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EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR BSDES WITH LOCALLY LIPSCHITZ COEFFICIENT

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Abstract

We deal with multidimensional backward stochastic differential equations (BSDE) with locally Lipschitz coefficient in both variables y, z and an only square integrable terminal data. Let L_N be the Lipschitz constant of the coefficient on the ball $B(0,N)$ of $R^d \times R^{dr}$. We prove that if $L_N = O(\sqrt{\log N})$, then the corresponding BSDE has a unique solution. Moreover, the stability of the solution is established under the same assumptions. In the case where the terminal data is bounded, we establish the existence and uniqueness of the solution also when the coefficient has an arbitrary growth (in y) and without restriction on the behaviour of the Lipschitz constant L_N.

Introduction

Let $(W_t)_{0\le t\le 1}$ be a r-dimensional Wiener process defined on a probability space $(\Omega, \mathcal{F}, P)$ and $(\mathcal{F}_t)_{0\le t\le 1}$ denotes the natural filtration of (W_t), such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$. Let ξ be an $\mathcal{F}_1$-measurable d-dimensional square integrable random variable. Let f be an $\mathbb{R}^d$-valued process defined on $\mathbb{R}_+ \times \Omega \times \mathbb{R}^d \times \mathbb{R}^{d\times r}$ with values in $\mathbb{R}^d$ such that for all $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times r}$, the map $(t,\omega) \longrightarrow f(t,\omega,y,z)$ is $\mathcal{F}_t$-progressively measurable. We consider the following BSDE,

$$(E^{f,\xi}) \qquad Y_t = \xi + \int_t^1 f(s, Y_s, Z_s)ds - \int_t^1 Z_s dW_s \qquad 0 \le t \le 1$$

Linear versions of the BSDE $(E^{f,\xi})$ appear as the equations for the adjoint process in stochastic control, as well as the model behind the Black & Scholes formula for the pricing and hedging of options in mathematical finance. It turned out recently that equation $(E^{f,\xi})$ is closely related to both elliptic and parabolic nonlinear partial differential equations of second order [16,17].

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The linear BSDEs can be solved more or less explicitly. When the coefficient f is uniformly Lipschitz, the BSDE $(E^{f,\xi})$ has a unique solution which can be constructed by using both Itô's representation theorem and a successive approximation procedure [15]. Further developments on the BSDEs with various applications to stochastic control, mathematical finance, partial differential equations and homogenization can be find in the lectures [5,6,12,14].

The comparison-theorem-technique is the essential tool to prove the existence of solutions to one dimensional BSDEs with continuous coefficient, see [7,10,11] and the references therein. The case where the coefficient is measurable has been treated in [4], by using a classical transformation which removes the drift. This transformation allows the authors of [4], to establish both existence and uniqueness of the solutions and to deal with the BSDE also involving a local time. It should be noted that the techniques used in dimension one do not work for multidimensional equations.

In multidimensional case, the improvements of the Lipschitz condition on f concern, generally, the variable y only (e.g. [3,8,13,14,18]). The coefficient f is usually assumed to be uniformly Lipschitz with respect to the variable z. Sometimes the Lipschitz condition in the variable y is replaced by monotonicity condition. Noticing that the techniques used in the Lipschitz case work in general for BSDEs with monotone coefficient. In all the previous papers, the assumptions on the coefficient are global, although are non-Lipschitz. The present work is the first one which consider multidimensional BSDEs with both local assumptions on the coefficient and an only square integrable terminal data. The solutions are usually constructed by successive approximations. Although this method is a powerful tool under global Lipschitz conditions on the coefficient, it fails when these assumptions are merely local. The second difficulty encountered in the locally Lipschitz case stays in the fact that the usual localization techniques by means of stopping times do not work in BSDE.

In this note we deal with multidimensional BSDEs with locally Lipschitz coefficient and a square integrable terminal data. We study the existence and uniqueness, as well as the stability of solutions. We show that if the coefficient f is locally Lipschitz in both variables y, z and the Lipschitz constant L_N in the ball $B(0, N)$ is such that $L_N = \mathcal{O}(\sqrt{\log N})$, then the corresponding BSDE $(E^{f,\xi})$ has a unique solution. The stability of the solution with respect to the data (f, ξ) is established under the same conditions. In the case where the terminal data ξ is bounded, we establish the existence and uniqueness of the solution also when the coefficient has an arbitrary growth (in y) and without any restrictive condition on the behaviour of the Lipschitz constant L_N. This last result remains valid also in the case where the coefficient f is bounded. The proofs of our results mainly consist to establishing an a priori estimate between two solutions (Y^1, Z^1), (Y^2, Z^2) with respectively the data (f^1, ξ^1), (f^2, ξ^2). We deduce the existence of solutions by approximating the coefficient f by a sequence of Lipschitz functions (f_n) via a suitable family of semi-norms and by using an appropriate (alternative) localization which seems to be more adapted to BSDEs than the usual localization by stopping time. Our method makes it possible to prove both existence and uniqueness, as well as the stability of the solution by using the same computations.

