Elect. Comm. in Probab. **13** (2008), 266–271

ELECTRONIC COMMUNICATIONS in PROBABILITY

A NOTE ON THE DISTRIBUTIONS OF THE MAXIMUM OF LINEAR BERNOULLI PROCESSES

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Submitted May 4, 2007, *accepted in final form* May 5, 2008

AMS 2000 Subject classification: 60J65
Keywords: Linear Bernoulli processes, distribution of the maximum

Abstract
We give a characterization of the family of all probability measures on the extended line $(-\infty, +\infty]$, which may be obtained as the distribution of the maximum of some linear Bernoulli process.

On a probability space $(\Omega, \mathbf{P})$ consider a linear process

$$X(t) = a_0(t) + \sum_{n=1}^{\infty} a_n(t)\xi_n, \quad t \in T, \tag{1}$$

generated by independent, identically distributed random variables ξ_n with $\mathbf{E}\xi_n = 0$, $\mathbf{E}\xi_n^2 = 1$. The coefficients $a_n(t)$ are assumed to be arbitrary functions on the parameter set T, satisfying $\sum_{n=1}^{\infty} a_n(t)^2 < +\infty$ for any $t \in T$, so that the series (1) is convergent a.s. Define

$$M = \sup_t X(t) \tag{2}$$

in the usual way as the essential supremum in the space of all random variables with values in the extended real line (identifying random variables that coincide almost surely; cf. Remark 4 below).

We consider the question on the characterization of the family $\mathcal{F}(L)$ of all possible distribution functions $F(x) = \mathbf{P}\{M \leq x\}$ of M, assuming that the common law L of ξ_n is given. In general, M may take the value $+\infty$ with positive probability, so its distribution is supported on $(-\infty, +\infty]$. Introduce also the collection $\mathcal{F}_0(L)$ of all possible distribution functions of M in (2), such that in the series (1), for all $t \in T$,

$$a_n(t) = 0, \quad \text{for all sufficiently large } n. \tag{3}$$

When ξ_n are standard normal, i.e., $L = N(0,1)$, we deal in (1) with an arbitrary Gaussian random process. As is well-known, for the distribution function F of M, $x_0 = \inf\{x \in \mathbf{R} : F(x) > 0\}$ may be finite, and then it is sometimes called a take-off point of the maximum of the Gaussian process. Moreover, F may have an atom at it. But anyway F is absolutely continuous and strictly increasing on $(x_0, +\infty)$, which follows from the log-concavity of Gaussian measures (cf. also [C], [HJ-S-D]).
A complete characterization of all possible distributions F in the Gaussian case may be derived from the Brunn-Minkowski-type inequality for the standard Gaussian measure γ_n on $\mathbf{R}^n$ due to A. Ehrhard [E]. It states that, for all convex (and in fact, for all Borell measurable, cf. [Bo2]) sets A and B in $\mathbf{R}^n$ of positive measure and for all $\lambda \in (0,1)$,

$$\Phi^{-1}\left(\gamma_n(\lambda A + (1-\lambda)B)\right) \;\geq\; \lambda\, \Phi^{-1}\left(\gamma_n(A)\right) + (1-\lambda)\, \Phi^{-1}\left(\gamma_n(B)\right),$$

where Φ^{-1} denotes the inverse to the standard normal distribution function on the line. This inequality immediately implies that, if F is non-degenerate, the function $U = \Phi^{-1}(F)$ must be concave on $\mathbf{R}$ in the generalized sense as a function with values in $[-\infty, +\infty)$. But the converse is true, as well.
Indeed, suppose $U = \Phi^{-1}(F)$ is concave on $\mathbf{R}$, and for simplicity let F be non-degenerate and do not assign a positive mass to the point $+\infty$. Then F is strictly increasing on $(x_0, +\infty)$, so is its inverse $F^{-1} : (F(x_0), 1) \to (x_0, +\infty)$. Moreover, the inverse function $U^{-1} = F^{-1}(\Phi)$ is convex and strictly increasing on $(U(x_0), +\infty)$. Put $M(x) = U^{-1}(x)$ for $x > U(x_0)$, and if x_0 is finite, $M(x) = x_0$ on $(-\infty, U(x_0)]$. Then M is convex and finite on the whole real line, and therefore admits a representation

