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A GAUSSIAN CORRELATION INEQUALITY AND ITS APPLICATIONS TO SMALL BALL PROBABILITIES

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Abstract:
We present a Gaussian correlation inequality which is closely related to a result of Schechtman, Schlumprecht and Zinn (1998) on the well-known Gaussian correlation conjecture. The usefulness of the inequality is demonstrated by several important applications to the estimates of small ball probability.

1 Introduction

The well-known Gaussian correlation conjecture states that for any two symmetric convex sets A and B in a separable Banach space E and for any centered Gaussian measure μ on E,

$$\mu(A \cap B) \geq \mu(A)\mu(B). \tag{1.1}$$

Various equivalent formulations, early history and recent progresses of the conjecture can be found in Schechtman, Schlumprecht and Zinn (1998). A special case of the conjecture, when one of the symmetric convex set is a slab of the form $\{x \in E : |f^*(x)| \leq 1\}$ for some linear functional f^* in the dual of E, was proved by Khatri (1967) and Šidák (1968) independently. The Khatri-Šidák result has many applications in probability and statistics (see Tong (1980)). Recently, it has became one of the most powerful tools in the lower bound estimates of the small ball probability which studies the behavior of

$$\log \mu(x : \|x\| \leq \varepsilon) = -\phi(\varepsilon) \quad \text{as} \quad \varepsilon \to 0\ . \tag{1.2}$$

for a given measure μ and a norm $\|\cdot\|$, see, for example, Shao (1993), Monrad and Rootzén (1995), and Talagrand (1993). For a recent comprehensive survey on small ball probability

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and its various applications, we refer the reader to Li and Shao (1999b). Other applications and connections between correlation type results and small ball probabilities can be found in Hitczenko, Kwapien, Li, Schechtman, Schlumprecht and Zinn (1998), Shao (1998), Li and Shao (1999a).

In this short note, we first present the following Gaussian correlation inequality which is closely related to a result of Schechtman, Schlumprecht and Zinn (1998) on the well-known Gaussian correlation conjecture given in (1.1).

Theorem 1.1 *Let μ be a centered Gaussian measure on a separable Banach space E. Then for any $0 < \lambda < 1$, any symmetric, convex sets A and B in E.*

$$\mu(A \cap B)\mu(\lambda^2 A + (1-\lambda^2)B) \geq \mu(\lambda A)\mu((1-\lambda^2)^{1/2}B).$$

In particular,

$$\mu(A \cap B) \geq \mu(\lambda A)\mu((1-\lambda^2)^{1/2}B).$$

The proof follows along the arguments of Proposition 3 in Schechtman, Schlumprecht and Zinn (1998) where the case $\lambda = 1/\sqrt{2}$ was proved. The full details are given in the next section. Below we give several important applications of the inequality to the estimates of small ball probability. Other applications of the inequality can be found in Li (1999a,b), Kuelbs and Li (1999), Li and Shao (1999a), Lifshits and Linde (1999). The varying parameter λ plays a fundamental role in all the applications we know so far. The main difference between the Khatri-Šidák inequality and our Theorem 1.1 in the applications to small ball probability is that the former only provides rate (upto a constant) and the later can preserve the rate together with the constant.

Our first application is to the small ball probability of the sum of two not necessarily independent joint Gaussian random vectors.

Theorem 1.2 *Let X and Y be any two joint Gaussian random vectors in a separable Banach space with norm $\|\cdot\|$. If*

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}(\|X\| \leq \varepsilon) = -C_X,$$

and

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}(\|Y\| \leq \varepsilon) = 0$$

with $0 < \gamma < \infty$ and $0 < C_X < \infty$. Then

$$\lim_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}(\|X+Y\| \leq \varepsilon) = -C_X.$$

Note that it is easy to show the above result if X and Y are independent. In fact, by Anderson's inequality and independent assumption,

$$\mathbb{P}(\|X+Y\| \leq \varepsilon) \leq \mathbb{P}(\|X\| \leq \varepsilon).$$

