Elect. Comm. in Probab. **14** (2009), 232–244

ELECTRONIC COMMUNICATIONS in PROBABILITY

FIRST EIGENVALUE OF ONE-DIMENSIONAL DIFFUSION PROCESSES

JIAN WANG
School of Mathematics and Computer Science, Fujian Normal University, Fuzhou, 350007, P.R. China
email: jianwang@fjnu.edu.cn

Submitted January 6, 2008, *accepted in final form* May 3, 2009

AMS 2000 Subject classification: 60J25, 60J27
Keywords: First Dirichlet eigenvalue, Hardy inequality, variational formula, transience, recurrence, diffusion operators

Abstract

We consider the first Dirichlet eigenvalue of diffusion operators on the half line. A criterion for the equivalence of the first Dirichlet eigenvalue with respect to the maximum domain and that to the minimum domain is presented. We also describe the relationships between the first Dirichlet eigenvalue of transient diffusion operators and the standard Muckenhoupt's conditions for the dual weighted Hardy inequality. Pinsky's result [17] and Chen's variational formulas [8] are reviewed, and both provide the original motivation for this research.

1 Introduction and Main Results

In this paper, we deal with explicit bounds of the first Dirichlet eigenvalue for diffusion operators on the half line $\mathbb{R}_+ := [0,\infty)$. The work is a continuation or a supplement of [17, 5, 6], and is also inspired by analogous research for birth-death processes in [7, 19]. Let $a(x)$ be positive everywhere on $(0,\infty)$. For any measurable function $b(x)$ on $(0,\infty)$, define $C(x) = \int_0^x b(u)/a(u)du$ for $x > 0$ and a measure $\mu(dx) = a(x)^{-1}e^{C(x)}dx$. Consider the diffusion operator with diffusion coefficient a and drift b

$$L := a(x)d^2/dx^2 + b(x)d/dx$$

on $\mathbb{R}_+$ with the Dirichlet boundary condition at $x = 0$. Then, L is a non-negative, self-adjoint operator on $(\mathbb{R}_+, \mathbb{L}^2(\mu))$, and it corresponds to a non-negative, Markovian symmetric and closable bilinear form (see [10])

$$D(f,g) = \int_0^\infty a(x)f'(x)g'(x)\mu(dx)$$

defined for $f, g \in C_0^\infty(\mathbb{R}_+)$, the space of smooth functions with compact support on $\mathbb{R}_+$. As usual, denote by $\|\cdot\|$ and $(\cdot,\cdot)$ the norm and the inner product on $\mathbb{L}^2(\mu)$, respectively. Let λ_0 be the first eigenvalue of Dirichlet diffusion operator $-L$. The classical Rayleigh-Ritz variational formula (c.f.

see [18, 16]) gives us

$$\lambda_0 = \inf_{f \in C_0^\infty(\mathbb{R}_+) \text{ with } f(0)=0} \frac{-(Lf, f)}{\|f\|^2}.$$

That is,

$$\lambda_0 = \inf\{D(f) : f \in C_0^\infty(\mathbb{R}_+), \|f\| = 1, f(0) = 0\}. \tag{1.1}$$

Our starting point is the explicit bounds up to multiplicative constant 4 for λ_0, taken from [17; Theorem 1].

Theorem 1.1. [Pinsky's Result] *If the operator L is recurrent, i.e.* $\int_0^\infty e^{-C(x)}dx = \infty$, *define*

$$\delta = \sup_{x>0} \int_0^x e^{-C(y)}dy \int_x^\infty a(y)^{-1} e^{C(y)}dy. \tag{1.2}$$

If L is transient, i.e. $\int_0^\infty e^{-C(x)}dx < \infty$, *let* $h(x) = \int_x^\infty e^{-C(u)}du$ *and define*

$$\delta = \sup_{x>0} \left(h(x)^{-1} - h(0)^{-1}\right) \int_x^\infty h(y)a(y)^{-1} e^{C(y)}dy. \tag{1.3}$$

Then

$$(4\delta)^{-1} \le \lambda_0 \le \delta^{-1}.$$

Theorem 1.1 shows that the bounds for λ_0 take two possible forms depending on whether $\int_0^\infty e^{-C(x)}dx$ is finite or infinite. For the clarity of exposition, we denote δ given in (1.2) and (1.3) by δ_1 and δ_2, respectively. As mentioned in [17; Remark 3], it is δ_1 (not δ_2) that coincides with the standard Muckenhoupt's constant for the weighted Hardy inequality (H1):

$$\int_0^\infty \left(\int_0^x f(t)dt\right)^2 u(x)dx \le C \int_0^\infty f(x)^2 v(x)dx \qquad \text{for all } f \ge 0, \tag{1.4}$$

where u and v are non-negative weighted functions on $\mathbb{R}_+$. Let C_1 be the optimal constant in (1.4). Then, [15] gives us

$$B_1 \le C_1 \le 4B_1,$$

where

$$B_1 = \sup_{x>0} \int_x^\infty u(t)dt \int_0^x v(t)^{-1}dt.$$

Set $u(x) = a(x)^{-1}e^{C(x)}$ and $v(x) = e^{C(x)}$. It follows that $\delta_1 = B_1$. Recently the Hardy inequality (1.4) has been extensively applied to studying the first non-trivial eigenvalue of diffusion operators and related functional inequalities (c.f. see [2, 8, 12, 14, 21]). For example, assume that the diffusion operator L is ergodic, i.e. $\int_0^\infty e^{-C(x)}dx = \infty$ and $\mu(\mathbb{R}_+) < \infty$. The other first eigenvalue $\widetilde{\lambda}_0$, slightly different from λ_0 in (1.1), is given by

$$\widetilde{\lambda}_0 = \inf\{D(f) : f \in C^1(\mathbb{R}_+), \|f\| = 1, f(0) = 0\}, \tag{1.5}$$

where $C^1(\mathbb{R}_+)$ denotes the space of continuously differential functions. Then, it has been proven in [8; Theorem 5.7 (2)] or [4; Theorem 1.1] that

$$(4\delta_1)^{-1} \le \widetilde{\lambda}_0 \le \delta_1^{-1}.$$

The same estimations hold for λ_0 and $\widetilde{\lambda}_0$ when the operator L is ergodic. However, since the class of admissible functions in (1.5) is larger than that in (1.1), it only follows that $\lambda_0 \geq \widetilde{\lambda}_0$. In view of these facts, it is natural to question that whether $\lambda_0 = \widetilde{\lambda}_0$ in this case. The answer is positive, and in fact we have a stronger assertion.

