Elect. Comm. in Probab. **11** (2006), 89–99

ELECTRONIC COMMUNICATIONS in PROBABILITY

A CONTROLLER AND A STOPPER GAME WITH DEGENERATE VARIANCE CONTROL

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Submitted 2 February 2006, *accepted in final form* 9 June 2006

AMS 2000 Subject classification: 93E20, 60G40
Keywords: stochastic games, optimal stopping, degenerate diffusions, saddle point

Abstract

We consider a zero sum, stochastic differential game which involves two players, *the controller* and *the stopper*. The stopper selects the stopping rule which halts the game. The controller chooses the diffusion coefficient of the corresponding state process which is allowed to degenerate. At the end of the game, the controller pays the stopper, the amount $E\int_0^\tau e^{-\alpha t}C(Z_x(t))dt$, where $Z_x(\cdot)$ represents the state process with initial position x and α is a positive constant. Here $C(\cdot)$ is a reward function where the set $\{x : C(x) > 0\}$ is an open interval which contains the origin. Under some assumptions on the reward function $C(\cdot)$ and the drift coefficient of the state process, we show that this game has a value. Furthermore, this value function is Lipschitz continuous, but it fails to be a C^1 function.

1 Introduction

We study a stochastic differential game in this article. This work is related to the controller and the stopper game studied by Karatzas and Sudderth in [4] and we use their formulation of the problem. Consider a weak solution to the one dimensional stochastic differential equation

$$X_x(t) = x + \int_0^t \mu(X_x(s))ds + \int_0^t u(s)dW(s) \tag{1.1}$$

where x is a real number, μ is a twice differentiable function defined on $\mathbf{R}$, $\{W(t) : t \geq 0\}$ is a standard Brownian motion adapted to a right continuous filtration $\{\mathfrak{F}_t : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. The σ-algebra $\mathfrak{F}$ contains all the null sets in Ω and the Brownian increments $W(t+s) - W(t)$ are independent of $\mathfrak{F}_t$ for all $t \geq 0$ and $s \geq 0$. The control process $u(\cdot)$ is progressively measurable with respect to the filtration $\{\mathfrak{F}_t\}$ and satisfies

$$0 \leq u(t) \leq \sigma_0, \tag{1.2}$$

[1]Research supported by Army Research Office Grant W911NF0510032

where σ_0 is a constant.

The quadruple $((\Omega, \mathfrak{F}, P), (\mathfrak{F}_t), W, u(\cdot))$ is an *admissible control system* if the corresponding state process X_x satisfies (1.1) together with the above assumptions. For convenience, we let $\mathcal{A}(x)$ be the class of state processes X_x available with the initial position x. Here, the reward function $C(\cdot)$ is a twice continuously differentiable defined on $\mathbf{R}$ and $C(0) > 0$.

In this zero sum stochastic differential game, there are two players, *the controller* and *the stopper*. Given any initial point x, *the controller* selects a state process X_x from $\mathcal{A}(x)$. The second player, *the stopper* selects a stopping rule τ to halt the evolution of the state process X_x. As described in [4], such a stopping rule τ is considered a mapping from the space of continuous functions $C[0, \infty)$ to the half line $[0, \infty]$ satisfying

$$\{f \in C[0, \infty) : \tau(f) \leq t\} \in \varphi_t^{-1}(\mathfrak{B}) \qquad 0 \leq t < \infty, \tag{1.3}$$

where $\mathfrak{B}$ is the Borel σ-algebra generated by the open sets in $C[0, \infty)$ and $\varphi_t : C[0, \infty) \to C[0, \infty)$ is the mapping

$$(\varphi_t f)(s) = f(t \wedge s) \qquad 0 \leq s < \infty.$$

Let $\mathcal{S}$ be the collection of all such stopping rules. If the stopper decides to end the game at $\tau(X_x)$, then the controller pays the stopper an amount of $\int_0^{\tau(X_x)} e^{-\alpha t} C(X_x(t))dt$ at that time. Here α is a positive constant. Thus, the controller would like to choose the control process $u(\cdot)$ to minimize the expected value $E \int_0^\tau e^{-\alpha t} C(X_x(t))dt$, while the stopper would like to select the stopping rule τ to maximize it. The positive discount factor α is useful, so that the expected pay-off will remain finite even for any degenerate diffusion process with state space contained in the set $[C > 0]$.

