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CONCENTRATION OF THE SPECTRAL MEASURE FOR LARGE MATRICES

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Abstract

We derive concentration inequalities for functions of the empirical measure of eigenvalues for large, random, self adjoint matrices, with not necessarily Gaussian entries. The results presented apply in particular to non-Gaussian Wigner and Wishart matrices. We also provide concentration bounds for non-commutative functionals of random matrices.

1 Introduction and statement of results

Consider a random $N \times N$ Hermitian matrix $\mathbf{X}$ with i.i.d. complex entries (except for the symmetry constraint) satisfying a moment condition. It is well known since Wigner [30] that the spectral measure of $N^{-1/2}\mathbf{X}$ converges to the semicircle law. This observation has been generalized to a large class of matrices, e.g. sample covariance matrices of the form $\mathbf{XRX}^*$ where $\mathbf{R}$ is a deterministic diagonal matrix ([20]), band matrices (see [5, 17, 22]), etc. For the Wigner case, this convergence has been supplemented by Central Limit Theorems, see [16] for the case of Gaussian entries and [18], [24] for the general case.
Our goal in this paper is to study deviations beyond the central limit theorem regime. For Gaussian entries, full large deviation principles have been derived in [4] (for the Wigner case) and [15] (for Wishart matrices, that is sample covariance matrices with $\mathbf{R} = I$). These papers are based on the explicit form of the joint law of the eigenvalues, and this technique does not seem to extend to either non-Gaussian entries or even to general sample covariance matrices. Still in the Gaussian case, [13] derived upper bounds of large deviations type for the spectral measure of band matrices, including certain sample covariance matrices. Her derivation is

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based on the tools in [9] who considered more general non-commutative functionals of Wigner matrices.
We propose here a different approach to the estimation of deviations from the typical behaviour, which is based on the concentration inequalities developed in recent years by M. Talagrand ([25]-[27]). While the adaptation of concentration inequalities to the setup considered here requires rather moderate technical effort, it provides a very powerful tool to study large random matrices which seems to have been overlooked in the rich literature on this subject. Unlike the approach of [9] and [13], there is no hope to obtain by this technique complementary lower bounds, and the constants determining the rates are clearly not optimal. On the other hand, the Gaussian assumption can be dispensed with, and the derivation is much easier. It is also worthwhile to note that the concentration inequalities we obtain are in the correct scale ; the speed of convergence of the probabilities of large deviations that we prove is the same as in the Gaussian Wigner case. In particular, in certain cases it yields all the tightness needed for a CLT statement.
In the rest of this section, we define precisely the model of random matrices we consider, state concentration bounds for these models, and show how these imply exponential decay of Wasserstein distance between the empirical spectral measure $\hat{\mu}^N$ and its mean. We also present a technical lemma, Lemma 1.2, which allows to transfer properties of a function $f : \mathbb{R} \mapsto \mathbb{R}$ to properties of the function $\int f d\hat{\mu}^N$ viewed as a function of the entries of the random matrix. In Section 1.2, we combine our basic concentration estimates with known results on the convergence of $\hat{\mu}^N$ in some classical random matrix models to deduce exponential convergence to an N-independent limit. Section 1.3 contains concentration results for a class of non-commutative functionals of an ensemble of independent random matrices. Finally, Section 2 is devoted to the proofs of our results.

1.1 Concentration for large inhomogeneous random matrices

Let $\mathcal{M}^{sa}_{N\times N}(\mathbb{C})$ be the set of complex entries $N \times N$ self-adjoint matrices. Let f be a real valued function on $\mathbb{R}$. f can as well be seen as a function from $\mathcal{M}^{sa}_{N\times N}(\mathbb{C})$ into $\mathcal{M}^{sa}_{N\times N}(\mathbb{C})$ if we set, for $M \in \mathcal{M}^{sa}_{N\times N}(\mathbb{C})$ so that $M = UDU^*$ for a diagonal real matrix D and a unitary matrix U,

$$f(M) = Uf(D)U^*$$

where $f(D)$ is the diagonal matrix with entries $(f(D_{11}), .., f(D_{NN}))$ and U^* denotes the (complex) adjoint of U.
Let tr denote the trace on $\mathcal{M}^{sa}_{N\times N}(\mathbb{C})$ given by $\mathrm{tr}(M) = \sum_{i=1}^N M_{ii}$, and set $\mathrm{tr}_N(M) := N^{-1}\mathrm{tr}(M)$. We shall first consider the concentration of the real valued random variable $\mathrm{tr}_N(f(\mathbf{X}_A))$ for inhomogeneous random matrices given by

$$\mathbf{X}_A = ((\mathbf{X}_A)_{ij})_{1\le i,j\le N}\,, \quad \mathbf{X}_A = \mathbf{X}_A^*, \quad (\mathbf{X}_A)_{ij} = \frac{1}{\sqrt{N}} A_{ij}\omega_{ij}$$

with

$$\begin{aligned} \omega &:= (\omega^R + i\omega^I) = (\omega_{ij})_{1\le i,j\le N} = (\omega^R_{ij} + \sqrt{-1}\omega^I_{ij})_{1\le i,j\le N} \quad , \quad \omega_{ij} = \bar{\omega}_{ji}, \\ A &= (A_{ij})_{1\le i,j\le N} \quad , \quad A_{ij} = \bar{A}_{ji}, \end{aligned}$$