Theorem 1.2. *For λ_0 and $\widetilde{\lambda}_0$ given by* (1.1) *and* (1.5) *respectively, we have $\lambda_0 = \widetilde{\lambda}_0$ iff*

$$\mu([0,\infty)) + \int_0^\infty e^{-C(x)} dx = \infty. \tag{1.6}$$

Particularly, (1.6) *is satisfied for recurrent diffusion operators on the half line $\mathbb{R}_+$.*

Theorem 1.2 explains the apparent gap between Muckenhoupt's conditions for the Hardy inequality (1.4) and Pinsky's result in Theorem 1.1. According to Feller's classification of boundary points for one-dimensional diffusion, (1.6) means that the boundary point ∞ is not regular ([11, 13]). Thus, in this case the diffusion operator L determines the process uniquely ([1]). Another viewpoint of (1.6) comes from the theory of Dirichlet forms. Let $X_t^{\min}$ be the minimal process generated by $(D, C_0^\infty(\mathbb{R}_+))$, and $X_t^{\max}$ be the maximal process generated by $(D, \mathscr{D}(D))$, where $\mathscr{D}(D) = \{f \in \mathbb{L}^2(\mu) : D(f) < \infty\}$. Then, (1.6) is equivalent to $\overline{C_0^\infty(\mathbb{R}_+)}^{\|\cdot\|_{D_1}} = \mathscr{D}(D)$, where the norm is $\|f\|_{D_1} = D(f) + \|f\|^2$. Therefore, $X_t^{\min}$ coincides with $X_t^{\max}$ under the condition (1.6). These explanations of (1.6) describe rough idea about the proof of the first assertion in Theorem 1.2. The second conclusion in Theorem 1.2 is a direct consequence of the fact that condition (1.6) is weaker than the non-explosive condition, i.e. $\int_0^\infty \mu([0,x)) e^{-C(x)} dx = \infty$ (c.f. see (2.2) below).

Next, we turn to the transient situation. Theorems 1.1 and Theorem 1.2 show that in transient settings the bounds about λ_0 are not the same as what the Muckenhoupt inequalities (1.4) yield. However, we will see that these bounds are closely connected with the dual Hardy inequality (H2):

$$\int_0^\infty \left(\int_x^\infty f(t) dt \right)^2 u(x) dx \leq C \int_0^\infty f(x)^2 v(x) dx \qquad \text{for all } f \geq 0. \tag{1.7}$$

The difference between (1.4) and (1.7) only lies on small change (i.e. the range of integral inside) in the left hand side of these two equalities, but the assertions are significantly distinct. Actually, let C_2 be the optimal constant in (1.7). Then, by [15],

$$B_2 \leq C_2 \leq 4B_2,$$

where

$$B_2 = \sup_{x>0} \int_0^x u(t) dt \int_x^\infty v(t)^{-1} dt.$$

The constant B_2 is completely different from B_1 associated with the Hardy inequality (1.4). Just like δ_1 in Theorem 1.1, we define $\delta_T = B_2$ by letting $u(x) = a(x)^{-1} e^{C(x)}$ and $v(x) = e^{C(x)}$, i.e.

$$\delta_T = \sup_{x>0} \int_0^x a(t)^{-1} e^{C(t)} dt \int_x^\infty e^{-C(t)} dt.$$

Then the following conclusion holds for λ_0 given by (1.1).

Theorem 1.3. *For transient diffusion operators L on the half line $\mathbb{R}_+$,*

$$\lambda_0 > 0 \quad \text{iff} \quad \delta_{\mathrm{T}} < \infty.$$

Furthermore, define

$$\lambda_{0,T} = \inf\{D(f) : f \in C_0^\infty(\mathbb{R}_+), \|f\| = 1\}. \tag{1.8}$$

We have

$$(4\delta_T)^{-1} \le \lambda_{0,T} \le \delta_T^{-1}. \tag{1.9}$$

The absence of the condition $f(0) = 0$ in the definition (1.8) indicates that $\lambda_{0,T}$ is in fact the first Neumann eigenvalue of transient diffusion operator L on $\mathbb{R}_+$; the proof of this equivalence being deferred to Section 2.2. As an alternative probabilistic viewpoint (c.f. see [7; Proposition 1.1]), $\lambda_{0,T}$ is the rate of the exponential decay of transient diffusion process, and it is the largest ε such that $\|P_t f\| \le \|f\| e^{-\varepsilon t}$ for all $f \in C_0^\infty(\mathbb{R}_+)$. Theorem 1.3 establishes the relationships among the dual Hardy inequality (1.7), the first Dirichlet eigenvalue λ_0 and the first eigenvalue $\lambda_{0,T}$ of transient diffusion operators. As a byproduct, Theorem 1.3 implies that $\lambda_0 > 0$ iff $\lambda_{0,T} > 0$, though it is true on general grounds that $\lambda_0 \ge \lambda_{0,T}$.