The paper is organized as follows. The notations, definitions and some assumptions are collected in section 1. The existence and uniqueness of the solution to BSDEs with locally Lipschitz coefficient are stated in section 2. The Theorem 2, of section 2, has been already announced in [1] with a sketched proof. The stability of the solutions is estabished in section 3. The BSDE with bounded terminal data and arbitrary growth coefficient are studied in section 4. The final Section 5, is devoted to some remarks on the possible extentions.

1 Definitions, assumptions and notations

We denote by $\mathcal{E}$ the set of $\mathbb{R}^d \times \mathbb{R}^{d\times r}$-valued processes (Y,Z) defined on $\mathbb{R}_+ \times \Omega$ which are $\mathcal{F}_t$-adapted and such that: $||(Y,Z)||^2 = E\big(\sup_{0\le t\le 1}|Y_t|^2 + \int_0^1 |Z_s|^2 ds\big) < +\infty$. The couple $(\mathcal{E}, ||.||)$ is then a Banach space.

Definition 1. *A solution of equation $(E^{f,\xi})$ is a couple (Y,Z) which belongs to the space $(\mathcal{E}, ||.||)$ and satisfies $(E^{f,\xi})$.*

We consider the following assumptions:

(H1) *f is continuous in (y,z) for almost all (t,ω).*
(H2) *There exist two constants, $M>0$ and $\alpha\in[0,1]$, such that,*
$|f(t,\omega,y,z)| \le M(1+|y|^\alpha+|z|^\alpha)$ *P-a.s., a.e. $t\in[0,1]$.*
(H3) *For every $N\in\mathbb{N}$, there exists a constant $L_N>0$ such that,*
$|f(t,\omega,y,z)-f(t,\omega,y',z')| \le L_N(|y-y'|+|z-z'|)$, *$P$-a.s., a.e. $t\in[0,1]$*
and $\forall y,y',z,z'$ such that $|y|\le N$, $|y'|\le N$, $|z|\le N$, $|z'|\le N$.
(H4) *there exists a constant $L>0$ such that,*
$|f(t,\omega,y,z)-f(t,\omega,y',z')| \le L(|y-y'|+|z-z'|)$, *$P$-a.s., a.e. $t\in[0,1]$.*

When the assumptions (H1), (H2) are satisfied, we can define a family of semi-norms $\big(\rho_n(f)\big)_{n\in\mathbb{N}}$ by,

$$\rho_n(f) = \Big(E\int_0^1 \sup_{|y|,|z|\le n} |f(s,y,z)|^2 ds\Big)^{\frac{1}{2}}$$

We denote by Lip_{loc} (resp. Lip) the set of functions f satisfying (H3) (resp. (H4)). $Lip_{loc,\alpha}$ denotes the subset of functions f which satisfy assumptions (H2), (H3).

2 BSDE with locally Lipschitz coefficient

Theorem 2. *Let $f\in Lip_{loc,\alpha}$ and ξ be a square integrable random variable. Assume moreover that there exists a positive constant L such that, $L_N = L+\sqrt{\log N}$. Then equation $(E^{f,\xi})$ has a unique solution.*

The following corollary gives a weaker condition on L_N in the case where f is uniformly Lipschitz in the variable z and locally Lipschitz with respect to the variable y.

Corollary 1. *Let* (H1), (H2) *be satisfied and ξ be a square integrable random variable. Assume that f is uniformly Lipschiz in the variable z and locally Lipschitz in the variable y and denote by L_N the local Lipschitz constant of f with respect to the variable y. Then equation $(E^{f,\xi})$ has a unique solution if $L_N\le L+\log N$, where L is some positive constant.*

To prove Theorem 2 and their corollaries we need the following lemmas.