We define the upper and lower value functions of this game by

$$\overline{V}(x) = \inf_{\tau \in \mathcal{S}} \sup_{X_x \in \mathcal{A}(x)} E \int_0^\tau e^{-\alpha t} C(X_x(t))dt \tag{1.4}$$

and

$$\underline{V}(x) = \sup_{X_x \in \mathcal{A}(x)} \inf_{\tau \in \mathcal{S}} E \int_0^\tau e^{-\alpha t} C(X_x(t))dt \tag{1.5}$$

respectively.

If $\overline{V}(x) = \underline{V}(x)$, then this game has a *value*, and in that case, we denote this common value function by $V(x)$. The discrete time controller and stopper game has been studied in [5]. They showed that the game has a value when the reward function is Borel-measurable and the state space is a Polish space. Our problem is closely related to the continuous time controller and stopper game studied by [4].

Throughout the article, we make the following assumptions (i), (ii) and (iii) about the drift coefficient $\mu(\cdot)$ and the reward function $C(\cdot)$. Here, μ' and μ'' denote the first and second derivatives of the function μ. C' denotes the first derivative of C.

(i) The function μ is twice continuously differentiable and it satisfies the following conditions:

$$\text{a)} \quad x\mu(x) < 0 \text{ for all } x \neq 0, \tag{1.6}$$

$$\text{b)} \quad \alpha - \mu'(x) > 0 \text{ for all } x, \text{ and} \tag{1.7}$$

$$\text{c)} \quad (\alpha - 2\mu'(x))(\alpha - \mu'(x)) > \mu(x)\mu''(x) \text{ for all } x. \tag{1.8}$$

(ii) The function C is twice continuously differentiable, $C(0) > 0$ and the set $\{x : C(x) > 0\}$ is an open interval. We denote it by

$$\{x : C(x) > 0\} = (r, s). \tag{1.9}$$

(iii)

$$\text{The function } \frac{C'(x)}{(\alpha - \mu'(x))} \text{ is decreasing on } \mathbf{R}. \tag{1.10}$$

Remarks.

1. In the case of linear drift, say, $\mu(x) = -\theta x$ where $\theta > 0$ is a positive constant, the above assumptions (1.6)-(1.8) are trivially true. Furthermore, (1.9) and (1.10) reduce to C being a twice differentiable concave function with $C(0) > 0$.
2. If μ is decreasing and $\alpha^2 > \mu(x)\mu''(x)$ for all x, then (1.7) and (1.8) are automatically satisfied.

Next, as in [4], we introduce the definition of a saddle point of the game.

Definition 1.1.
A pair (τ^*, Z^*) in $\mathcal{S} \times \mathcal{A}(x)$ is called a *saddle point* of the game, if

$$\begin{aligned} E\int_0^{\tau} e^{-\alpha t} C(Z^*(t))dt &\leq E\int_0^{\tau^*(Z^*)} e^{-\alpha t} C(Z^*(t))dt \\ &\leq E\int_0^{\tau^*(X)} e^{-\alpha t} C(X(t))dt \end{aligned} \tag{1.11}$$

for every τ in $\mathcal{S}$ and every $X(\cdot)$ in $\mathcal{A}(x)$.

The existence of a saddle point clearly implies that the game has a value and in this case,

$$\overline{V}(x) = \underline{V}(x) = E\int_0^{\tau^*(Z^*)} e^{-\alpha t} C(Z^*(t))dt.$$

We intend to characterize a saddle point and to derive the explicit form of the value function for this game.

For early work on stochastic differential games, we refer to chapter 17 of [2]. There, stochastic differential games are discussed for non-degenerate diffusion processes in which the control variables occur in the drift coefficient. In [1], two player, zero sum stochastic differential games in finite time horizon are considered in the viscosity solutions framework. There, both players control the drift and diffusion coefficients in the presence of a running cost function and a terminal cost function. They assume that the drift and diffusion coefficients of the state process as well as both cost functions are bounded. They show the existence of the value function and characterize it as the unique viscosity solution to the corresponding "Bellman-Isaacs" partial differential equation.

In our problem, the cost function is unbounded and the drift coefficient of the state process also can be unbounded. Here, the game has a value and this value function is Lipschitz continuous, but it fails to be a C^1 function.

This paper is organized as follows. In section 2, we introduce a class of degenerate diffusion processes. Then we assume the existence of a function Q_0 which is a solution to a second order differential equation with overdetermined boundary data. Using this function Q_0, we prove the main theorem of the article. There, we obtain a saddle point and the corresponding optimal state process which belongs to the above described class of degenerate diffusion processes. We discuss the properties of the value function of an optimal stopping problem in section 3. This will be used in the construction of Q_0. Section 4 is devoted to the construction of the function Q_0 with the desired properties as required in section 2. In the case of linear drift coefficient $\mu(x) = -\theta x$ where θ is a positive number, the computations related to the construction of Q_0 in Theorem 4.1 simplify a great deal.