On the other hand, for any $0 < \delta < 1$, we have by the independent assumption

$$\begin{aligned} \mathbb{P}(\|X+Y\| \leq \varepsilon) &\geq \mathbb{P}(\|X\| \leq (1-\delta)\varepsilon, \|Y\| \leq \delta\varepsilon) \\ &= \mathbb{P}(\|X\| \leq (1-\delta)\varepsilon) \cdot \mathbb{P}(\|Y\| \leq \delta\varepsilon). \end{aligned}$$

Thus

$$\liminf_{\varepsilon \to 0} \varepsilon^\gamma \log \mathbb{P}(\|X+Y\| \leq \varepsilon) \geq -(1-\delta)^{-\gamma} C_X$$

and the result in the independent case follows by taking $\delta \to 0$. Now without independent assumption, our Theorem 1.1 still allows us to obtain both upper and lower estimates. The very simple proof is given in the next section. It is also interesting to point out that, without using the correlation inequality, we can only obtain the correct rate (without the exact constant) by the precise link, discovered in Kuelbs and Li (1993) and completed in Li and Linde (1998), between the function $\phi(\varepsilon)$ in (1.2) and the metric entropy of the unit ball K_μ of the Hilbert space H_μ generated by μ.
As a direct consequence of Theorem 1.2, we have the following for any Gaussian "bridge".

Corollary 1.1 *Let $\{X(t), 0 \le t \le 1\}$ be a $\mathbb{R}^d$-valued, $d \ge 1$, continuous Gaussian random variable. Assume for some norm $\|\cdot\|$ on $C([0,1],\mathbb{R}^d)$ that*

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}\left(\|X(t)\| \le \varepsilon\right) = -C_X$$

with $0 < \gamma < \infty$ and $0 < C_X < \infty$. Then

$$\lim_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}\left(\|X(t) - tX(1)\| \le \varepsilon\right) = -C_X.$$

Our next application extends the previous known small ball results for Brownian motion under weighted sup-norms over the finite interval to those over the infinite interval. Let $W(t)$, $t \ge 0$, be the standard Brownian motion. If $f : (0,T] \mapsto (0,\infty)$ satisfies either of the conditions (H1): $\inf_{0<t\le T} f(t) > 0$ or (H2): $f(t)$ is nondecreasing in a neighborhood of 0. Then,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{f(t)} \le \varepsilon\right) = -\frac{\pi^2}{8}\int_0^T f^{-2}(t)dt. \tag{1.3}$$

This result was proved by Mogulskii (1974) under essentially condition (H1) and by Berthet and Shi (1998) under condition (H2). The critical case, when $\int_0^T f^{-2}(t)dt = \infty$, and connections with Gaussian Markov processes were treated in Li (1998). Here we extend (1.3) to sup over the whole positive half line.

Theorem 1.3 *Let $g : (0,\infty) \mapsto (0,\infty]$ satisfies the conditions:*
(i). $\inf_{0<t<\infty} g(t) > 0$ or $g(t)$ is nondecreasing in a neighborhood of 0.
(ii). $\inf_{0<t<\infty} t^{-1}g(t) > 0$ or $t^{-1}g(t)$ is nonincreasing for t sufficiently large;
Then,

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) = -\frac{\pi^2}{8}\int_0^\infty g^{-2}(t)dt. \tag{1.4}$$

Here we use the convention $1/\infty = 0$ and hence we can recover (1.3) from (1.4) by taking $g(t) = f(t)$ for $t \le T$ and $g(t) = \infty$ for $t > T$. Results similar to Theorem 1.3 for Brownian motion under weighted L_p-norm, $1 \le p \le \infty$, are studied in Li (1999a).