The proofs of our theorems are presented in the next section. Here, Chen's variational formulas (c.f. [4, 5, 6]) for the first eigenvalue of ergodic diffusion processes are reviewed. We also use them to improve Theorem 1.1. Although we restrict ourselves on the half line in this paper, the corresponding results for the whole line, higher-dimensional situations and Riemannian manifolds follow from similar ideas in [17, 5, 6].

2 Proofs

2.1 Improvement of Theorem 1.1 and Proof of Theorem 1.2

We begin with the proof of Theorem 1.2.

Proof of Theorem 1.2. Set $\mathscr{D}(D) = \{f \in C^1(\mathbb{R}^+) : D(f) + \|f\| < \infty\}$. Then, it holds that

$$\widetilde{\lambda}_0 = \inf\{D(f) : f \in \mathscr{D}(D), \|f\| = 1 \text{ and } f(0) = 0\}.$$

To prove our conclusion, it suffices to verify that

$$\overline{C_0^\infty(\mathbb{R}^+)}^{\|\cdot\|_{D_1}} = \mathscr{D}(D). \tag{2.1}$$

Adopting the standard Feller's notations (c.f. see [11, 13]) for one-dimensional diffusion operator, $\mu(dx)$ and $s(x) := \int_0^x e^{-C(u)} du$ are called speed measure and scale function, respectively. Moreover, L can be expressed as

$$Lf = \frac{d}{d\mu}\frac{df}{ds} = D_\mu D_s f,$$

and the boundary behavior of the corresponding process is also described by speed measure and scale function. Particularly, ∞ is said to be not regular if and only if

$$\int_0^\infty s(x) d\mu(dx) + \int_0^\infty \mu([0,x]) s'(x) dx = \infty. \tag{2.2}$$

Now, denote by $\mathscr{H}$ the complement of $\overline{C_0^\infty(\mathbb{R}^+)}^{\|\cdot\|_{D_1}}$ in the Hilbert space $(\mathscr{D}(D), \|\cdot\|_{D_1})$. Following the arguments of [10; Example 1.2.2], $\mathscr{H} = \emptyset$ if and only if ∞ is not a regular point. Therefore, (2.1) and (2.2) are equivalent.

Next, we claim that (1.6) and (2.2) are also equivalent. On the one hand, if

$$\int_0^\infty s(x)d\mu(dx) + \int_0^\infty \mu([0,x])s'(x)dx = \infty,$$

then

$$2\left[\mu([0,\infty)) + \int_0^\infty e^{-C(x)}dx\right]^2 \geq \int_0^\infty s(x)d\mu(dx) + \int_0^\infty \mu([0,x])s'(x)dx = \infty,$$

and so

$$\mu([0,\infty)) + \int_0^\infty e^{-C(x)}dx = \infty.$$

On the other hand, assume that

$$\mu([0,\infty)) + \int_0^\infty e^{-C(x)}dx = \infty.$$

If

$$\int_0^\infty s(x)d\mu(dx) = \infty,$$

then

$$\int_0^\infty s(x)d\mu(dx) + \int_0^\infty \mu([0,x])s'(x)dx = \infty;$$

if

$$\int_0^\infty s(x)d\mu(dx) < \infty,$$

then, the integration by parts formula yields

$$\int_0^\infty \mu([0,x])s'(x)dx = \mu([0,x])s(x)\Big|_0^\infty - \int_0^\infty s(x)\mu(dx) = \mu([0,\infty))s(\infty) - \int_0^\infty s(x)d\mu(dx) = \infty,$$

and so it also holds that

$$\int_0^\infty s(x)d\mu(dx) + \int_0^\infty \mu([0,x])s'(x)dx = \infty.$$

The required assertion follows by the above facts. □

According to Theorem 1.2, in recurrent settings we can use Chen's variational formulas to improve the estimations for λ_0 in Theorem 1.1. For instance, define four classes of functions:

$$\mathscr{F}_{II} = \{f \in C(\mathbb{R}_+) : f(0) = 0 \text{ and } f > 0\},$$

$$\widetilde{\mathscr{F}}_{II} = \{f \in C(\mathbb{R}_+) : f(0) = 0, \text{ there exists } x_0 \text{ such that } f = f(\cdot \wedge x_0) \text{ and } f|_{(0,x_0]} > 0\},$$

$$\mathscr{F}_I = \{f \in C^1(\mathbb{R}_+) : f(0) = 0 \text{ and } f' > 0\},$$

$$\widetilde{\mathscr{F}}_I = \{f \in C^1(\mathbb{R}_+) : f(0) = 0, \text{ there exists } x_0 \text{ such that } f = f(\cdot \wedge x_0), f \in C^1([0,x_0]) \text{ and } f'|_{(0,x_0)} > 0\}.$$

Then, [4, 5, 6] give us the following two powerful variational formulas for λ_0 given by (1.1).