2 A Class of Degenerate Diffusion Processes and a Saddle Point

We begin with a class of degenerate diffusion processes used in a stochastic control problem in [6]. The construction of such processes is described in page 5 of [6], but it is an immediate application of the proof of Theorem 7.2 in chapter 4 of Ikeda and Watanabe (p.208-214, [3]). For each open interval (a, b) with $a < 0 < b$, consider a weak solution to

$$X_x(t) = x + \int_0^t \mu(X_x(s))ds + \int_0^t \sigma_0 I_{(a,b)}(X_x(s))dW(s), \tag{2.1}$$

where x is a real number, μ is a twice differentiable function on $\mathbf{R}$ which satisfies (1.6), $\{W(t) : t \geq 0\}$ is a standard Brownian motion adapted to a right continuous filtration $\{\mathfrak{F}_t : t \geq 0\}$ on a probability space $(\Omega, \mathfrak{F}, P)$. For a discussion on the existence of a weak solution and the uniqueness in law for such processes, we refer to [6].

To describe the saddle point, first we assume the existence of a function Q_0 and an open interval (a^*, b^*) so that $a^* < 0 < b^*$ and the following conditions are satisfied:

$$\text{(i)} \quad \frac{\sigma^2}{2}Q_0''(x) + \mu(x)Q_0'(x) - \alpha Q_0(x) + C(x) = 0 \text{ for all } x \text{ in } (a^*, b^*). \tag{2.2}$$

$$\text{(ii)} \quad Q_0(a^*) = Q_0(b^*) = 0, \quad Q_0''(a^*) = Q_0''(b^*) = 0 \text{ and } Q_0(x) > 0 \text{ for all } x \text{ in } (a^*, b^*). \tag{2.3}$$

It will be shown in section 5 that such a function Q_0 and the interval (a^*, b^*) satisfying $a^* < 0 < b^*$ exist. In the following lemma, we obtain two properties of the function Q_0 under the assumptions (2.2) and (2.3). They will be used in the derivation of a saddle point in the proof of Theorem 2.2.

Lemma 2.1. *Assume the existence of a function Q_0 and a finite interval (a^*, b^*) so that $a^* < 0 < b^*$ and satisfying (2.2) and (2.3). Then*

$$\begin{aligned}&\text{(i)} \quad a^* \le r < 0 < s \le b^*, \text{ where the interval } (r,s) \text{ is defined in (1.9), and} && (2.4)\\ &\text{(ii)} \quad Q_0''(x) \le 0 \text{ for all } x \text{ in the interval } [a^*, b^*]. && (2.5)\end{aligned}$$

Proof. Since the coefficients of the differential equation (2.2) are C^1 functions on $\mathbf{R}$, the function Q_0 which satisfy (2.2) and (2.3) can be extended to $\mathbf{R}$ so that it satisfies (2.2) everywhere. Consider this extended function Q_0 and evaluate (2.2) at a^* and b^*. Using (2.3), then we obtain $\mu(a^*)Q_0'(a^*) + C(a^*) = 0$ and $\mu(b^*)Q_0'(b^*) + C(b^*) = 0$. By (2.3), $Q_0 > 0$ on (a^*, b^*) and hence $Q_0'(a^*) \ge 0$ and $Q_0'(b^*) \le 0$. Since $a^* < 0 < b^*$, using (1.6), we obtain $\mu(a^*) > 0 > \mu(b^*)$. Consequently, $C(a^*) \le 0$ and $C(b^*) \le 0$. Therefore, using (1.9), we obtain $a^* \le r < 0 < s \le b^*$ and part (i) follows.

To prove part (ii), we intend to use the maximum principle in differential equations (page 7, [8]). By differentiating the differential equation in (2.2) when $a^* < x < b^*$, we obtain

$$(\alpha - \mu'(x))Q_0'(x) = \frac{\sigma^2}{2}Q_0'''(x) + \mu(x)Q_0''(x) + C'(x). \tag{2.6}$$