Lemma 1. -(i)- *Let (Y,Z) be a solution of equation $(E^{f,\xi})$. If f satisfies* (H2) *then there exists a positive constant $K=K(M,\xi)$ which depends only on M and $E(|\xi|^2)$ such that for every $t\in[0,1]$,*

$$E(|Y_t|^2)\le K \qquad \textit{and} \qquad E\int_0^1 |Z_s|^2 ds \le K$$

-(ii)- Let ξ^1, ξ^2 be two d-dimensional square integrable random variables which are $\mathcal{F}_1$-measurable. Let f_1 f_2 be such that, f_1 satisfies (H1), (H2), (H3) *and f_2 verifies* (H1), (H2). *Let (Y^1, Z^1)* [resp. (Y^2, Z^2)] *be a solution of the BSDE (E^{f^1,ξ^1})* [resp. (E^{f^2,ξ^2}))]. *Then for every $N > 1$, every $\beta > 0$ and every $t \in [0,1]$ the following estimates hold*

$$E\int_t^1 |Z_s^1 - Z_s^2|^2 ds \leq K(M, \xi^1, \xi^2)\big(E(|\xi^1 - \xi^2|^2) + [E\int_t^1 |Y_s^1 - Y_s^2|^2 ds]^{\frac{1}{2}}\big).$$

and

$$E(|Y_t^1 - Y_t^2|^2) \leq \big[E(|\xi^1 - \xi^2|^2) + \frac{\rho_N^2(f^1 - f^2)}{L_N^2} + \frac{K(M, \xi^1, \xi^2)}{L_N^2 N^{2(1-\alpha)}}\big] \exp[2(1-t)L_N^2 + 1]$$

where $K(M, \xi^1, \xi^2)$ is a constant which depends from M, $E(|\xi^1|^2)$ and $E(|\xi^2|^2)$.

Proof. Since $|x|^\alpha \leq 1 + |x|$ for each $\alpha \in [0,1]$, assertion (i) follows then from standard arguments of BSDEs. The first inequality of assertion (ii) follows from Itô's formula and Schwarz inequality. We shall prove the second inequality of (ii). Let $<,>$ denote the inner product in $\mathbb{R}^d$. By Itô's formula we have,

$$|Y_t^1 - Y_t^2|^2 + \int_t^1 |Z_s^1 - Z_s^2|^2 ds = |\xi^1 - \xi^2|^2 + 2\int_t^1 < Y_s^1 - Y_s^2, \quad f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2) > ds$$
$$-2\int_t^1 < Y_s^1 - Y_s^2, \quad (Z_s^1 - Z_s^2) dW_s >$$

Let β be a strictly positive number. For a given $N > 1$, let L_N be the Lipschitz constant of f^1 in the ball $B(0, N)$, $A_N := \{(s, \omega); \quad |Y_s^1|^2 + |Z_s^1|^2 + |Y_s^2|^2 + |Z_s^2|^2 \geq N^2\}$, $\overline{A}_N := \Omega \setminus A_N$ and denote by χ_E the indicator function of the set E. Taking expectation in the last identity, we show that

$$\begin{aligned}
&E(|Y_t^1 - Y_t^2|^2) + E\int_t^1 |Z_s^1 - Z_s^2|^2 ds \leq \\
&\leq E(|\xi^1 - \xi^2|^2) + 2E\int_t^1 |Y_s^1 - Y_s^2||f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2))| ds \\
&\leq E(|\xi^1 - \xi^2|^2) + \beta^2 E\int_t^1 |Y_s^1 - Y_s^2|^2 ds \\
&\quad + \frac{1}{\beta^2} E\int_t^1 |f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)|^2 \chi_{A_N} ds \\
&\quad + \frac{1}{\beta^2} E\int_t^1 \big(|f^1(s, Y_s^1, Z_s^1) - f^2(s, Y_s^2, Z_s^2)|^2\big) \chi_{\overline{A}_N} ds \\
&\leq E(|\xi^1 - \xi^2|^2) + \beta^2 E\int_t^1 |Y_s^1 - Y_s^2|^2 ds \\
&\quad + \frac{2M^2}{\beta^2} E\int_t^1 (1 + |Y_s^1|^\alpha + |Z_s^1|^\alpha)^2 \chi_{A_N} ds \\
&\quad + \frac{2M^2}{\beta^2} E\int_t^1 (1 + |Y_s^2|^\alpha + |Z_s^2|^\alpha)^2 \chi_{A_N} ds
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{\beta^2}\rho_N^2(f^1-f^2)+\frac{2}{\beta^2}L_N^2E\int_t^1(|Y_s^1-Y_s^2|^2+|Z_s^1-Z_s^2|^2)ds\\
\leq\ & E(|\xi^1-\xi^2|^2)+(\beta^2+\frac{2L_N^2}{\beta^2})E\int_t^1|Y_s^1-Y_s^2|^2ds\\
&+\frac{6M^2}{\beta^2}E\int_t^1(1+|Y_s^1|^{2\alpha}+|Z_s^1|^{2\alpha})\chi_{A_N}ds\\
&+\frac{6M^2}{\beta^2}E\int_t^1(1+|Y_s^2|^{2\alpha}+|Z_s^2|^{2\alpha})\chi_{A_N}ds\\
&+\frac{2}{\beta^2}\rho_N^2(f^1-f^2)+\frac{2L_N^2}{\beta^2}E\int_t^1|Z_s^1-Z_s^2|^2ds
\end{aligned}
$$