Corollary 1.2 *For $p > 1$,*

$$\lim_{\varepsilon\to 0} \varepsilon^{2(p-1)/p} \log \mathbb{P}\left(\sup_{t\ge 0}(|W(t)| - t^{p/2}) \le \varepsilon\right) = -\frac{\pi^2}{8}\cdot k_p$$

where

$$k_p = \int_0^\infty \frac{dt}{(1+t^{p/2})^2} = \begin{cases} 1 & \text{if } p = 2 \\ 2p^{-1}\left(1 - 2p^{-1}\right)\pi \cdot \csc(2\pi/p) & \text{if } p > 1,\ p \ne 2 \end{cases} \tag{1.5}$$

The above result can be easily seen by using the following identity in law, given in Song and Yor (1987) and Revuz and Yor (1994), page 23,

$$\sup_{t\geq 0}(|W(t)| - t^{p/2}) \stackrel{d}{=} \sup_{t\geq 0}\left(\frac{|W(t)|}{1+t^{p/2}}\right)^{p/(p-1)} \tag{1.6}$$

In particular, when $p = 2$, we have in fact

$$\sup_{t\geq 0}(|W(t)| - t) \stackrel{d}{=} \sup_{t\geq 0}\left(\frac{|W(t)|}{1+t}\right)^{2} \stackrel{d}{=} \sup_{0\leq t\leq 1} B^2(t)$$

where $B(t)$ is the Brownian bridge. The evaluation of the integral in (1.5) follows from Gradshteyn and Ryzhik (1994), page 334. It is of some interests to note that the general relation, see Song and Yor (1987),

$$\mathbb{P}\left(\psi\left(\sup_{0\leq t\leq 1}|W(t)|\right)\leq x\right) \geq \mathbb{P}\left(\sup_{t\geq 0}\left(|W(t)| - \phi(t^{1/2})\right)\leq x\right)$$

does not provide sharp estimate as $x \to 0$ (missing the constant at the log level) for $\phi(t) = t^p$, $p > 1$, where $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ is the Young function and $\psi(t) = \sup_{s\geq 0}(ts - \phi(s))$ is the conjugate function.

Next we mention the corresponding results in higher dimensions. Let $\{W_d(t);\, t\geq 0\}$ denote standard d–dimensional Brownian motion ($d \geq 1$), and "$\|\cdot\|$" the usual Euclidean norms in $\mathbb{R}^d$.

Theorem 1.4 *If g is a positive function satisfying conditions (i) and (ii) of Theorem 1.3, then*

$$\lim_{\varepsilon\to 0}\varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<\infty}\frac{\|W_d(t)\|}{g(t)}\leq\varepsilon\right) = -\frac{j^2_{(d-2)/2}}{2}\int_0^\infty g^{-2}(t)dt$$

where $j_{(d-2)/2}$ is the smallest positive root of the Bessel function $J_{(d-2)/2}$ and $j_{-1/2} = \pi/2$.

Our next application is related to the theory of empirical processes where the Brownian bridge plays an important role. To see how our next result can be applied to weighted empirical processes, we refer to Csáki (1994).

Theorem 1.5 *Let $\{B_d(t);\, 0\leq t\leq 1\}$ be a standard $\mathbb{R}^d$-valued Brownian bridge, with $d\geq 1$. Assume that $\inf_{a\leq t\leq b} g(t) > 0$ for all $0 < a \leq b < 1$. If $\inf_{0\leq t\leq 1} g(t) > 0$ or both $(1-t)^{-1}g(t)$ and $(1-t)^{-1}g(1-t)$ are nondecreasing in a neighborhood of 0, then*

$$\lim_{\varepsilon\to 0}\varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<1}\frac{\|B_d(t)\|}{g(t)}\leq\varepsilon\right) = -\frac{j^2_{(d-2)/2}}{2}\int_0^\infty g^{-2}(t)dt$$

where $j_{(d-2)/2}$ is the smallest positive root of the Bessel function $J_{(d-2)/2}$ and $j_{-1/2} = \pi/2$.