Let $P(x) = Q_0''(x)$ on $[a^*, b^*]$. By differentiating (2.6) again and rearranging the terms, we derive

$$\frac{\sigma^2}{2}P''(x) + \mu(x)P'(x) - (\alpha - 2\mu'(x))P(x) + \mu''(x)Q_0'(x) + C''(x) = 0,$$

for $a^* < x < b^*$. Next, we use (2.6) to replace $Q_0'(x)$ in the above equation and obtain

$$\begin{aligned}&\frac{\sigma^2}{2}P''(x) + \mu(x)P'(x) - (\alpha - 2\mu'(x))P(x)\\ &\qquad + \frac{\mu''(x)}{(\alpha - \mu'(x))}[\frac{\sigma^2}{2}P'(x) + \mu(x)P(x) + C'(x)] + C''(x) = 0,\end{aligned}$$

for $a^* < x < b^*$. This can be rewritten as

$$\frac{\sigma^2}{2}P''(x) + b(x)P'(x) - r(x)P(x) + H(x) = 0$$

for $a^* < x < b^*$. Here, $b(x) = \mu(x) + \dfrac{\sigma^2}{2}\frac{\mu''(x)}{(\alpha-\mu'(x))}$, $r(x) = [(\alpha - 2\mu'(x)) - \frac{\mu(x)\mu''(x)}{(\alpha-\mu'(x))}]$ and $H(x) = \frac{C''(x)(\alpha-\mu'(x))+\mu''(x)C'(x)}{(\alpha-\mu'(x))}$ on the interval $[a^*, b^*]$. We use the assumptions (1.7) and (1.8) to observe that the functions $b(.), r(.)$ and $H(.)$ are well defined and continuous on $[a^*, b^*]$. Moreover, $r(x) > 0$ by (1.6) and by (1.10), $\frac{d}{dx}[\frac{C'(x)}{(\alpha-\mu'(x))}] \le 0$ on $[a^*, b^*]$.
Notice that $\frac{H(x)}{(\alpha-\mu'(x))} = \frac{d}{dx}[\frac{C'(x)}{(\alpha-\mu'(x))}]$. Therefore, using (1.7) and (1.10), we can conclude that $H(x) \le 0$ on (a^*, b^*). Since, $P(a^*) = P(b^*) = 0$, now we can apply the maximum principle for differential equations (page 7, [8]) to conclude $P(x) \le 0$ on $[a^*, b^*]$. Hence, part (ii) follows and this completes the proof of the lemma. □

To describe our candidate for a saddle point, first we introduce the process Z_x^*, which is a weak solution to

$$Z_x^*(t) = x + \int_0^t \mu(Z_x^*(s))ds + \int_0^t \sigma_0 I_{(a^*,b^*)}(Z_x^*(s))dW(s) \tag{2.7}$$

where I_A represents the indicator function of the set A. Notice that the corresponding control process $u^*(t) = I_{(a^*,b^*)}(Z_x^*(t))$ is a feed-back type control. Next, we introduce a stopping rule τ^* on $C[0,\infty)$ defined by

$$\begin{aligned} \tau^*(f) &= \inf\{t \geq 0 : f(t) \leq a^* \text{ or } f(t) \geq b^*\} \\ &= +\infty, \text{ if the above set is empty.} \end{aligned} \tag{2.8}$$

Now, we are ready to show that the above pair (τ^*, Z_x^*) is a saddle point.

Theorem 2.2. *Assume the existence of a finite interval (a^*, b^*) and a function Q_0 satisfying (2.2) and (2.3). Then the pair (τ^*, Z_x^*) is a saddle point. Furthermore, the stochastic game has a value function, is given by*

$$V(x) = \begin{cases} Q_0(x) & \text{if } a^* < x < b^*, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Proof. We intend to verify the two inequalities in (1.11) in the following proof. Notice that the domain of the infinitesimal generator of the Z_x^* process consists of all C^2 functions f which satisfy $f''(a^*) = f''(b^*) = 0$ (for details, see [6].)

Let x be in $[a^*, b^*]$. Then,we can apply Itô's lemma to $Q_0(Z_x^*(t))e^{-\alpha t}$ and use (2.2) and (2.3) to obtain

$$Q_0(x) = E\int_0^{\tau^*(Z^*)} e^{-\alpha t} C(Z^*(t))dt \tag{2.10}$$

Next, we take any stopping time τ and apply Itô's lemma to obtain

$$0 \leq E[Q_0(Z_x^*(\tau))e^{-\alpha\tau}] = Q_0(x) - E\int_0^{\tau} e^{-\alpha t} C(Z^*(t))dt.$$

Hence,

$$E\int_0^{\tau} e^{-\alpha t} C(Z^*(t))dt \leq Q_0(x). \tag{2.11}$$

This verifies the first inequality of (1.11) when the initial point x is inside the interval $[a^*, b^*]$.

