

so $\omega \in A$, or $\overline{D_{\alpha\psi}} \subseteq A$. Therefore, for $\alpha < \alpha^*$,

$$\mathbb{P}[\Omega_{\alpha\psi}] \leq \mathbb{P}[\overline{D_{\alpha\psi}}] \leq \mathbb{P}[A] < 1,$$

as required for (i).
To prove (ii), note from (33), that $\bar{I} < \infty$, a.s. Hence, by (40), as $\gamma > 1$,

$$\lim_{\alpha\to 0} \mathbb{P}[\overline{D_{\alpha\psi}}] = \lim_{\alpha\to 0} \mathbb{P}\left[\bar{I} \geq \frac{\alpha\psi(0)}{\alpha^\gamma \|\psi\|^\gamma \bar{L}}\right] = 0.$$

Since $\mathbb{P}[\Omega_{\alpha\psi}] \leq \mathbb{P}[\overline{D_{\alpha\psi}}]$, (39) now follows.
It remains to justify (41). Now, suppose that $\omega \in \Omega_\psi$ and for $t \leq \tau_\psi + \tau$ define

$$L(t) = \begin{cases} \frac{f(X(t-\tau))}{X(t-\tau)^\gamma}, & X(t-\tau) \neq 0 \\ L, & X(t-\tau) = 0. \end{cases}$$

By (28) and (29), $t \mapsto L(t)$ is continuous, strictly positive and bounded, with $0 < L(t) \leq \bar{L}$. Also define

$$P(t) = L(t)\varphi(t-\tau)^\gamma \varphi(t)^{-1}, \quad 0 \leq t \leq \tau_\psi + \tau.$$

For $0 \leq t < \tau_\psi + \tau$, we have $X(t-\tau) > 0$, so

$$Z'(t) = -P(t)Z(t-\tau)^\gamma, \;\; 0 < t < \tau_\psi + \tau. \tag{42}$$

For $t = \tau_\psi + \tau$, $X(t-\tau) = 0$. Then $Z(t-\tau) = 0$, so $Z'(t) = 0$. But $-P(t)Z(t-\tau)^\gamma = 0$, so $Z'(t) = -P(t)Z(t-\tau)^\gamma$ once more. Hence $Z'(t) < 0$ for $t < \tau_\psi + \tau$. Thus $Z(t) \leq \psi(0)$ for all $t \in [0, \tau_\psi + \tau]$. Another way of writing this is to say that $Z(t-\tau) \leq \psi(0)$ for all $t \in [\tau, 2\tau + \tau_\psi]$.

Also, $Z(t-\tau)=\psi(t-\tau)\le\|\psi\|$ for all $t\in[0,\tau]$. So $Z(t-\tau)\le\|\psi\|$ for all $t\in[0,\tau_\psi+2\tau]$, which implies that $Z(t-\tau)\le\|\psi\|$ for all $t\in[0,\tau_\psi]$. Using this, and (42), we get

$$\begin{aligned}\psi(0)=-Z(\tau_\psi)+Z(0)&=\int_0^{\tau_\psi}P(s)Z(s-\tau)^\gamma\,ds\le\int_0^{\tau_\psi}P(s)\|\psi\|^\gamma\,ds\\&\le\bar{L}\|\psi\|^\gamma\int_0^{\tau_\psi}\varphi(s-\tau)^\gamma\varphi(s)^{-1}\,ds\le\bar{L}\|\psi\|^\gamma\int_0^{\infty}\varphi(s-\tau)^\gamma\varphi(s)^{-1}\,ds.\end{aligned}$$

Therefore

$$\frac{\psi(0)}{\bar{L}\|\psi\|^\gamma}\le\int_0^\infty\varphi(s-\tau)^\gamma\varphi(s)^{-1}ds=I(\omega),$$

as required. □

Remark 4.5. On a first viewing, it is perhaps not immediately apparent why the random variable $\bar{I}$ is introduced, as one might expect to be able to prove the results of theorem 4.4 with

$$D'_\psi=\left\{\omega\in\Omega_1^*:I(\omega)<\frac{\psi(0)}{\|\psi\|^\gamma\bar{L}}\right\},\quad\text{and}\quad A'=\{\omega\in\Omega_1^*:I(\omega)>C\}.$$

It is not automatic that $\lim_{\alpha\to0}\mathbb{P}[D'_{\alpha\psi}]=0$, as the random variable I depends on α, because it depends on X through the initial data $(\alpha\psi)$. However, the random variable $\bar{I}$ has the same distribution as a random variable independent of the initial data, and therefore independent of the scaling factor α. Indeed,

$$\mathbb{P}[D_{\alpha\psi}]=\mathbb{P}\left[\bar{I}'\ge\frac{\psi(0)}{\alpha^{\gamma-1}\|\psi\|^\gamma\bar{L}}\right]$$

where the random variable $\bar{I}'$ is given by

$$\bar{I}'=\frac{1}{\underline{h}^2}\left\{\int_0^{\tau\bar{h}^2}e^{\frac{\sigma^2}{2}u-\sigma B'(u)}\,du+e^{\frac12\sigma^2\bar{h}^2\tau}\int_0^\infty e^{-\frac12\sigma^2(\gamma-1)u+\sigma\gamma B'(u)-\sigma\min_{u\le w\le u+\tau}B'(w)}\,du\right\}$$

and B' is any standard Brownian motion. As $\bar{I}'$ is independent of the initial data,

$$\lim_{\alpha\to0}\mathbb{P}\left[\bar{I}'\ge\frac{\psi(0)}{\alpha^{\gamma-1}\|\psi\|^\gamma\bar{L}}\right]=0,$$

so $\mathbb{P}[\overline{D_{\alpha\psi}}]\to0$ as $\alpha\to0$.

Remark 4.6. If $\bar{I}$ is supported on $(0,\infty)$ then there is a positive probability of nonoscillation for any positive and continuous initial function ψ. $\bar{I}$ is an integral function of Brownian motion, and the distributions of some similar functionals are known. For example, Dufresne has shown in [4] that if B^* is a standard Brownian motion, $a>0$ and $\sigma\ne0$, then the random variable

$$L=\int_0^\infty e^{-as+\sigma B^*(s)}ds$$

is a.s. fintie, continuous and supported on $(0,\infty)$. In fact, Dufresne has determined the probability density of L.

While this is not direct evidence for $\bar{I}$ to be supported on $[0,\infty)$, the similar functional forms of L and $\bar{I}$ mean that the possibility cannot be automatically ruled out.

Remark 4.7. For any positive initial function ψ, a knowledge of the distribution of $\bar{I}$ allows us to construct explicitly the scaling factor α^* which guarantees the existence of non-oscillatory solutions with positive probability.

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