This result was mentioned in Berthet and Shi (1998), without proof, under a slightly stronger condition. Here we can see easily that Theorem 1.5 follows from Theorem 1.4 by observing the relation in law

$$\sup_{0<t<1}\frac{\|B_d(t)\|}{g(t)} \stackrel{d}{=} \sup_{0<t<1}\frac{\|(1-t)W_d(t(1-t)^{-1})\|}{g(t)} \stackrel{d}{=} \sup_{0<t<\infty}\frac{\|W_d(t)\|}{(1+t)g(t(1+t)^{-1})}.$$

For other applications to Chung's functional laws and general moving boundaries, the results in Berthet and Shi (1998) under weighted sup-norm over finite intervals can all be extended to the corresponding results over the infinite intervals. We omit the details here.

2 Proofs of Theorems

Proof of Theorem 1.1. Based on a classic finite dimensional approximation procedure which can be found in Chapter 4 of Ledoux (1996), we only need to show theorem 1.1 for $\mu = \mu_n$, the standard Gaussian product measure on $E = \mathbb{R}^n$. For notational convenience, let $\eta = (1-\lambda^2)^{1/2}$.
Using the rotational invariance of the measure $\mu_n \times \mu_n$ for $(x, y) \mapsto (\lambda x + \eta y, \eta x - \lambda y)$, we have

$$\begin{aligned} \mu_n(\lambda A)\mu_n(\eta B) &= \int 1_A(\lambda^{-1}x)1_B(\eta^{-1}y)\mu_n(dx)\mu_n(dy) \\ &= \int 1_A(x+\lambda^{-1}\eta y)1_B(x-\eta^{-1}\lambda y)\mu_n(dx)\mu_n(dy) \\ &= \int \mu_n\left((A-\lambda^{-1}\eta y)\cap(B+\eta^{-1}\lambda y)\right)\mu_n(dy). \end{aligned} \tag{2.1}$$

Note that for $y \in \mathbb{R}^n$, $(A-\lambda^{-1}\eta y)\cap(B+\eta^{-1}\lambda y)$ is not empty if and only if there exists $z \in \mathbb{R}^n$ for which $\lambda z + \eta y \in \lambda A$ and $\eta z - \lambda y \in \eta B$ and that can only happen if $z \in \lambda^2 A + \eta^2 B$. Thus the integrand

$$I_n(y) = \mu_n\left((A-\lambda^{-1}\eta y)\cap(B+\eta^{-1}\lambda y)\right)$$

can only be non-zero on $\lambda^2 A + \eta^2 B = \lambda^2 A + (1-\lambda^2)B$. Next, by the Prékopa-Leindler theorem on log concave functions, see Schechtman, Schlumprecht and Zinn (1998) for related details, the integrand

$$I_n(y) = \int 1_A(x+\lambda^{-1}\eta y)\cdot 1_B(x-\eta^{-1}\lambda y)\mu_n(dx)$$

is log concave since both the indicator function of convex set and the density of μ_n are log concave, and the product of log concave functions are log concave. Further, a symmetric log concave function is maximized at zero, and hence the integrand $I_n(y)$ is dominated by the value at $y = 0$ which is $\mu_n(A\cap B)$. Putting things together, the integral in (2.1) is bounded by $\mu(A\cap B)\mu(\lambda^2 A + (1-\lambda^2)B)$ and the proof is finished.

Proof of Theorem 1.2. For the lower bound, we have by the correlation inequality with any $0 < \delta < 1$, $0 < \lambda < 1$,

$$\begin{aligned} \mathbb{P}(\|X+Y\| \le \varepsilon) &\ge \mathbb{P}(\|X\| \le (1-\delta)\varepsilon,\, \|Y\| \le \delta\varepsilon) \\ &\ge \mathbb{P}(\|X\| \le \lambda(1-\delta)\varepsilon)\cdot\mathbb{P}\left(\|Y\| \le (1-\lambda^2)^{1/2}\delta\varepsilon\right). \end{aligned}$$