Theorem 2.1. **[Chen's Formulas]** *For recurrent diffusion operator L,*

$$\inf_{f\in\widetilde{\mathscr{F}}_{II}}\sup_{x>0} II(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I}}\sup_{x>0} I(f)(x)^{-1} \geq \lambda_0 \geq \sup_{f\in\mathscr{F}_{I}}\inf_{x>0} I(f)(x)^{-1} = \sup_{f\in\mathscr{F}_{II}}\inf_{x>0} II(f)(x)^{-1}, \tag{2.3}$$

where

$$I(f)(x) = \frac{e^{-C(x)}}{f'(x)}\int_x^\infty \left[fe^C/a\right](u)du, \qquad II(f)(x) = \frac{1}{f(x)}\int_0^x dy e^{-C(y)}\int_y^\infty \left[fe^C/a\right](u)du.$$

Moreover, both inequalities in (2.3) *become equalities if a and b are continuous.*

Next, we turn to the transient situation. To handle this case, [17] employs the h-transformed operator $L_h := h^{-1}\circ L\circ h$, where $h = \int_\cdot^\infty e^{-C(u)}du$. When written out, we get

$$L_h = h^{-1}\circ L\circ h = a(x)d/dx^2 + (b(x) + 2h(x)^{-1}a(x)h'(x))d/dx := a(x)d/dx^2 + b^*(x)d/dx.$$

Letting

$$\widetilde{C}(x) = \int_0^x b^*(u)/a(u)du = C(x) + 2\ln(h(x)/h(0)),$$

one has

$$\int_0^\infty e^{-\widetilde{C}(u)}du = h(0)^{-2}\int_0^\infty e^{-C(u)}h(u)^{-2}du = \infty.$$

Thus, the diffusion operator L_h with diffusion coefficient a and the new drift b^* is recurrent. Note that the spectrum is variant under h-transforms. So, $\lambda_0 = \lambda_0^h$, where λ_0^h is the first Dirichlet eigenvalue of L_h on $\mathbb{R}_+$ with an absorbing boundary at $x = 0$. That is, the transformed operator L_h reduces transient cases into recurrent ones. It immediately follows that $(4\delta_2^*)^{-1} \leq \lambda_0 \leq \delta_2^{*-1}$, where

$$\delta_2^* = \sup_{x>0}\int_0^x e^{-C^*(y)}dy\int_x^\infty a(y)^{-1}e^{C^*(y)}dy.$$

Furthermore,

$$\begin{aligned}\delta_2^* &= \sup_{x>0} h^2(0)\int_0^x e^{-C(y)}h(y)^{-2}dy\cdot h(0)^{-2}\int_x^\infty a(y)e^{C(y)}h^2(y)dy\\ &= \sup_{x>0}(h(x)^{-1} - h(0)^{-1})\int_x^\infty a(y)e^{C(y)}h^2(y)dy,\end{aligned} \tag{2.4}$$

which is just δ_2 defined in (1.3).

Again we use Chen's variational formulas to refine the estimations for λ_0 in Theorem 1.1. Theorem 2.1 along with the remark above yields that the following statement for λ_0 given by (1.1).

Theorem 2.2. *For transient diffusion operator L,*

$$\inf_{f\in\widetilde{\mathscr{F}}_{II}}\sup_{x>0} II^*(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I}}\sup_{x>0} I^*(f)(x)^{-1} \geq \lambda_0 \geq \sup_{f\in\mathscr{F}_{I}}\inf_{x>0} I^*(f)(x)^{-1} = \sup_{f\in\mathscr{F}_{II}}\inf_{x>0} II^*(f)(x)^{-1}, \tag{2.5}$$

where

$$I^*(f)(x)=\frac{e^{-C^*(x)}}{f'(x)}\int_x^\infty \big[fe^{C^*}/a\big](u)du, \qquad II^*(f)(x)=\frac{1}{f(x)}\int_0^x dy e^{-C^*(y)}\int_y^\infty \big[fe^{C^*}/a\big](u)du,$$

$C^*(x)=C(x)+2\ln h(x)$, $h(x)=\int_x^\infty e^{-C(u)}du$ and $C(x)=\int_0^x b(u)/a(u)du$. Moreover, both inequalities in (2.5) *become equalities if a and b are continuous.*

The power of (2.3) and (2.5) are the following: (1) the variational formulas with single integral yield the Muckenhoupt's estimations in Theorem 1.1 (see [5; Theorem 2.1] for ergodic cases); (2) the variational formulas containing double integrals allow for more sharp estimates on λ_0 (see [6; Theorem 1.2] for ergodic cases).

For the completeness, we will prove that the formula (2.5) implies the second assertion in Theorem 1.1. By similar arguments, Theorem 2.1 also improves the first assertion in Theorem 1.1. Firstly, the proof of [5; Theorem 1.1] yields

$$\sup_{f\in\mathscr{F}_I}\inf_{x>0} I^*(f)(x)^{-1}\ge (4\delta_2)^{-1}. \tag{2.6}$$

In fact, take $f(x)=\big(\int_0^x e^{-C^*(u)}du\big)^{1/2}$. Then, $f\in\mathscr{F}_I$ and

$$I^*(f)(x)=2\left(\int_0^x e^{-C^*(u)}du\right)^{1/2}\int_x^\infty \left(\int_0^z e^{-C^*(u)}du\right)^{1/2} a(z)^{-1}e^{C^*(z)}dz.$$

By the integration by parts formula and (2.4),

$$\begin{aligned}
&\int_x^\infty \left(\int_0^z e^{-C^*(u)}du\right)^{1/2} a(z)^{-1}e^{C^*(z)}dz\\
&\qquad = -\int_x^\infty \left(\int_0^z e^{-C^*(u)}du\right)^{1/2} d\left(\int_z^\infty a(u)^{-1}e^{C^*(u)}du\right)\\
&\qquad \le \left(\int_0^x e^{-C^*(u)}du\right)^{1/2}\int_x^\infty a(u)^{-1}e^{C^*(u)}du\\
&\qquad\quad +\frac{1}{2}\int_x^\infty \left(\int_z^\infty a(u)^{-1}e^{C^*(u)}du\right)\left(\int_0^z e^{-C^*(u)}du\right)^{-1/2} e^{-C^*(z)}dz\\
&\qquad \le \delta_2\left(\int_0^x e^{-C^*(u)}du\right)^{-1/2}+\frac{\delta_2}{2}\int_x^\infty \left(\int_0^z e^{-C^*(u)}du\right)^{-3/2} e^{-C^*(z)}dz\\
&\qquad = \delta_2\left(\int_0^x e^{-C^*(u)}du\right)^{-1/2}-\delta_2\int_x^\infty d\left(\int_0^z e^{-C^*(u)}du\right)^{-1/2}\\
&\qquad \le 2\delta_2\left(\int_0^x e^{-C^*(u)}du\right)^{-1/2}.
\end{aligned}$$

Thus,

$$I^*(f)(x)\le 2\left(\int_0^x e^{-C^*(u)}du\right)^{1/2}\cdot 2\delta_2\left(\int_0^x e^{-C^*(u)}du\right)^{-1/2}=4\delta_2.$$

The required assertion (2.6) follows.