$$M(x) = \sup_{t \in T}\, [a_0(t) + a_1(t)x], \quad x \in \mathbf{R},$$

for some coefficients $a_0(t)$, $a_1(t)$. By the construction, M has the distribution function F under the measure γ_1, as was required.
Thus, a given non-degenerate distribution function F belongs to $\mathcal{F}(N(0,1))$, if and only if the function $\Phi^{-1}(F)$ is concave. A similar characterization holds true, when ξ_n's have a shifted one-sided exponential distribution with mean zero. Then, F represents the distribution function of M for some coefficients $a_n(t)$, if and only if the function $\log F$ is concave. This follows from the log-concavity of the multidimensional exponential distribution (which is a particular case of Prékopa's theorem [P]; cf. also [Bo1] for a general theory of log-concave measures).
In both above examples, for the "if" part it sufficies to consider simple linear processes $X(t) = a_0(t) + a_1(t)\xi_1$. Hence, $\mathcal{F}_0(L) = \mathcal{F}(L)$. The situation is completely different, when ξ_n have a symmetric Bernoulli distribution L, i.e., taking the values ± 1 with probability $\frac{1}{2}$. This may be seen from:

Theorem 1. *Any distribution function F, such that $F(x) = 0$, for some $x \in \mathbf{R}$, may be obtained as the distribution function of the supremum M of some linear Bernoulli process X in* (1) *with coefficients, satisfying the property* (3).

In turn, the condition (3) ensures that all random variables $X(t)$ in (1) are bounded from below, so is the random variable M in (2). Therefore, the distribution F of M must be one-sided. Thus, we have a full description of the family $\mathcal{F}_0(L)$ in the Bernoulli case. Removing the condition (3), we obtain a larger family $\mathcal{F}(L)$; however, it is not clear at all how to characterize it.

One should also mention that in the homogeneous case $a_0(t) = 0$, much is known about various properties of M in terms of L, but the characterization problem is more delicate, and it seems no description or even conjecture are known in all above cases.
For the proof of Theorem 1 one may assume that $\Omega = \{-1,1\}^\infty$ is the infinite dimensional discrete cube, equipped with the product Bernoulli measure $\mathbf{P}$. An important property of Ω, which will play the crucial role, is that it represents the collection of all extreme points in the cube $K = [-1,1]^\infty$. More precisely, we apply the following statement.

Lemma 2. *Any lower semi-continuous function $f : \{-1,1\}^\infty \to (-\infty,+\infty]$ is representable as*

$$f(x) \ = \ \sup_{t \in T} \bigg[a_0(t) + \sum_{n=1}^\infty a_n(t) x_n \bigg], \qquad x = (x_1, x_2, \dots), \tag{4}$$

for some family of the coefficient functions $a_n(t)$, defined on a countable set T and satisfying the property (3).

Note any function of the form (4) is lower semi-continuous.

Proof. First, more generally, let K be a non-empty, compact convex set in a locally convex space E, and denote by Ω the collection of all extreme points of K. A function $f : \Omega \to (-\infty,+\infty]$ is representable as

$$f(x) = \sup_t f_t(x), \qquad x \in \Omega, \tag{5}$$

for some family $(f_t)_{t\in T}$ of continuous, affine functions on E, if and only if

a) f is lower semi-continuous on Ω;
b) f is bounded from below.