Thus

$$\liminf_{\varepsilon\to 0} \varepsilon^\gamma \log \mathbb{P}(\|X+Y\| \le \varepsilon) \ge -(\lambda(1-\delta))^{-\gamma} C_X$$

and the lower bound follows by taking $\delta \to 0$ and $\lambda \to 1$.
For the upper bound, we have again by the correlation inequality with any $0 < \delta < 1$, $0 < \lambda < 1$,

$$\begin{aligned} \mathbb{P}\left(\|X\| \le \frac{\varepsilon}{(1-\delta)\lambda}\right) &\ge \mathbb{P}\left(\|X+Y\| \le \frac{\varepsilon}{\lambda},\, \|Y\| \le \delta\cdot\frac{\varepsilon}{(1-\delta)\lambda}\right) \\ &\ge \mathbb{P}(\|X+Y\| \le \varepsilon)\cdot\mathbb{P}\left(\|Y\| \le (1-\lambda^2)^{1/2}\delta\frac{\varepsilon}{(1-\delta)\lambda}\right). \end{aligned}$$

Thus

$$\limsup_{\varepsilon\to 0} \varepsilon^{\gamma} \log \mathbb{P}\left(\|X+Y\| \le \varepsilon\right) \le -(\lambda(1-\delta))^{\gamma} C_X$$

and the upper bound follows by taking $\delta \to 0$ and $\lambda \to 1$.

Proof of Theorem 1.3. Without loss of generality, we assume $\int_0^\infty g^{-2}(t)dt$ exists and is finite. The upper estimate follows easily from (1.3) by observing

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) &\le \limsup_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \varepsilon\right) \\ &= -\frac{\pi^2}{8}\int_0^T g^{-2}(t)dt\end{aligned}$$

for any $T>0$. Taking $T \to \infty$ gives the desired upper bound.
For the lower bound, we have by the correlation inequality with any $0<\lambda<1$ and $T>0$,

$$\begin{aligned}\mathbb{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) &= \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \varepsilon\ ,\ \sup_{T\le t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) \\ &\ge \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \lambda\varepsilon\right) \cdot \mathbb{P}\left(\sup_{T\le t<\infty} \frac{|W(t)|}{g(t)} \le (1-\lambda^2)^{1/2}\varepsilon\right).\end{aligned} \tag{2.2}$$

For the second term in the equation above, we have by using the time inversion representation $\{W(t), t>0\} = \{tW(1/t), t>0\}$ in law

$$\mathbb{P}\left(\sup_{T<t<\infty} \frac{|W(t)|}{g(t)} \le (1-\lambda^2)^{1/2}\varepsilon\right) = \mathbb{P}\left(\sup_{0<t\le 1/T} \frac{|W(t)|}{tg(1/t)} \le (1-\lambda^2)^{1/2}\varepsilon\right) \tag{2.3}$$

Combining (2.2) and (2.3), we obtain by (1.3),

$$\begin{aligned}\liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t<\infty} \frac{|W(t)|}{g(t)} \le \varepsilon\right) &\ge \liminf_{\varepsilon\to 0} \varepsilon^2 \log \mathbb{P}\left(\sup_{0<t\le T} \frac{|W(t)|}{g(t)} \le \lambda\varepsilon\right) \\ &\quad + \liminf_{\varepsilon\to 0} \varepsilon^2 \mathbb{P}\left(\sup_{0<t\le 1/T} \frac{|W(t)|}{tg(1/t)} \le (1-\lambda^2)^{1/2}\varepsilon\right) \\ &= -\lambda^{-2}\frac{\pi^2}{8}\int_0^T \frac{dt}{g^2(t)} - (1-\lambda^2)^{-1}\frac{\pi^2}{8}\int_0^{1/T} \frac{dt}{t^2 g^2(1/t)} \\ &= -\lambda^{-2}\frac{\pi^2}{8}\int_0^T \frac{dt}{g^2(t)} - (1-\lambda^2)^{-1}\frac{\pi^2}{8}\int_T^{\infty} \frac{dt}{g^2(t)}.\end{aligned}$$

Taking $T \to \infty$ first and then $\lambda \to 1$, we obtain the desired lower estimate and thus finish the whole proof.

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