Secondly, we claim that

$$\delta_2^{-1} \geq \inf_{f\in\widetilde{\mathscr{F}}_I} \sup_{x>0} I^*(f)(x)^{-1}. \tag{2.7}$$

For fixed $x > 0$, take $f(y) := f_x(y) = \int_0^{x\wedge y} e^{-C^*(u)}du$. Then $f \in \widetilde{\mathscr{F}}_I$. For any $y < x$, $f'(y) = e^{-C^*(y)}$, and

$$\begin{aligned}I^*(f)(y) &= \int_y^\infty \int_0^{x\wedge u} e^{-C^*(s)}ds a(u)^{-1}e^{C^*(u)}du \\ &\geq \int_0^x e^{-C^*(u)}du \int_x^\infty a(u)^{-1}e^{C^*(u)}du.\end{aligned}$$

Noting that $f'(y) = 0$ for $y \geq x$,

$$\inf_{y>0} I^*(f)(y) = \inf_{0<y<x} I^*(f)(y) \geq \int_0^x e^{-C^*(u)}du \int_x^\infty a(u)^{-1}e^{C^*(u)}du.$$

Since x is arbitrary,

$$\sup_{f\in\widetilde{\mathscr{F}}_I} \inf_{y>0} I^*(f)(y) \geq \sup_{x>0} \int_0^x e^{-C^*(u)}du \int_x^\infty a(u)^{-1}e^{C^*(u)}du = \delta_2,$$

which gives us (2.7). Therefore, according to (2.6) and (2.7), the required conclusion follows.

We end this subsection with an illustration of the improvements offered by Theorem 2.1 over the bounds provided in Theorem 1.1. Let $a(x) = 1$ and $b(x) = -x$. The corresponding diffusion is recurrent and it is an easy exercise that $\lambda_0 = 1$ with eigenfunction $f(x) = x$. From [17; Remark 1] (or see [5; Example 3.9]) we know

$$\delta_1 = \sup_{x>0} \int_0^x e^{t^2/2}dt \int_x^\infty e^{-t^2/2}dt \approx 0.4788.$$

Now, take $f(x) = \big(\int_0^x e^{u^2/2}du\big)^{1/2}$. Then,

$$\delta_{II}' := \sup_{x>0} II(f)(x) = \sup_{x>0} \left(\int_0^x e^{u^2/2}du\right)^{-1/2} \int_0^x e^{y^2/2} \int_y^\infty e^{-t^2/2}\left(\int_0^t e^{u^2/2}du\right)^{1/2} dt\,dy \approx 1.0928.$$

On the other hand, for every $x > 0$, take $f_x(y) := \int_0^{x\wedge y} e^{u^2/2}du$. Then,

$$\begin{aligned}\delta_{II}'' &:= \sup_{x>0}\inf_{z>0} II(f_x)(z) = \sup_{x>0}\inf_{z>0} \left(\int_0^{z\wedge x} e^{u^2/2}du\right)^{-1} \int_0^z e^{y^2/2} \int_y^\infty e^{-t^2/2}\left(\int_0^{x\wedge t} e^{u^2/2}du\right)dt\,dy \\ &= \sup_{x>0}\left\{\left[\left(\int_0^x e^{u^2/2}du\right)^{-1} \int_0^x e^{-u^2/2}\left(\int_0^u e^{t^2/2}dt\right)^2 du\right] + \int_x^\infty e^{-t^2/2}dt \int_0^x e^{u^2/2}du\right]\right\} \\ &\approx 0.9285.\end{aligned}$$

Therefore, for this example Theorem 1.1 gives us

$$\delta^{-1} = 2.0886 > \lambda_0 = 1 > (4\delta)^{-1} = 0.5221;$$

while Theorem 2.1 yields that

$$\delta_{II}''^{-1} = 1.0770 > \lambda_0 = 1 > \delta_{II}'^{-1} = 0.9150.$$

2.2 Proof of Theorem 1.3 and Extensions

We begin by proving that the variational formula (1.8) produces the familiar Neumann eigenvalue (c.f. see [9, 20, 3])

$$\lambda_{0,T}^* = \inf\{D(f) : f \in C_0^\infty(\mathbb{R}_+), \|f\| = 1, f'(0) = 0\}. \tag{2.8}$$

Proof. Clearly, $\lambda_{0,T}^* \geq \lambda_{0,T}$. For any $f \in C_0^\infty(\mathbb{R}_+)$ with $\|f\| = 1$, there exists $z > 0$ such that $\mathrm{supp} f \subset [0,z]$, $f(z) = 0$ and $f'|_{[z,\infty)} = 0$. Define

$$f^*(x) = \int_x^\infty |f'|(t)dt \qquad \text{for } x > 0.$$

Then, $f^* \in C_0^\infty(\mathbb{R}_+)$ and $f^* \geq f$. We get

$$\frac{D(f^*)}{\|f^*\|^2} = \frac{\int_0^\infty a(x)f'(x)^2\mu(dx)}{\left(\int_0^\infty f^*(x)^2\mu(dx)\right)^2} \leq \frac{\int_0^\infty a(x)f'(x)^2\mu(dx)}{\left(\int_0^\infty f(x)^2\mu(dx)\right)^2} = D(f).$$