We use Hölder's inequality and Chebychev's inequality to obtain,

$$
\begin{aligned}
&E(|Y_t^1-Y_t^2|^2)+E\int_t^1|Z_s^1-Z_s^2|^2ds\\
\leq\ & E(|\xi^1-\xi^2|^2)+(\beta^2+\frac{2L_N^2}{\beta^2})E\int_t^1|Y_s^1-Y_s^2|^2ds\\
&+\frac{K(M,\xi^1,\xi^2)}{\beta^2N^{2(1-\alpha)}}+\frac{2}{\beta^2}\rho_N^2(f^1-f^2)\\
&+\frac{2L_N^2}{\beta^2}E\int_t^1|Z_s^1-Z_s^2|^2ds
\end{aligned}
$$

where $K(M,\xi^1,\xi^2)$ is a constant which depends from M, $E(|\xi^1|^2)$ and $E(|\xi^2|^2)$ and which may change from line to line. We choose β such that $\frac{2L_N^2}{\beta^2}=1$, then we use Gronwall's lemma to get,

$$
E(|Y_t^1-Y_t^2|^2)\leq\big[E(|\xi^1-\xi^2|^2)+\frac{\rho_N^2(f^1-f^2)}{L_N^2}+\frac{K(M,\xi^1,\xi^2)}{L_N^2N^{2(1-\alpha)}}\big]\exp[2(1-t)L_N^2+1]. \blacksquare
$$

Lemma 2. *Let f be a function which satisfies* (H1), (H2). *Then there exists a sequence of functions f_n such that,*
-(i) -$a)$ *For each n, $f_n\in Lip_\alpha$.*
-$b)$ $\sup_n|f_n(t,\omega,y,z)|\leq|f(t,\omega,y,z)|\leq M(1+|y|^\alpha+|z|^\alpha)$ *P-a.s., a.e. $t\in[0,1]$.*
-(ii)- *For every N, $\rho_N(f_n-f)\longrightarrow 0$ as $n\longrightarrow\infty$.*

Proof. Let ψ_n be a sequence of smooth functions with support in the ball $B(0,n+1)$ and such that $\psi_n=1$ in the ball $B(0,n)$. It is not difficult to see that the sequence (f_n) of truncated functions, defined by $f_n=f\psi_n$, satisfies all the properties quoted in Lemma 2. $\blacksquare$

Lemma 3. *Let f and ξ be as in Theorem 2. Let (f_n) be the sequence of functions associated to f by Lemma 2 and denote by (Y^n,Z^n) the solution of equation (E^{f_n}). Then there exists a constant $K=K(M,\xi)$ which depends only on M and $E(|\xi|^2)$ such that,*
-$a)$ $\sup\limits_n E(|\ Y_t^n\ |^2)\leq K$.
-$b)$ $\sup\limits_n E\int_0^1|Z_s^n|^2ds\leq K$.

Proof. It goes as that of Lemma 1 (i). K can have the following form,
$K=\max\{[E(|\xi|^2)+9]\exp(3M^2+1),\big[E(|\xi|^2)+9\big]\big[2+(1+12M^2)\exp(3M^2+1)\big]\}.$ $\blacksquare$

Lemma 4. *Let f and ξ be as in theorem 2. Let (f_n) be the sequence of functions associated to f by Lemma 2 and denote by (Y^n, Z^n) the solution of equation $(E^{f_n,\xi})$. Then there exists a process $(Y,Z) \in \mathcal{E}$ such that* $\lim_{n\to\infty} ||(Y^n,Z^n) - (Y,Z)|| = 0$.

