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THE CHANCE OF A LONG LIFETIME FOR BROWNIAN MOTION IN A HORN-SHAPED DOMAIN

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Abstract
By means of a simple conditioning/comparison argument, we derive the chance of a long lifetime for Brownian motion in a horn-shaped domain.

1 Introduction

Recently several studies of killed Brownian motion in unbounded domains have appeared:

1. Consider the parabolic-type domain in $\mathbb{R}^d$, $d \geq 2$,

$$\mathcal{P}_p = \{(x_1, \dots, x_d) \in \mathbb{R}^d : \; x_d > 1 + A[x_1^2 + \cdots + x_{d-1}^2]^{p/2}\} \tag{1}$$

where $A > 0$ and $p > 1$. Bañuelos et al. (2001), Li (2003) and Lifshits and Shi (2002) studied the asymptotic behavior of the lifetime of killed Brownian motion in $\mathcal{P}_p$.

2. An *exterior domain* in $\mathbb{R}^d$, $d \geq 2$, is any domain with a compact complement. Collet et al. (2000) studied the long time behavior of the transition density of killed Brownian motion (Dirichlet heat kernel) in an exterior domain. They also derived the asymptotic behavior of the lifetime of killed Brownian motion for such a domain in two dimensions.

3. Let K be a closed proper subset of a hyperplane in $\mathbb{R}^d$, $d \geq 2$. The set $\mathbb{R}^d \backslash K$ is known as a *Benedicks domain*. Collet et al. (1999) and (2003) proved a ratio limit theorem for the Dirichlet heat kernel in a Benedicks domain. As a consequence of some of their estimates, they were also able to obtain asymptotics for the lifetime of killed Brownian motion in the domain.

4. M. van den Berg (2003) showed how subexponential behavior of the lifetime of killed Brownian motion in an unbounded domain implies subexponential behavior of the Dirichlet heat kernel. By combining his results with those of Lifshits and Shi (2002), he was able to derive asymptotics for the Dirichlet heat kernel in parabolic-type domains.

5. In cylindrical coordinates (r, θ, z) in $\mathbb{R}^3$, consider the horn-shaped region

$$H = \{(r, \theta, z) \colon\ 1 + z^2 < r\}.$$

Note H is obtained by revolving the parabolic region $\{(y, z) \in \mathbb{R}^2 \colon\ 1 + z^2 < y\}$ about the z-axis. Let τ_H be the exit time of Brownian motion from H. Collet et al. (2006) proved a ratio limit theorem for the Dirichlet heat kernel in H. As an added bonus from their proof, they were also able to show

$$\lim_{t \to \infty} t^{-1/3} \log P_x(\tau_H > t) = -C_{1/2} \tag{2}$$

where

$$C_{1/2} = \frac{3\pi^2}{8}.$$

Here P_x is probability associated with Brownian motion started at x. (Note: there is a minor error in the proof of this theorem concerning the exact value of $C_{1/2}$. Collet et al. (2006) study the operator $\frac{1}{2}\Delta$ and use results of van den Berg (2003). The subtle error results from the fact that van den Berg considers the operator Δ. Once this is accounted for, the correct value of $C_{1/2}$ is as stated above.)

The method of Collet et al. works for more general parabolic-type regions $\{(r, z) \colon\ 1 + |z|^p < r\}$, $p > 1$, so it seems (2) ought to take on the corresponding form

$$\lim_{t \to \infty} t^{(1-p)/(1+p)} \log P_x(\tau_H > t) = -B_p$$

where

$$B_p = (1 + p) \left[\frac{\pi^{1+2p}}{2^{3p+3}(p-1)^{p-1}} \left(\frac{\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p}{2})} \right)^2 \right]^{1/(1+p)}.$$

In this note we extend (2) to higher dimensions and more general parabolic-type horn-shaped domains. By a conditioning and comparison argument, our proof sidesteps the difficult estimates needed to derive the ratio limit theorem. Before stating the main result, we fix the notation.

For $d \geq 2$ let $(r, z, \theta) \in (0, \infty) \times \mathbb{R} \times S^{d-1}$ denote the cylindrical coordinates of a nonzero point $x = (\tilde{x}, x_{d+1}) \in \mathbb{R}^d \times \mathbb{R}$:

$$r = |\tilde{x}|, \quad z = x_{d+1}, \quad \theta = \frac{\tilde{x}}{r}.$$

Given $p > 1$ and $A > 0$, consider the horn-shaped domain

$$H_p = \{(r, z, \theta) \colon\ 1 + A|z|^p < r\}.$$

Denote by τ_p the exit time of Brownian motion from H_p. Our main result is the following theorem.