Note the the function f^* is decreasing, so $f^{*\prime}(0) = 0$. This fact along the inequality above gives us $\lambda_{0,T}^* \leq D(f)$, thanks to the definition (2.8) of $\lambda_{0,T}^*$. Since f is arbitrary, it follows that $\lambda_{0,T}^* \leq \lambda_{0,T}$. Therefore, $\lambda_{0,T}^* = \lambda_{0,T}$. □

Now, we are ready to prove Theorem 1.3.
Proof of Theorem 1.3. (1) Define

$$\delta_2^* = \sup_{x>0} h(x)^{-1} \int_x^\infty h^2(y)a(y)^{-1}e^{C(y)}dy.$$

Then, it holds that

$$\delta_2^*/2 \leq \delta_T \leq 2\delta_2^*. \tag{2.9}$$

Let $g(x) = \mu([0,x)) = \int_0^x e^{C(y)}dy$. Thanks to the facts that $h' \leq 0$ and $h(\infty) = 0$, for $x > 0$,

$$\begin{aligned}\int_x^\infty h^2(y)a(y)^{-1}e^{C(y)}dy &= \int_x^\infty h^2(y)dg(y) \leq h^2(\infty)g(\infty) - 2\int_x^\infty g(y)h(y)h'(y)dy \\ &\leq \left(\sup_{x>0} h(x)g(x)\right)\left(h(\infty) - 2\int_x^\infty h'(y)dy\right) = 2h(x)\left(\sup_{x>0} h(x)g(x)\right).\end{aligned}$$

That is,

$$h(x)^{-1}\int_x^\infty h^2(y)a(y)^{-1}e^{C(y)}dy \leq 2\delta_T.$$

This yields the first inequality in (2.9) upon taking the supremum with respect to $x > 0$. For any

$x > 0$,

$$\begin{aligned}\int_0^x a(y)^{-1}e^{C(y)}dy &= \int_0^x a(y)^{-1}e^{C(y)}h^2(y)h(y)^{-2}dy\\ &= -\int_0^x h(y)^{-2}d\left(\int_y^\infty a(z)^{-1}e^{C(z)}h^2(z)dz\right)\\ &\le h(0)^{-2}\int_0^\infty a(z)^{-1}e^{C(z)}h^2(z)dz + \int_0^x \left(\int_y^\infty a(z)^{-1}e^{C(z)}h^2(z)dz\right)dh(y)^{-2}\\ &\le \left(\sup_{x>0} h(x)^{-1}\int_x^\infty h^2(y)a(y)^{-1}e^{C(y)}dy\right)\left(h(0)^{-1} + \int_0^x h(y)dh(y)^{-2}\right)\\ &\le 2h(x)^{-1}\left(\sup_{x>0} h(x)^{-1}\int_x^\infty h^2(y)a(y)^{-1}e^{C(y)}dy\right).\end{aligned}$$

Thus,

$$h(x)\int_0^x a(y)^{-1}e^{C(y)}dy \le 2\delta_2^*.$$

The second required inequality in (2.9) also follows.
Now, if $\delta_2 = \infty$, then $\delta_2^* = \infty$. Assume that $\delta_2^* = \infty$. If $\int_0^\infty h(y)^2 a(y)^{-1}e^{C(y)}dy = \infty$, then $\delta_2 = \infty$; if $\int_0^\infty h(y)^2 a(y)^{-1}e^{C(y)}dy < \infty$, then

$$\begin{aligned}\delta_2 &\ge \sup_{x>1}\left(h(x)^{-1} - h(0)^{-1}\right)\int_x^\infty h(y)^2 a(y)^{-1}e^{C(y)}dy\\ &\ge \left(1 - h(1)/h(0)\right)\sup_{x>1} h(x)^{-1}\int_x^\infty h(y)^2 a(y)^{-1}e^{C(y)}dy = \infty,\end{aligned}$$

by using the decreasing property of h. Hence, the qualities δ_2 and δ_2^* are equivalent. The first required conclusion follows by this assertion and (2.9).
(2) According to the definition of $\lambda_{0,T}$, the Muckenhoupt's condition for (1.7) gives us $\lambda_{0,T} \ge (4\delta)^{-1}$. We now prove that $\lambda_{0,T} \le \delta^{-1}$. Fix $m < n$. Take $f(x) = \int_m^n e^{-C(z)}dz$, $x \in [0,m]$; $f(x) = \int_x^n e^{-C(z)}dz$, $x \in [m,n]$ and $f(x) = 0$, $x > n$. Then,

$$\|f\|^2 = \mu([0,m])\left(\int_m^n e^{-C(u)}du\right)^2 + \int_m^n \left(\int_x^n e^{-C(z)}dz\right)^2 \mu(dx)$$

and

$$D(f) = \int_m^n e^{-2C(x)}\mu(dx) = \int_m^n e^{-C(x)}dx.$$

Hence,

$$\lambda_{0,T} \le D(f)/\|f\|^2 \le \left(\mu([0,m])\int_m^n e^{-C(u)}du\right)^{-1}.$$

Letting $n \to \infty$ and then taking infimum with respect to $m > 0$, the second required assertion follows. The proof is complete. $\square$

Remark 2.3. As shown by part (2) in the proof above, the admissible function $f \in C_0^\infty(\mathbb{R}_+)$ for the definition of $\lambda_{0,T}$ only satisfies that $f(\infty)=0$ not $f(0)=0$. This is the crucial distinction between $\lambda_{0,T}$ and $\widetilde{\lambda}_0$ given by (1.5). Just due to this difference, one only links $\lambda_{0,T}$ with the dual Hardy inequality (1.7), but connects $\widetilde{\lambda}_0$ with the other Hardy inequality (1.4).