Proof. For the simplicity we assume $L=0$. Observe that for each $n \geq (N+1)$, $|f_n(t,y,z) - f_n(t,y',z')| \leq L_N(|y-y'|+|z-z'|)$ on the ball $B(0,N)$. Assume first that $L_N \leq \sqrt{(1-\alpha)\log N}$. Then applying Lemma 1 to $(Y^1,Z^1,f^1,\xi^1) = (Y^n,Z^n,f_n,\xi)$, $(Y^2,Z^2,f^2,\xi^2) = (Y^m,Z^m,f_m,\xi)$ and passing to the limit successively on n,m,N one gets Lemma 4. Assume now that $L_N \leq \sqrt{\log N}$. Let δ be a strictly positive number such that $\delta > (1-\alpha)$. Let $([t_{i+1},t_i])$ be a subdivision of $[0,1]$ such that $|t_i - t_{i+1}| \leq \delta$. Appliying Lemma 1 in all the subintervals $[t_{i+1},t_i]$ we get Lemma 4. ∎

Proof of Theorem 2. The uniqueness follows from the Lemma 1 by letting $f^1 = f^2 = f$ and $\xi^1 = \xi^2 = \xi$). We shall prove the existence of solutions. Thanks to Lemma 4 , there exists $(Y,Z) \in \mathcal{E}$ such that $||(Y^n,Z^n) - (Y,Z)|| \to 0$ as $n \to \infty$. Thus, we immediately have

$$(1) \qquad \lim_{n\to\infty} E(\sup_{0\leq s\leq 1} | Y^n_s - Y_s |^2) = 0 \qquad \text{and} \qquad \lim_{n\to\infty} E\int_0^1 |Z^n_s - Z_s|^2 ds$$

It remains to prove that $\int_t^1 f_n(s,Y^n_s,Z^n_s)ds$ converges to $\int_t^1 f(s,Y_s,Z_s)ds$ *in probability*. Let $N > 1$ and denote by L_N the Lipschitz constant of f in the ball $B(0,N)$. We put $A^N_n := \{(s,\omega);\quad |Y^n_s|+|Z^n_s|+|Y_s|+|Z_s| \geq N\}$ and $\overline{A}^N_n := \Omega \setminus A^N_n$. Arguing as in the proof of Lemma 1 then using Lemma 3 and Fatou's Lemma, we show that

$$E|\int_t^1 f_n(s,Y^n_s,Z^n_s)ds - \int_t^1 f(s,Y_s,Z_s)ds| \leq I_1(n) + L_N I_2(n) + \frac{K(M,\xi)}{N}$$

where

$$I_1(n) = E\int_0^1 \sup_{|y|,|z|\leq N} |f_n(s,y,z) - f(s,y,z)|ds.$$

$$I_2(n) = E\int_0^1 |Y^n_s - Y_s|ds + E\int_0^1 |Z^n_s - Z_s|ds.$$

and $K(M,\xi)$ is a constant which depends only on M and $E(|\xi|)$

Lemma 2 shows that $lim_{n\to\infty} I_1(n) = 0$. We shall prove that $lim_{n\to\infty} I_2(n) = 0$. From the identity (1) we have $lim_{n\to\infty} E\int_0^1 |Z^n_s - Z_s|ds = 0$. We use equality (1), Lemma 3, Fatou's Lemma and the Lebesgue dominated convergence Theorem to show that $lim_{n\to\infty} E\int_0^1 |Y^n_s - Y_s|ds = 0$. This proves that equation $(E^{f,\xi})$ has at least one solution. Theorem 2 is proved. ∎

Proof of Corollary 1. For $\alpha = 1$, the problem will be reduced to the classical case. We shall treat the case $\alpha < 1$. We assume $L=0$ for simplicity. Let L' be the (uniform) Lipschitz constant of f with respect to the variable z. For a given $N > 1$, let L_N denote the Lipschitz constant (in y) of f in the ball $B(0,N)$. We define $A^N_{n,m} := \{(s,\omega);\quad |Y^n_s|^2 + |Z^n_s|^2 + |Y^m_s|^2 + |Z^m_s|^2 \geq N^2\}$, $\overline{A}^N_{n,m} := \Omega \setminus A^N_{n,m}$. By Itô's formula, we have

$$E(|Y^n_t - Y^m_t|^2) + E\int_t^1 |Z^n_s - Z^m_s|^2 ds = I_1(n,m) + I_2(n,m) + I_3(n,m) + I_4(n,m)$$

where,

$$I_1(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f_n(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s^n) > \chi_{\overline{A}_{n,m}^N}\, ds$$

$$I_2(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^m) > \chi_{\overline{A}_{n,m}^N}\, ds$$

$$I_3(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f(s,Y_s^m,Z_s^m) - f_m((s,Y_s^m,Z_s^m) > \chi_{\overline{A}_{n,m}^N}\, ds$$

$$I_4(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f_n(s,Y_s^n,Z_s^n) - f_m(s,Y_s^m,Z_s^m) > \chi_{A_{n,m}^N}\, ds$$