Theorem 1.1. *The exit time τ_p satisfies*

$$\lim_{t \to \infty} t^{(1-p)/(1+p)} \log P_x(\tau_p > t) = -C_{p,A} \tag{3}$$

where

$$C_{p,A} = (1 + p) \left[\frac{\pi^{1+2p} A^2}{2^{3p+3}(p-1)^{p-1}} \left(\frac{\Gamma(\frac{p-1}{2})}{\Gamma(\frac{p}{2})} \right)^2 \right]^{1/(1+p)}.$$

Notice the constant $C_{p,A}$ is independent of the dimension. In dimension $d = 2$, the results of Lifshits and Shi (2002) show that for the parabolic-type region

$$\mathcal{P}_p = \{(x, y):\ y > 1 + A|x|^p\},$$

the exit time $\eta_p(B)$ of two-dimensional Brownian motion B from $\mathcal{P}_p$ satisfies

$$\lim_{t\to\infty} t^{(1-p)/(1+p)} \log P_x(\eta_p(B) > t) = -C_{p,A}. \tag{4}$$

There is some interesting intuition connected with the equality of the limits in (3) and (4). Let X_2 be a d-dimensional Bessel process ($d \geq 3$) and let B_2 be one-dimensional Brownian motion. Let $\tau_{(a,\infty)}(X_2)$ and $\tau_{(a,\infty)}(B_2)$ be the exit times of X_2 and B_2, respectively, from the interval (a, ∞), $a \geq 1$. It is known (Feller (1971)) that for some $c_1 > 0$,

$$P_y(\tau_{(a,\infty)}(B_2) > t) \sim c_1 t^{-1/2} \quad \text{as} \quad t \to \infty,$$

where $f \sim g$ means $f/g \to 1$. It is easy to show for $d \geq 3$ that

$$P_y(\tau_{(a,\infty)}(X_2) = \infty) = \left(\frac{y}{a}\right)^{2-d}.$$

Thus adding a drift $\frac{d-1}{x_2}$ to one-dimensional Brownian motion significantly alters the chance of a long lifetime in (a, ∞), even to the extent that there is a nonzero chance the process never dies.

Next consider two-dimensional Brownian motion $B = (B_1, B_2)$ in the parabolic-type region $\mathcal{P}_p$ and let $X = (X_1, X_2)$ be the process resulting from addition of a vertical drift $\frac{d-1}{x_2}$. That is, X is associated with the differential operator

$$\frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}\frac{d-1}{x_2}\frac{\partial}{\partial x_2}.$$

In this two-dimensional case there are competing effects: first, the vertical drift $\frac{d-1}{x_2}$ tends to push X away from the boundary, trying to significantly increase the chance of a long lifetime. The recurrence of the horizontal component fights this effect. The natural question is to ask which effect dominates the other, if at all. Since the influence of the vertical drift on the vertical component is so strong, as suggested by the one-dimensional case described above, it is tempting to conjecture the overall chance of a long lifetime in $\mathcal{P}_p$ is increased because of the vertical drift $\frac{d-1}{x_2}$.

Since the Laplacian in $\mathbb{R}^{d+1}$ expressed in cylindrical coordinates is

$$\frac{\partial^2}{\partial r^2} + \frac{d-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{S^{d-1}} + \frac{\partial^2}{\partial z^2},$$

it is clear by symmetry that the lifetime of Brownian motion in the horn H_p is the same as the lifetime of X in $\mathcal{P}_p$. Thus (3) and (4) tell us the effect from the horizontal component tends to strongly cancel out the effect of the vertical component in the sense that the addition of a vertical drift of $\frac{d-1}{x_2}$ to a two-dimensional Brownian motion does not change the chance of a long lifetime, at least up to logarithmic equivalence. Any effect must be very fine indeed.

2 Proof of Theorem 1.1

With $X = (X_1, X_2)$ as defined at the end of the introduction, it suffices to prove

$$\lim_{t\to\infty} \log P(\eta_p(X) > t) = -C_{p,A},$$

where $\eta_p(X)$ is the first exit time of X from $\mathcal{P}_p$.