If x is outside the interval $[a^*, b^*]$, then $\tau^*(Z_x^*) = 0$ and $E\int_0^{\tau^*(Z^*)} e^{-\alpha t} C(Z^*(t))dt = 0$. The process Z_x^* is deterministic until it enters the interval $[a^*, b^*]$ and we let T_0 be the first entrance time to the interval $[a^*, b^*]$. Then

$$\begin{aligned} E\int_0^{\tau} e^{-\alpha t} C(Z^*(t))dt =& E\int_0^{\tau\wedge T_0} e^{-\alpha t} C(Z^*(t))dt \\ &+ E\int_{\tau\wedge T_0}^{\tau} e^{-\alpha t} C(Z^*(t))dt. \end{aligned}$$

By Lemma 2.1, the cost function C is negative outside the interval $[a^*, b^*]$. Hence, the first term is less than or equal to zero. Similar to the proof of (2.11), we can show that the second term is also less than or equal to zero. Thus,

$$E\int_0^{\tau} e^{-\alpha t}C(Z^*(t))dt \le E\int_0^{\tau^*(Z^*)} e^{-\alpha t}C(Z^*(t))dt = 0.$$

This verifies the first inequality of (1.11) when x is outside $[a^*, b^*]$.

Now it remains to verify the second inequality of (1.11). Let the initial point x be inside $[a^*, b^*]$ and X_x be any process which satisfies (1.1) with the corresponding control $u(\cdot)$. We apply Itô's lemma and obtain

$$E\int_0^{\tau^*(X)} e^{-\alpha t}(\frac{u(t)^2}{2}Q_0'' + \mu(X_x(t))Q_0' - \alpha Q_0)(X_x(t))dt$$
$$= Q_0(x) - E\left[Q_0(X_x(\tau^*(X)))e^{-\alpha\tau^*(X)}\right] \quad (2.12)$$

where τ^* is defined in (2.8). By (2.5), $Q_0'' \le 0$ on $[a^*, b^*]$. Therefore,

$$\frac{\sigma_0^2}{2}Q_0''(X_x(t)) \le \frac{u(t)^2}{2}Q_0''(X_x(t)).$$

By using (2.2), we obtain

$$E\left[Q_0(X_x(\tau^*(X)))e^{-\alpha\tau^*(X)}\right] \ge Q_0(x) - E\int_0^{\tau^*(X)} e^{-\alpha t}C(X_x(t))dt.$$

Notice that if $\tau^*(X)$ is finite, then $Q_0(X_x(\tau^*(X))) = 0$ and
$Q_0(X_x(\tau^*(X)))e^{-\alpha\tau^*(X)} = 0$ when $\tau^*(X)$ is infinite.
Hence, $E[Q_0(X_x(\tau^*(X)))e^{-\alpha\tau^*(X)}] = 0$ and

$$E\int_0^{\tau^*(X)} e^{-\alpha t}C(X_x(t))dt \ge Q_0(x).$$

Using this together with (2.10), we obtain the second inequality of (1.11) when x is in $[a^*, b^*]$.

When x is outside the interval $[a^*, b^*]$, clearly, $\tau^*(X) = 0$ and $\tau^*(Z^*) = 0$. Consequently, $E\int_0^{\tau^*(X)} e^{-\alpha t}C(X_x(t))dt = 0$ and

$$E\int_0^{\tau^*(Z^*)} e^{-\alpha t}C(Z_x^*(t))dt = 0$$

. Hence the second inequality of (1.11) follows.

Therefore, we can conclude that the pair (τ^*, Z_x^*) is a saddle point and $\underline{V}(x) = \overline{V}(x)$ for all x, where $\overline{V}(x)$ and $\underline{V}(x)$ are defined in (1.4) and (1.5) respectively. Consequently, this stochastic game has a value function and it is given by (2.9). This completes the proof. $\square$

3 An Optimal Stopping Problem

In this section, we consider an optimal stopping problem which is related to the construction of the function Q_0. We begin with the process $\{Y_x(t) : t \geq 0\}$ which satisfies

$$Y_x(t) = x + \int_0^t \mu(Y_x(s))ds + \sigma_0 W(t), \tag{3.1}$$

for $t \geq 0$. Here, $\{W(t) : t \geq 0\}$ is a Brownian motion adapted to a Brownian filtration $\{\mathfrak{F}_t\}$ in a probability space $(\Omega, \mathfrak{F}, P)$. We introduce the infinitesimal generator $\mathcal{G}$ of (3.1) by

$$\mathcal{G} = \frac{\sigma_0^2}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} \tag{3.2}$$

and for $\alpha > 0$, we define the differential operator $\mathcal{G} - \alpha$ by

$$\mathcal{G} - \alpha = \frac{\sigma_0^2}{2}\frac{d^2}{dx^2} + \mu(x)\frac{d}{dx} - \alpha. \tag{3.3}$$