To conclude this section, we give a stronger conclusion (i.e. the variational formula) for $\lambda_{0,T}$ than that in Theorem 1.3. Similar to Theorem 2.1, we need other four classes of functions:

$$\begin{aligned}
\mathscr{F}_{II_T} &= \{f \in C(\mathbb{R}_+) : f > 0 \text{ and } f(\infty)=0\},\\
\widetilde{\mathscr{F}}_{II_T} &= \{f \in C(\mathbb{R}_+) : \text{ there exists } x_0 \text{ such that } f = f(\cdot \wedge x_0), f|_{[0,x_0)} > 0 \text{ and } f(x_0)=0\},\\
\mathscr{F}_{I_T} &= \{f \in C^1(\mathbb{R}_+) : f' < 0 \text{ and } f(\infty)=0\},\\
\widetilde{\mathscr{F}}_{I_T} &= \{f \in C^1(\mathbb{R}_+) : \text{ there exists } x_0 \text{ such that } f = f(\cdot \wedge x_0), f \in C^1([0,x_0]), f'|_{(0,x_0)} < 0 \text{ and } f(x_0)=0\}.
\end{aligned}$$

Theorem 2.4. *The following Chen-type variational formulas are satisfied for $\lambda_{0,T}$:*

$$\inf_{f\in\widetilde{\mathscr{F}}_{II_T}} \sup_{x>0} II_T(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I_T}} \sup_{x>0} I_T(f)(x)^{-1} \geq \lambda_{0,T} \geq \sup_{f\in\mathscr{F}_{I_T}} \inf_{x>0} I_T(f)(x)^{-1} = \sup_{f\in\mathscr{F}_{II_T}} \inf_{x>0} II_T(f)(x)^{-1}, \tag{2.10}$$

where

$$I_T(f)(x) = -\frac{e^{-C(x)}}{f'(x)}\int_0^x \left[fe^C/a\right](u)du, \qquad II_T(f)(x) = \frac{1}{f(x)}\int_x^\infty dy\, e^{-C(y)}\int_0^y \left[fe^C/a\right](u)du.$$

Theorem 2.4 is a dual form of Theorem 2.1. Here, all test functions f in $\mathscr{F}_{II_T}$, $\mathscr{F}_{I_T}$, $\widetilde{\mathscr{F}}_{II_T}$ and $\widetilde{\mathscr{F}}_{I_T}$ are decreasing, while test functions in Theorem 2.1 are increasing.

Proof of Theorem 2.4. For any $l>0$, denote

$$\lambda_{0,T}^l = \inf\left\{D(f) : f \in C_0^\infty(\mathbb{R}_+), \|f\|=1 \text{ and } f|_{[l,\infty)}=0\right\}.$$

It is easy to check that

$$\lim_{l\to\infty} \lambda_{0,T}^l = \lambda_{0,T}. \tag{2.11}$$

Note that $\lambda_{0,T}^l$ is just the first eigenvalue of $-L$ with Neumann boundary at $x=0$ and Dirichlet boundary at $x=l$. So $\lambda_{0,T}^l$ can be written as

$$\lambda_{0,T}^l = \inf\left\{D^l(f) : f \in C_0^1([0,l]), \mu^l(f^2)=1 \text{ and } f(l)=0\right\},$$

where $D^l(f) = \int_0^l f'(x)^2 e^{C(x)}dx$ and $\mu^l(f) = \int_0^l f(x)\mu(dx)$. For any $f \in C_0^1([0,l])$ with $f(l)=0$, define $g(x)=f(l-x)$. Then

$$\lambda_{0,T}^l = \inf\left\{\int_0^l g'(x)^2 e^{-C(l-x)}dx : g \in C_0^1([0,l]), \int_0^l g(x)^2 a(l-x)^{-1}e^{C(l-x)}dx = 1 \text{ and } g(0)=0\right\}.$$

That is, $\lambda_{0,T}^l$ is also the first eigenvalue of

$$\overleftarrow{L} := a(l-x)dx/dx^2 + b(l-x)d/dx$$

with Dirichlet boundary at $x = 0$ and Neumann boundary at $x = l$. Note that all the assertions in Theorem 2.1 still hold for $[0, D]$ with Neumann boundary at $x = D$ if $D < \infty$ (c.f. see [4, 5, 6]). Then,

$$\inf_{f\in\widetilde{\mathscr{F}}_{II}}\sup_{x>0}\overleftarrow{II}(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}_{I}}\sup_{x>0}\overleftarrow{I}(f)(x)^{-1} \ge \lambda_{0,T}^{l} \ge \sup_{f\in\mathscr{F}_{I}}\inf_{x>0}\overleftarrow{I}(f)(x)^{-1} = \sup_{f\in\mathscr{F}_{II}}\inf_{x>0}\overleftarrow{II}(f)(x)^{-1},$$

where $\overleftarrow{I}$ (resp. $\overleftarrow{II}$) is the operator I(resp. II) in (2.3) by replacing $a(x)$ and $b(x)$ with $a(l-x)$ and $b(l-x)$, respectively. Changing variables yields the following Chen-type variational formulas for $\lambda_{0,T}^{l}$:

$$\inf_{f\in\widetilde{\mathscr{F}}^{l}_{II_T}}\sup_{x>0}II_T(f)(x)^{-1} = \inf_{f\in\widetilde{\mathscr{F}}^{l}_{I_T}}\sup_{x>0}I_T(f)(x)^{-1} \ge \lambda_{0,T}^{l} \ge \sup_{f\in\mathscr{F}^{l}_{I_T}}\inf_{x>0}I_T(f)(x)^{-1} = \sup_{f\in\mathscr{F}^{l}_{II_T}}\inf_{x>0}II_T(f)(x)^{-1}, \tag{2.12}$$