Lower Bound. For some two-dimensional Brownian motion $B = (B_1, B_2)$ we can write for $t < \eta_p(X)$

$$\begin{aligned} dX_1(t) &= dB_1(t) \\ dX_2(t) &= dB_2(t) + \frac{d-1}{2X_2(t)}dt. \end{aligned}$$

Then by the Comparison Theorem (Ikeda and Watanabe (1989))

$$P_y(\eta_p(B) > t) \le P_y(\eta_p(X) > t). \tag{5}$$

We will never use the processes X and B simultaneously within the same probability, so we will abuse the notation P_y to indicate the process inside, whatever it might be, starts at y. Combining (5) with (4), we get

$$-C_{p,A} \le \liminf_{t\to\infty} t^{(1-p)/(1+p)} \log P_y(\eta_p(X) > t). \tag{6}$$

Upper Bound. This is the heart of our argument. We must distinguish 2 cases: $d \ge 3$ and $d = 2$. First assume $d \ge 3$. Define

$$\begin{aligned} L_1 &= \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2} + \frac{1}{2}\frac{d-1}{x_2}\frac{\partial}{\partial x_2} \\ L_2 &= \frac{1}{2}\frac{\partial^2}{\partial x_1^2} + \frac{1}{2}\frac{\partial^2}{\partial x_2^2}. \end{aligned}$$

Then X and B from above are the processes associated with L_1 and L_2, respectively. Next, for $x = (x_1, x_2)$ set

$$\begin{aligned} \beta &= \frac{d-1}{2}, \\ V(x) &= -\frac{\beta(\beta-1)}{2}\frac{1}{x_2^2}, \\ h(x) &= x_2^\beta, \quad \text{and} \\ L &= L_2 + V. \end{aligned}$$

Notice h is L-harmonic: $Lh = 0$. The *h-transform* of L is defined to be

$$L^h f = \frac{1}{h}L(hf).$$

A simple computation shows

$$L^h = L_1. \tag{7}$$

Since $d \geq 3$, V is nonpositive and consequently there is a diffusion Y_t associated with L. Then if $p_X(t,x,y)$ and $p_Y(t,x,y)$ are the transition densities of X and Y, respectively, by (7) X is Y conditioned by h and

$$p_X(t,x,y) = p_Y(t,x,y)h(y)/h(x)$$

(Pinsky (1995), Theorem 4.1.1). By the Feynman–Kac formula, for any $\varepsilon > 0$,

$$\begin{aligned}
P_x(\eta_p(X) > t) &= \int_0^\infty p_X(t,x,y)dy \\
&= \frac{1}{h(x)} \int_0^\infty p_Y(t,x,y)h(y)dy \\
&= \frac{1}{h(x)} E_x[h(Y_t)I(\eta_p(Y) > t)] \\
&= \frac{1}{h(x)} E_x\left[\exp\left(\int_0^t V(B(s))ds\right) h(B(t))I(\eta_p(B) > t)\right] \\
&\leq \frac{1}{h(x)} E_x[h(B(t))I(\eta_p(B) > t)] \\
&\leq \frac{1}{h(x)} \left[E_x[h^{(1+\varepsilon)/\varepsilon}(B(t))]\right]^{\varepsilon/(1+\varepsilon)} [P_x(\eta_p(B) > t)]^{1/(1+\varepsilon)} \\
&= \frac{1}{h(x)} [E_x[(B_2(t))^{\beta(1+\varepsilon)/\varepsilon}]^{\varepsilon/(1+\varepsilon)} [P_x(\eta_p(B) > t)]^{1/(1+\varepsilon)} \\
&= \frac{1}{h(x)} t^{\beta/2} H\left(\frac{x}{\sqrt{t}}\right) [P_x(\eta_p(B) > t)]^{1/(1+\varepsilon)}
\end{aligned}$$

where

$$H(w) = \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty (u+w)^{\beta(1+\varepsilon)/\varepsilon} e^{-u^2/2} du\right]^{\varepsilon/(1+\varepsilon)}.$$

Taking the natural logarithm, dividing by $t^{(1-p)/(1+p)}$, letting $t \to \infty$ and using (4), we get

$$\limsup_{t\to\infty} t^{(1-p)/(1+p)} \log P_x(\eta_p(X) > t) \leq -\frac{1}{1+\varepsilon} C_{p,A}.$$

Then let $\varepsilon \to 0$ and combine with (6) to get the desired limiting behavior for the case $d \geq 3$. As for $d = 2$, let Z and X be the processes associated with L_1 for $d = 3$ and $d = 2$ respectively. Then by the Comparison Theorem,

$$P_x(\eta_p(X) > t) \leq P_x(\eta_p(Z) > t).$$

By the case $d = 3$, we get the desired upper bound for the case $d = 2$. $\square$

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