Let $\mathcal{D}$ be the collection of all $\{\mathfrak{F}_t\}$ stopping times. We consider the optimal stopping problem with the value function $U(x)$ given by

$$U(x) = \sup_{\tau \in \mathcal{D}} E \int_0^\tau e^{-\alpha t} C(Y_x(t))dt. \tag{3.4}$$

Since $C(\cdot)$ is positive on the interval (r, s), $U(\cdot)$ is also strictly positive on (r, s). It is known that(see [7]) this value function U is a C^1 function and it satisfies the variational inequality

$$\max\{(\mathcal{G} - \alpha)U(x) + C(x), -U(x)\} = 0, \tag{3.5}$$

for almost every x in $\mathbf{R}$.

It is also known that, (see [7]), there is a finite interval $[c^*, d^*]$ so that the set $[U(x) > 0]$ is equal to the open interval (c^*, d^*). Moreover, $U(c^*) = U(d^*) = 0$, $(\mathcal{G} - \alpha)U(x) + C(x) = 0$ for $c^* < x < d^*$, $U'(c^*) = U'(d^*) = 0$ and $U(x) = 0$ outside the interval (c^*, d^*). Thus, "the principle of smooth fit" holds for U at the points c^* and d^*. Also, $U''(c^*+) > 0$ and $U''(d^*-) > 0$.

Furthermore, if Q_{ab} is a positive function on $[a, b]$ which satisfies

$$\begin{aligned} (\mathcal{G} - \alpha)Q_{ab}(x) + C(x) = 0, \text{ for } a < x < b, \\ Q_{ab}(a) = Q_{ab}(b) = 0, \quad Q_{ab}(x) > 0 \text{ on } (a, b). \end{aligned} \tag{3.6}$$

Then by Itô's lemma, we can conclude that

$$Q_{ab}(x) = E \int_0^{\tau_{ab}} e^{-\alpha t} C(Y_x(t))dt,$$

for $a \leq x \leq b$, where τ_{ab} is the first exit time from the interval $[a, b]$. Hence $0 < Q_{ab}(x) \leq U(x)$ for all x in (a, b). Consequently,

$$[a, b] \subseteq [c^*, d^*]. \tag{3.7}$$

4 Construction of Q_0

Let $\mathcal{G}$ be the differential operator introduced in (3.2). We consider the collection of the functions Q_{ab} along with their corresponding intervals $[a, b]$ satisfying the following conditions:

(i) The interval $[a, b]$ satisfy

$$[r, s] \subseteq [a, b] \subseteq [c^*, d^*]\,. \tag{4.1}$$

(ii) The function Q_{ab} is a solution to the boundary value problem

$$\begin{aligned} &(\mathcal{G} - \alpha)Q_{ab}(x) + C(x) = 0, \quad \text{for } a < x < b, \\ &Q_{ab}(a) = Q_{ab}(b) = 0. \end{aligned} \tag{4.2}$$

(iii) The function Q_{ab} also satisfies

$$Q_{ab}(x) > 0 \quad \text{on } (a, b), \quad Q''_{ab}(a) \le 0, \quad \text{and} \quad Q''_{ab}(b) \le 0. \tag{4.3}$$

Theorem 4.1. *There is a finite interval $[a^*, b^*]$ and an associated function $Q_{a^*b^*}$ which satisfy* (4.1), (4.2) *and* (4.3) *above together with the following additional property:*

$$Q''_{a^*b^*}(a^*) = Q''_{a^*b^*}(b^*) = 0 \quad \textit{and} \quad Q''_{a^*b^*}(x) \le 0$$

on the interval (a^*, b^*).

Remark.
For convenience, we relabel this function $Q_{a^*b^*}$ as Q_0 and use it as required in Theorem 2.1.

Proof. We set

$$\mathcal{C} \;=\; \big\{[a, b] \;:\; [a, b] \text{ satisfies (4.1) and there exists } Q_{ab} \text{ satisfying (4.2) and (4.3)}\big\} \tag{4.4}$$

First we show that $[r, s]$ is in $\mathcal{C}$. Since $C(\cdot)$ is strictly positive in (r, s), we can apply the maximum principle for differential equations (see [8], page 7) to Q_{rs} to conclude $Q_{rs}(x) > 0$ on (r, s). Also, by the boundary point lemma ([8] page 7), $Q'_{rs}(r) > 0$ and $Q'_{rs}(s) < 0$. Hence, using the differential equation for Q_{rs} it follows that $Q''_{rs}(r) < 0$ and $Q''_{rs}(s) < 0$. Therefore, the interval $[r, s]$ is in $\mathcal{C}$.