We shall estimate successively $I_1(n,m)$, $I_2(n,m)$, $I_3(n,m)$, $I_4(n,m)$. Let β_1, β_2 be a strictly positive numbers. It is easy to see that,

$$I_1(n,m) \le E\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_n - f) \tag{2}$$

$$I_3(n,m) \le E\int_t^1 |Y_s^n - Y_s^m|^2 ds + \rho_N^2(f_m - f) \tag{3}$$

$$I_2(n,m) \le (2L_N + \beta_1^2 L')E\int_t^1 |Y_s^n - Y_s^m|^2 \chi_{\overline{A}_{n,m}^N}\, ds + \frac{L'}{\beta_1^2}E\int_t^1 |Z_s^n - Z_s^m|^2 ds \tag{4}$$

We use assumption (H2), Hölder's inequality, Chebychev's inequality and Lemma 3 to show that

$$I_4(n,m) \le \beta_2^2 E\int_t^1 |Y_s^n - Y_s^m|^2 \chi_{A_{n,m}^N}\, ds + \frac{K(M,\xi)}{\beta_2^2 N^{2(1-\alpha)}} \tag{5}$$

where $K(M,\xi)$ is a constant which depends only on M and $E(|\xi|^2)$ and which may change from a line to another.
We choose $\beta_1^2 = L'$ and $\beta_2^2 = 2L_N$ then we use (2), (3), (4), (5) and Gronwall's lemma to obtain,

$$E(|Y_t^n - Y_t^m|^2) \le \left[\rho_N^2(f_n - f) + \rho_N^2(f_m - f) + \frac{K(M,\xi)}{(L_N)N^{2(1-\alpha)}}\right] \exp(2L_N)\exp(L'^2 + 2)$$

Passing to the limit successively on n, m, N and using the Burkholder-Davis-Gundy inequality, we show that (Y^n, Z^n) is a Cauchy sequence in the Banach space $(\mathcal{E}, ||.||)$. The end of the proof goes as that of Theorem 2. Corollary 1 is proved. ■

3 Stability of the solutions

In this section, we prove a stability result for the solution with respect to the data (f,ξ). Roughly speaking, if f_n converges to f in the metric defined by the family of semi-norms (ρ_N) and ξ_n converges to ξ in $L^2(\Omega)$ then (Y^n, Z^n) converges to (Y,Z) in $(\mathcal{E}, ||.||)$. Let (f_n) be a sequence of functions which are $\mathcal{F}_t$-progressively measurable for each n. Let (ξ_n) be a sequence of random variables which are $\mathcal{F}_1$-measurable for each n and such that $E(|\xi_n|^2) < \infty$.
Throughout this section we will assume that for each n, the BSDE (E^{f_n,ξ_n}) corresponding to the data (f_n, ξ_n) has a (not necessarily unique) solution. Each solution of the equation (E^{f_n})

will be denoted by (Y^n, Z^n). We suppose also that the following assumptions (H3), (H4), (H5) are fulfilled,

(H3) *For every* N, $\rho_N(f_n - f) \longrightarrow 0$ *as* $n \to \infty$.
(H4) $E(|\xi_n - \xi|^2) \longrightarrow 0$ *as* $n \to \infty$.
(H5) *There exist two constants,* $M > 0$ *and* $\alpha \in [0,1]$*, such that,*
$\sup_n |f_n(t,\omega,y,z)| \leq M(1 + |y|^\alpha + |z|^\alpha)$ *P-a.s., a.e.* $t \in [0,1]$.

Theorem 6. *Let* f *and* ξ *be as in Theorem 2. Assume that* (H3), (H4), (H5) are satisfied. *Then* (Y^n, Z^n) *converges to* (Y,Z) *in the space* $(\mathcal{E}, ||.||)$.

Proof. For $\alpha = 1$, the result is classic. We shall treat the case $\alpha < 1$. Applying Lemma 1 to $(Y^1, Z^1, f^1, \xi^1) = (Y, Z, f, \xi)$, $(Y^2, Z^2, f^2, \xi) = (Y^n, Z^n, f_n, \xi^n)$ and passing to the limits successively on n, N one gets Theorem 6. ■

IV) BSDEs with bounded terminal data.