where the operators I_T, II_T are defined in Theorem 2.4; moreover,

$$\begin{aligned}
\mathscr{F}^{l}_{II_T} &= \{f \in C([0,l]) : f > 0 \text{ and } f(l) = 0\},\\
\widetilde{\mathscr{F}}^{l}_{II_T} &= \{f \in C([0,l]) : \text{ there exists } x_0 \text{ such that } f = f(\cdot\wedge x_0), f|_{[0,x_0)} > 0 \text{ and } f(x_0) = 0\},\\
\mathscr{F}^{l}_{I_T} &= \{f \in C^1([0,l]) : f' < 0 \text{ and } f(l) = 0\},\\
\widetilde{\mathscr{F}}^{l}_{I_T} &= \{f \in C^1([0,l]) : \text{ there exists } x_0 \text{ such that } f = f(\cdot\wedge x_0), f \in C^1([0,x_0]), f'|_{(0,x_0)} < 0 \text{ and } f(x_0) = 0\}.
\end{aligned}$$

Now, the required assertion follows by combining (2.11) with (2.12) and letting $l \to \infty$. □

By using Theorem 2.4, one can also present the second proof of (1.9). We only give a sketch here. Firstly, applying $f(x) = \int_x^{\infty} e^{-C(u)}du$ to $\inf_{x>0} I_T(f)(x)^{-1}$, one has $\lambda_{0,T} \ge (4\delta_T)^{-1}$ by following the proof of (2.6) with some modifications. For $\lambda_{0,T} \le \delta_T^{-1}$, instead applying the test function f in part (2) of the proof of Theorem 1.3 to $\inf_{x>0} I_T(f)(x)^{-1}$, the required assertion follows by the same argument of (2.7). So, Theorem 2.4 implies (1.9) in Theorem 1.3.

Acknowledgement The author would like to thank an anonymous referee and the editor for great help in earlier drafts of the paper.

References

[1] C. Albanese and A. Kuznetsov, *Transformations of Markov processes and classification scheme for solvable driftless diffusions*, http://arxiv.org/0710.1596, 2007.

[2] S. G. Bobkov and F. Götze, *Exponential integrability and transportation cost related to logarithmic Sobolev inequalities*, J. Funct. Anal. **163** (1999), no. 1, 1–28. MR1682772

[3] S. G. Bobkov and F. Götze, *Hardy type inequalities via Riccati and Sturm-Liouville equations*, Sobolev Spaces in Mathematics I: Sobolev Type Inequalities, Inter. Math. Ser. **8**, (2009), 69–86.

[4] M. F. Chen, *Analystic proof of dual variational formula for the first eigenvalue in dimension one*, Sci. Chin. Ser. A **42** (1999), no. 8, 805–815. MR1738551

[5] M. F. Chen, *Explicit bounds of the first eigenvalue*, Sci. Chin. Ser. A **43** (2000), no. 10, 1051–1059. MR1802148

[6] M. F. Chen, *Variational formulas and approximation theorems for the first eigenvalue in dimension one*, Sci. Chin. Ser. A **44** (2001), no. 4, 409–418. MR1831443

[7] M. F. Chen, *Exponential decay of birth-death processes*, Preprint 2008.

[8] M. F. Chen, *Eigenvalues, Inequalities and Ergodic Theory,* Springer-Verlag London, Ltd., London, 2005. MR2105651

[9] M. F. Chen and F. Y. Wang, *Estimation of spectral gap for elliptic operators,* Trans. Amer. Math. Soc. **349** (1997), no. 3, 1239–1267. MR1401516

[10] M. Fukushima, Y. Oshima and M. Takeda, *Dirichlet Forms and Symmetric Markov Processes*,de Gruyter, Stud. Math. **19**, Berlin, 1994. MR1303354

[11] K. Ito and Jr. H. P. McKean, *Diffusion Processes and Their Sample Paths,* Springer-Verlag, Berlin, Heidelberg and New York, 1965. MR0199891

[12] Y. H. Mao, *Nash inequalities for Markov processes in dimension one,* Acta Math. Sin. Eng. Ser. **18** (2002), no. 1, 147–156. MR1894847

[13] P. Mandl, *Analytical Treatment of One-dimensional Markov Processes,* Springer-Verlag, Berlin, Heidelberg and New York, 1968. MR0247667

[14] L. Miclo, *An example of application of discrete Hardy's inequalities*, Markov Processes Relat. Fields **5** (1999), no. 3, 319–330. MR1710983

[15] B. Muckenhoupt, *Hardy inequality with weights*, Studia. Math. **44** (1972), 31–38. MR0311856

[16] R. D. Nussbaum and Y. Pinchover, *On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications*, Journal d'Analyse Mathématique **59** (1992), 161–177. MR1226957

[17] R. G. Pinsky, *Explicit and almost explicit spectral calculations for diffusion operators,* J. Funct. Anal. **256** (2009), 3279–3312.

[18] M. Reed. and B. Simon, *Methods of Modern Mathematical Physics*, Analysis of Operators, **4** Academic Press, New York, 1978. MR0493421

[19] J. Wang, *First Dirichlet eigenvalue of transient birth-death processes*, Preprint 2008.

[20] F. Y. Wang, *Application of coupling methods to the Neumann eigenvalue problem*, *Probab. Theory Relat. Fields* **98** (1994), no. 3, 299–306. MR1262968

[21] F. Y. Wang, *Functional Inequalities, Markov Semigroups and Spectral Theory,* Science Press, Beijing/New York, 2004.