Now let $([a_n, b_n])$ be an increasing sequence of nested intervals in $\mathcal{C}$. Then (a_n) is a decreasing sequence and (b_n) is an increasing sequence. We let $l = \lim_n a_n$ and $m = \lim_n b_n$. Then l and m are finite and the interval $[l, m]$ satisfies (4.1).

Throughout the remaining arguments, we will use the following facts from the theory of differential equations.

a) Two distinct solutions of the differential equation $(\mathcal{G} - \alpha)Q(x) + C(x) = 0$ cannot meet more than once.

b) The solution $Q_{ab}(x)$ and its first and second derivatives are jointly continuous in (a, b, x).

By elementary analysis, it can be shown that $Q_{a_n b_n}$ is increasing to Q_{lm}, $\lim_n Q'_{a_n b_n}(x) = Q'_{lm}(x)$ and $\lim_n Q''_{a_n b_n}(x) = Q''_{lm}(x)$ for all x in (l, m). It is easy to verify that Q_{lm} satisfies (4.2) and (4.3). Thus, $[l, m]$ is in $\mathcal{C}$. Therefore, each nested increasing sequence in $\mathcal{C}$ has an upper bound in $\mathcal{C}$. By Zorn's lemma, $\mathcal{C}$ has a maximal element. We label it $[a^*, b^*]$. Our next step is to to show that $Q''_{a^*b^*}(a^*) = Q''_{a^*b^*}(b^*) = 0$.

First, notice that from (3.7), it follows that if $[a, b]$ is in $\mathcal{C}$, then $[a, b] \subseteq [c^*, d^*]$. In the following argument, we intend to show that $[a, b]$ is a proper subset of $[c^*, d^*]$.

Suppose that $[a, b] = [c^*, d^*]$, then, by the uniqueness of the solution to (4.2), $Q_{ab}(x) = U(x)$ for all x in $[c^*, d^*]$, where U is given in (3.4). But, by the discussion below (3.5), we have $U''(c^*+) > 0$ and $U''(d^*-) > 0$. Hence, Q_{ab} does not satisfy (4.3). This is a contradiction and we conclude that $[a, b] \neq [c^*, d^*]$. Now suppose $a = c^*$, $b < d^*$. Then, $Q_{c^*d^*}(x) \geq Q_{ab}(x) \geq 0$ for all x in $[a, b]$, since $Q_{c^*d^*}$ is same as the value function U of the optimal stopping problem (3.4). But $Q'_{c^*d^*}(c^*) = 0$, hence $Q'_{ab}(a) = 0$. Then, $Q_{c^*d^*}$ and Q_{ab} both satisfy the same differential equation with the same initial conditions at the point $a = c^*$. Hence, they are the same and this implies that $b = d^*$. This is a contradiction. A similar contradiction can be obtained in the case $a > c^*$, $b = d^*$. Consequently, $c^* < a < b < d^*$ for any $[a, b]$ in $\mathcal{C}$.

Next, if $[a, b]$ is in $\mathcal{C}$, then by (4.3), $Q''_{ab}(a) \leq 0$ and $Q''_{ab}(b) \leq 0$. Suppose that $Q''_{ab}(a) \leq 0$ and $Q''_{ab}(b) < 0$ when $[a, b]$ is in $\mathcal{C}$. We will now show that there is an $\epsilon > 0$ and a $\delta > 0$ so that $[a - \delta, b + \epsilon]$ is in $\mathcal{C}$. The corresponding result can also be obtained when $Q''_{ab}(a) < 0$ and $Q''_{ab}(b) \leq 0$ and $[a, b]$ is in $\mathcal{C}$ with an analogous proof.