Let $<,>$ denote the inner product in $\mathbb{R}^d$ and consider the following assumptions,

(H6) *There exists a constant* $M > 0$ *such that,*
$\xi \leq M$ *P-a.s.*
(H7) *There exists a constant* $M > 0$ *such that, for every* y *and* z,
$< y, f(t,\omega,y,z) >\leq M(1 + |y|^2 + |y||z|)$ *P-a.s., a.e.* $t \in [0,1]$.
(H8) *There exists a constant* $M > 0$ *and a positive continuous function* $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ *such that for every* y *and* z,
$|f(t,\omega,y,z)| \leq M(1 + \varphi(|y|) + |z|)$ *P-a.s., a.e.* $t \in [0,1]$.
(H9) *For every* $N \in \mathbb{N}$*, there exists a constant* $L_N > 0$ *such that,*
$|f(t,\omega,y,z) - f(t,\omega,y',z)| \leq L_N(|y - y'|)$,
P-a.s., a.e. $t \in [0,1]$ and for all
y, y', z such that $|y| \leq N$, $|y'| \leq N$.
(H10) *there exists a constant* $L' > 0$ *such that, for every* y, z, z',
$|f(t,\omega,y,z) - f(t,\omega,y,z')| \leq L'(|z - z'|)$, *P-a.s., a.e.* $t \in [0,1]$.

Proposition 7. *Let* (H6)–(H10) *be satisfied. Then equation* $(E^{f,\xi})$ *has a unique solution. Moreover the solution is stable in the sense of Theorem 6.*

To prove proposition 7, we need the following lemmas.

Lemma 8. *Let* f *be a function which satisfies* (H7)–(H10). *Then there exists a sequence of functions* (f_n) *such that,*
-(i)- *For each* n, f_n *is globally Lipschitz in* (y,z) *a.e.* t *and* P*-a.s.*ω.
-(ii)- *There exists a constant* $K(M) > 0$ *such that for each* (y,z),
$\sup_n < y, f_n(t,\omega,y,z) >\leq K(M)(1 + |y|^2 + |y||z|)$ *P-a.s. and a.e.* $t \in [0,1]$.
-(iii)- *For every* (y,z), $\sup_n |f_n(t,\omega,y,z|) \leq M(1 + \varphi(|y|) + |z|)|$ *P-a.s., a.e.* $t \in [0,1]$.
-(iv)- *For every* N, $\rho_N(f_n - f) \longrightarrow 0$ *as* $n \longrightarrow \infty$.

Proof. Let $\psi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq \psi_n \leq 1$, $\psi_n(u) = 1$ for $|u| \leq n$ and $\psi_n(u) = 0$ for $|u| \geq n+1$. Likewise we define the sequence ψ'_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. It is not difficult to see that the sequence f_n defined by $f_n(t,y,z) := f(t,y,z)\psi_n(y)\psi'_n(z)$ satisfies all the assertions of Lemma 8. ■

Lemma 9. *Let f and ξ be as in Proposition 7. Let (f_n) be the sequence of functions associated to f by Lemma 8 and denote by (Y^n, Z^n) the solution of equation (E^{f_n}). Then there exist two positive constants $K = K(M)$ such that,*

$$\sup_n \big(\sup_{0\leq t\leq 1} \mid Y_t^n \mid \big) \leq K \qquad \text{and} \qquad \sup_n E\int_0^1 \mid Z_s^n \mid^2 ds \leq K$$

Proof. It follows by using Itô's formula, the conditional expectation, Gronwall's Lemma and Lemma 8. ■

Proof of Proposition 7. - (i)- Let l, N be two strictly positive numbers. We define $A_{n,m}^N := \{(s,\omega);\quad |Y_s^n|^2 + |Y_s^m|^2 \geq N^2\}$, $\overline{A}_{n,m}^N := \Omega \setminus A_{n,m}^N$ and $B_{n,m}^l := \{(s,\omega);\quad |Z_s^n|^2 + |Z_s^m|^2 \geq l^2\}$, $\overline{B}_{n,m}^l := \Omega \setminus B_{n,m}^l$. By Itô's formula, we have

$$E(|Y_t^n - Y_t^m|^2) + E\int_t^1 |Z_s^n - Z_s^m|^2 ds = I_1(n,m) + I_2(n,m) + I_3(n,m)$$

where,

$$I_1(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n) > ds$$

$$I_2(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f(s, Y_s^n, Z_s^n) - f(s, Y_s^m, Z_s^m) > ds$$

$$I_3(n,m) = 2E\int_t^1 < Y_s^n - Y_s^m, \quad f(s, Y_s^m, Z_s^m) - f_m((s, Y_s^m, Z_s^m) > ds$$

We shall estimate $I_1(n,m)$, $I_2(n,m)$, $I_3(n,m)$. Let n and N be such that, $n \geq N \geq \sup_n(|Y_t^n|)$. We then have,

$$I_1(n,m) = 2E\int_t^1 |Y_s^n - Y_s^m||f(Y^n, Z^n)||(\psi_n(Z^n) - 1)|(\chi_{\overline{B}_{n,m}^l} + \chi_{B_{n,m}^l})ds$$

We use assumption (H8), Lemma 8 and Chebychev's inequality to get,

$$I_1(n,m) \leq K(M,\varphi)\big(\sup_{|z|\leq l} |\psi_n(z) - 1| + \frac{1}{l^2}\big)$$

where $K(M,\varphi)$ is a constant which depends only on M and φ and which can changes from a line to another.