This characterization follows from a theorem, usually attributed to Hervé [H]; see E. M. Alfsen [A], Proposition 1.4.1, and historical remarks. Namely, a point x is an extreme point of K, if and only if $\bar{g}(x) = g(x)$, for any lower semi-continuous function g on K, where $\bar{g}$ denotes the lower envelope of g (i.e., the maximal convex, lower semi-continuous function on K, majorized by g).
Clearly, the equality (5) defines a function with properties $a) - b)$. For the opposite direction one may use an argument, contained in the proof of Corollary 1.4.2 of [A]. If f is bounded and lower semi-continuous on Ω, put $g(x) = \liminf_{y\to x} f(y)$ for $x \in \mathrm{clos}(\Omega)$ and $g = \sup_\Omega f$ on $K \setminus \mathrm{clos}(\Omega)$. Then g is lower semi-continuous on K and $g = f$ on Ω. By Hervé's theorem, $\bar{g}(x) = g(x) = f(x)$, for all $x \in \Omega$. Since $\bar{g}$ is also convex on K, one may apply to it the classical theorem on the existence of the representation

$$\bar{g}(x) = \sup_t f_t(x), \quad x \in K,$$

for some family $(f_t)_{t\in T}$ of continuous, affine functions on E (cf. e.g. [A], Proposition 1.1.2, or [M], Chapter 11). Thus, restricting this representation to Ω, we arrive at (5). Finally, if f is unbounded from above, write $f = \sup_n \min\{f,n\}$ and apply (5) to the sequence $\min\{f,n\}$.
In case of the infinite dimensional discrete cube, the right-hand side of (5) may further be specified. Indeed, any continuous, affine function g on $E = \mathbf{R}^\infty$ has the form $g(x_1,x_2,\dots) = a_0 + \sum_{n=1}^\infty a_n x_n$ with finitely many non-zero coefficients. Therefore, (5) is reduced to the

relation (4) with some coefficient functions $a_n = a_n(t)$, that are defined on non-empty, perhaps, uncountable set T and satisfy the property (3).
The latter implies that the sets $T_N = \{t \in T : a_n(t) = 0, \text{ for all } n > N\}$ are non-empty for all $N \geq N_0$ with a sufficiently large N_0. Define

$$f_N(x) \;=\; \sup_{t \in T_N} \left[a_0(t) + \sum_{n=1}^{\infty} a_n(t) x_n \right] \;=\; \sup_{t \in T_N} \left[a_0(t) + \sum_{n=1}^{N} a_n(t) x_n \right], \tag{6}$$

so that $f = \sup_{N \geq N_0} f_N$. Since for each point $v = (x_1, \dots, x_N)$ in the finite dimensional discrete cube $\{-1,1\}^N$, the second supremum in (6) is asymptotically attained for some sequence of indices in T_N, one may choose a countable subset $T_N(v)$ of T_N, such that

$$\sup_{t \in T_N} \left[a_0(t) + \sum_{n=1}^{N} a_n(t) x_n \right] \;=\; \sup_{t \in T_N(v)} \left[a_0(t) + \sum_{n=1}^{N} a_n(t) x_n \right].$$

Therefore, the set $T_N' = \cup_{v \in \{-1,1\}^N} T_N(v)$ is also countable, is contained in T_N, and by (6),

$$f_N(x) \;=\; \sup_{t \in T_N'} \left[a_0(t) + \sum_{n=1}^{\infty} a_n(t) x_n \right], \qquad \text{for all } x \in \{-1,1\}^{\infty}.$$

As a result, the supremum in (4) may be restricted to the countable set $\cup_N T_N'$.
Finally, let us note Ω is compact, so the property *b*) is automatically satisfied, when *a*) holds. This yields Lemma 2.