We must first extend the function Q_{ab} to $\mathbf{R}$ so that it satisfies the differential equation in (4.2) everywhere. Then we evaluate Q_{ab} in (4.2) at the points a and b to obtain

$$\mu(a)Q'_{ab}(a) + C(a) \geq 0 \quad \text{and} \quad \mu(b)Q'_{ab}(b) + C(b) > 0. \tag{4.5}$$

Since $[r, s] \subseteq [a, b]$, $\mu(a) > 0$ and $\mu(b) < 0$, we can conclude $Q'_{ab}(a) \geq 0$ and $Q'_{ab}(b) < 0$. Hence, we can pick a small $\epsilon_1 > 0$ so that $Q_{ab}(b+\epsilon) < 0$ for each $0 < \epsilon < \epsilon_1$. Next, consider the solution $Q_{a,b+\epsilon}$ to (4.2) on the interval $(a, b+\epsilon)$. Since $Q_{ab}(x) > 0$ on (a, b) and since two distinct solutions to the differential equation can meet only once, we obtain $Q_{a,b+\epsilon}(x) > Q_{ab}(x)$ for all $a < x \leq b$. Consequently, $Q'_{a,b+\epsilon}(a) \geq Q'_{ab}(a)$. But, by the uniqueness of the solutions to the initial value problem corresponding to (4.2), we conclude $Q'_{a,b+\epsilon}(a) \neq Q'_{ab}(a)$. Hence, $Q'_{a,b+\epsilon}(a) > Q'_{ab}(a) \geq 0$.

By evaluating (4.2) for $Q_{a,b+\epsilon}$ at the point $x = a$, and by (4.5), we obtain

$$\frac{\sigma^2}{2} Q''_{a,b+\epsilon}(a) = -[\mu(a)Q'_{a,b+\epsilon}(a) + C(a)] < -[\mu(a)Q'_{ab}(a) + C(a)] \leq 0.$$

Thus, $Q''_{a,b+\epsilon}(a) < 0$ for each $0 < \epsilon < \epsilon_1$. But, as a function of (p, q, x), $Q''_{pq}(x)$ is continuous in (p, q, x) and $Q''_{ab}(b) < 0$. Therefore, we can choose an $\epsilon > 0$ so that $Q''_{a,b+\epsilon}(b + \epsilon) < 0$. Consequently, $[a, b + \epsilon]$ is in $\mathcal{C}$. Using the joint continuity of $Q_{pq}(x), Q'_{pq}(x)$ and $Q''_{pq}(x)$ in the variables (p, q, x), now we can find a small $\delta > 0$ so that $Q''_{a-\delta,b+\epsilon}(a - \delta) < 0$ and $Q''_{a-\delta,b+\epsilon}(b + \epsilon) < 0$. For such a $\delta > 0$, $Q_{a-\delta,b+\epsilon}$ and $Q_{a,b+\epsilon}$ can meet only at $b + \epsilon$ and hence $Q_{a-\delta,b+\epsilon}(x) > Q_{a,b+\epsilon}(x) > Q_{ab}(x)$ for all x in $(a, b + \epsilon)$. Consequently, $[a - \delta, b + \epsilon]$ is also in $\mathcal{C}$.

Since $[a^*, b^*]$ is a maximal element in $\mathcal{C}$, therefore it follows that $Q''_{a^*b^*}(a^*) = Q''_{a^*b^*}(b^*) = 0$. This completes the proof. $\square$

Acknowledgements. I would like to thank an anonymous referee for many helpful comments. This work is supported in part by the Army Research Office under the grant no. W 911NF0510032. I am grateful for their support.

References

[1] W.H. Fleming and P.E. Souganidis, *On the existence of value functions of two-player zero-sum stochastic differential games,* Indiana Univ. Math. Journal 38,(1989), pp.293-314.

[2] A. Friedman, *Stochastic differential equations and applications, volume 2,* (1976) Academic Press, New York.

[3] N. Ikeda and S. Watanabe, *Stochastic differential equations and diffusion processes* (1989) North-Holland, Amsterdam, New York.

[4] I. Karatzas and W. D. Sudderth, *The controller-and-stopper game for a linear diffusion,* Annals of Probability, 29 (2001) pp. 1111-1127.

[5] A.P. Maitra and W. D. Sudderth, *The gambler and the stopper*, Statistics, Probability and Game Theory: Papers in Honor of David Blackwell (T.S. Ferguson, L.S. Shapely and J. B. MacQueen, Editors) (1996) pp.191-208 Springer, Berlin.

[6] D. Ocone and A. Weerasinghe, *Degenerate variance control of a one dimensional diffusion process*, SIAM J. Control and Opt. 39 (2000) pp. 1-24.

[7] D. Ocone and A. Weerasinghe, *Degenerate variance control in the one dimensional stationary case,* Elec. J. of Probab. 8 (2003), pp.1-27.

[8] M. Protter and H. F. Weinberger, *Maximum principles in differential equations*, (1984), Springer Verlag, New York.