By similar arguments, we show that

$$I_3(n,m) \leq K(M,\varphi)\big(\sup_{|z|\leq l} |\psi_m(z) - 1| + \frac{1}{l^2}\big)$$

We successively use assumption (H8), Lemma 8, Hölder's inequality and Chebychev's inequality to show that

$$\begin{aligned} I_2(n,m) \quad &\leq \quad (2L_N + L'^2)E\int_t^1 |Y_s^n - Y_s^m|^2 ds + \frac{L'}{L'}E\int_t^1 |Z_s^n - Z_s^m|^2 ds \\ &\qquad +2E\int_t^1 |Y_s^n - Y_s^m||f(Y_s^n, Z_s^n) - f(Y_s^m, Z_s^n)|(\chi_{\overline{A}_{n,m}^N} + \chi_{A_{n,m}^N})ds \\ &\leq \quad (2L_N + L'^2)E\int_t^1 |Y_s^n - Y_s^m|^2 ds + E\int_t^1 |Z_s^n - Z_s^m|^2 ds \end{aligned}$$

Using these last estimates of $I_1(n,m)$, $I_2(n,m)$, $I_3(n,m)$ and the Gronwall Lemma, we obtain

$$E(|Y_t^n - Y_t^m|^2) \leq K(M,\varphi)\big[\sup_{|z|\leq l}(|\psi_n(z)-1|) + \sup_{|z|\leq l}(|\psi_m(z)-1|) + \frac{1}{l^2}\big]\exp(2L_N)\exp(L'^2)$$

Passing to the limit, first on n,m and next on l, we show that (Y^n, Z^n) is a Cauchy sequence in the Banach space $(\mathcal{E}, ||.||)$. The end of the proof goes as that of Theorem 2. Proposition 7 is proved. ∎

V) Remarks.

1) *BSDEs with monotone coefficient in Y and locally Lipschitz in Z.*

We consider the following assumptions,

(H11) *there exists a constant $\mu \in \mathbb{R}$ such that,*
$< y - y', f(t,\omega,y,z) - f(t,\omega,y',z) > \leq \mu|y-y'|^2$ *P-a.s., a.e. $t \in [0,1]$*

(H12) *For every $N \in \mathbb{N}$, there exists a constant $L_N > 0$ such that,*
$|f(t,\omega,y,z) - f(t,\omega,y,z')| \leq L_N|z-z'|$, *$P$-a.s., a.e. $t \in [0,1]$ and $\forall y,z$ such that $|y| \leq N$, $|z'| \leq N$, $|z| \leq N$.*

Arguing as in the proof of Corollary 1 one can establish the following result which is an extension of the Darling-Pardoux result [3] to the locally Lipschitz case.

If f satisfies the assumptions (H1), (H2), (H11), (H12). *Then equation $(E^{f,\xi})$ has a unique solution. Moreover the solution is stable in the sense of Theorem 6.*

2) Our method works under assumptions considered in [13]. That is, the results established in [13] can be proved by our techniques. Indeed, in these cases we can approximate the coefficient f, uniformly in $(y,z) \in \mathbb{R}^d \times \mathbb{R}^{d\times r}$, by a sequence (f_n) of uniformly Lipschitz functions.

3) Observe that the condition $L_N = \mathcal{O}(\sqrt{\log N})$ allows to f a super-linear growth such $|z|\sqrt{|\log|z||}$ or $|y|\sqrt{|\log|y||}$. Hence we can think that the BSDE $(E^{f,\xi})$ has a unique solution under the following conditions
$L_N = \mathcal{O}(\sqrt{\log N})$ and $|f| \leq M(1 + |z|\sqrt{|\log|z||} + |y|\sqrt{|\log|y||})$.

4) Modifying the construction of the sequence (f_n), It seems possible to prove that all the previous results can be extended to the case where f is locally, μ_N-monotone in y and L_N-Lipschitz in z, on the ball $B(0,N)$ of $\mathbb{R}^d \times \mathbb{R}^{d\times r}$. The supplementary assumption which could be required in this case seems to be: $(2\mu_N^+ + L_N^2) = \mathcal{O}(\log N)$.

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