Proof of Theorem 1. According to Lemma 2, we need to show that distributions of lower semi-continuous functions f on $\{-1,1\}^{\infty}$ under the Bernoulli measure $\mathbf{P}$ fill the family of all one-sided distributions on $(-\infty, +\infty]$. In fact, it is enough to consider the functions of the special form $f(x) = \varphi(Q(x))$, where

$$Q(x) = \sum_{n=1}^{\infty} \frac{x_n + 1}{2^{n+1}}, \qquad x = (x_1, x_2, \dots) \in \{-1,1\}^{\infty},$$

and where $\varphi : [0,1] \to (-\infty, +\infty]$ is an arbitrary non-decreasing, left (or, equivalently, lower semi-) continuous function. It is allowed that for some point $p \in [0,1]$, φ jumps to the value $+\infty$, and then we require that $\lim_{s \to p} \varphi(s) = +\infty$, as part of the lower semi-continuity assumption.
The map Q is continuous and pushes forward $\mathbf{P}$ to the normalized Lebesgue measure λ on the unit interval $[0,1]$. Hence, f is lower semi-continuous, and its distribution under $\mathbf{P}$ coincides with the distribution of φ under λ.
It remains to see that, for any one-sided probability measure μ on $(-\infty, +\infty]$, there is an admissible φ with the distribution μ under λ. Let us recall the standard argument (cf. e.g. [Bi], Theorem 14.1). Introduce the distribution function $F(u) = \mu((-\infty, u])$, $-\infty < u \leq +\infty$, and define its "inverse"

$$\varphi(s) = \min\{u : F(u) \geq s\}, \quad 0 < s \leq 1.$$

Also put $\varphi(0) = \lim_{s \to 0} \varphi(s)$. Clearly, φ is non-decreasing. Given a sequence $s_n \uparrow s$, $0 < s_n < s \leq 1$, take minimal values u_n, u, such that $F(u_n) \geq s_n$, $F(u) \geq s$. We have $u_n \uparrow u'$, for some

$u' \le u$. Since $F(u') \ge s_n$, for all n, we get $F(u') \ge s$ and hence $u' \ge u$. This shows that φ is left continuous. Finally, given $s \in (0,1]$ and $\alpha > \varphi(0)$, by the definition, $\varphi(s) \le \alpha \Leftrightarrow F(u) \ge s$, for some $u \le \alpha$. Hence,

$$\{s \in (0,1] : \varphi(s) \le \alpha\} = \{s \in (0,1] : F(u) \ge s, \text{ for some } u \le \alpha\} = (0, F(\alpha)].$$

Thus, φ has the distribution function F under λ. The proof is now complete.

Remark 3. The statement of Theorem 1 remains to hold in case of arbitrary independent random variables ξ_n, taking two values, say, a_n and b_n with probabilities p_n and q_n, satisfying

$$\prod_{n=1}^{\infty} \max\{p_n, q_n\} = 0.$$

In this case, the joint distribution $\mathbf{P}$ of ξ_n's represents a product probability measure on $\prod_{n=1}^{\infty}\{a_n, b_n\}$ without atoms. Let $a_n = -1$ and $b_n = 1$ (without loss of generality). Then, the map Q in the proof of Theorem 1 pushes $\mathbf{P}$ forward to a non-atomic probability measure λ on $[0,1]$, and a similar argument works.

Remark 4. The set $S = S(\Omega, \mathbf{P})$ of all random variables with values in the extended line $(-\infty, +\infty]$ represents a lattice with ordering $X \le Y$ a.s. Given an arbitrary non-empty collection $\{X(t)\}_{t \in T}$ in S, there is a unique element M in S, called the essential (or structural) supremum of the family $\{X(t)\}_{t \in T}$, with the properties that

a) $X(t) \le M$ (a.s), for all $t \in T$;
b) If for all $t \in T$ we have $X(t) \le M'$ (a.s.), $M' \in S$, then $M \le M'$ (a.s.)

It is a well-known general fact that M can be represented as a pointwise supremum $M = \sup_n X(t_n)$ a.s., for some sequence t_n in T (cf. e.g. [K-A]). In particular, the supremum in (2) may always be taken over all t's from a countable subset of T.

Acknowledgement. I am grateful to a referee for valuable comments and references.

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