$\{\omega_{ij}, 1 \le i \le j \le N\}$ are independent complex random variables with laws $\{P_{ij}, 1 \le i \le j \le$

$N\}$, P_{ij} being a probability measure on $\mathbb{C}$ with

$$P_{ij}(\omega_{ij} \in \bullet) = \int \mathbb{1}_{u+iv \in \bullet} P_{ij}^R(du) P_{ij}^I(dv)\,,$$

and A is a non-random complex matrix with entries $\{A_{ij}, 1 \leq i \leq j \leq N\}$ uniformly bounded by, say, a. When needed, we shall write $\mathbf{X}_A = \mathbf{X}_A(\omega)$. We let $\Omega_N = \{(\omega_{ij}^R, \omega_{ij}^I)\}_{1 \leq i \leq j \leq N}$, and denote by $\mathbb{P}^N$ the law $\mathbb{P}^N = \otimes_{1 \leq i \leq j \leq N} (P_{ij}^R \otimes P_{ij}^I)$ on Ω_N, with $P_{ii}^I = \delta_0$, and set $\mathbb{E}^N$ to be the corresponding expectation. The rather general form of the matrix $\mathbf{X}_A$ is chosen such that the concentration inequalities developed in this section allow much flexibility in their application to classical models (see Section 1.2 for details).

We shall also use the following notations and definitions.

For a compact set K, denote by $|K|$ its diameter, that is the maximal distance between two points of K. For a Lipschitz function $f : \mathbb{R}^k \mapsto \mathbb{R}$, we define the Lipschitz constant $|f|_{\mathcal{L}}$ by

$$|f|_{\mathcal{L}} = \sup_{x,y} \frac{|f(x) - f(y)|}{||x - y||},$$

where here and throughout, $||\cdot||$ denotes the Euclidean norm on $\mathbb{R}^k$.

We say that a measure ν on $\mathbb{R}$ satisfies the logarithmic Sobolev inequality with (not necessarily optimal) constant c if, for any differentiable function f,

$$\int f^2 \log \frac{f^2}{\int f^2 d\nu} d\nu \leq 2c \int |f'|^2 d\nu.$$

Recall that a measure ν satisfying the logarithmic Sobolev inequality possesses sub-Gaussian tails ([19]). Recall also that the Gaussian law [12], any probability measure ν absolutely continuous with respect to Lebesgue measure satisfying the Bobkov and Götze [6] condition (including $\nu(dx) = Z^{-1} e^{-|x|^\alpha} dx$ for $\alpha \geq 2$), as well as any distribution absolutely continuous with respect to them possessing a bounded above and below density, satisfies the logarithmic Sobolev inequality [19, Section 7.1], [14, Propriété 4.5].

Our main result is

Theorem 1.1 *a) Assume that the $(P_{ij}, i \leq j, i, j \in \mathbb{N})$ are uniformly compactly supported, that is that there exists a compact set $K \subset \mathbb{C}$ so that for any $1 \leq i \leq j \leq N$, $P_{ij}(K^c) = 0$. Assume f is convex and Lipschitz. Then, for any $\delta > \delta_0(N) := 8|K|\sqrt{\pi}a|f|_{\mathcal{L}}/N > 0$,*

$$\mathbb{P}^N \left(|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \mathbb{E}^N [\mathrm{tr}_N(f(\mathbf{X}_A))]| \geq \delta \right) \leq 4 \exp \Big\{ - \frac{N^2(\delta - \delta_0(N))^2}{16|K|^2 a^2 |f|_{\mathcal{L}}^2} \Big\}.$$

b) If the $(P_{ij}^R, P_{ij}^I, 1 \leq i \leq j \leq N)$ satisfy the logarithmic Sobolev inequality with uniform constant c, then for any Lipschitz function f, for any $\delta > 0$,

$$\mathbb{P}^N \left(|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \mathbb{E}^N [\mathrm{tr}_N(f(\mathbf{X}_A))]| \geq \delta \right) \leq 2 \exp \Big\{ - \frac{N^2 \delta^2}{8ca^2 |f|_{\mathcal{L}}^2} \Big\}.$$

Remark: Note that the concentration results stated in Theorem 1.1 above hold also in the case that $P_{ij}^I = \delta_0$ for all i, j, i.e. for inhomogeneous *real* Wigner matrices. Studying separately this case roughly improves the constants in the exponents by a factor 2. This comment applies to the other theorems in this paper.

The above theorem is based on Talagrand's results [26], and Ledoux [19]. To obtain it, we have to see $\mathrm{tr}_N f(\mathbf{X}_A)$ as a function of the entries $\{\omega_{ij}^R, \omega_{ij}^I, 1 \le i \le j \le N\}$ and prove the following lemma, which is at the heart of our results. We note that Lemma 1.2 is rather classical (in fact, L. Pastur has pointed out to us that part (a) is referred to as "Klein's lemma" in [21]), we provide a proof below for completeness.

Lemma 1.2 *a) If f is a real valued convex function on $\mathbb{R}$, it holds that $(\omega^R, \omega^I) \mapsto \mathrm{tr}(f(\mathbf{X}_A(\omega)))$ is convex.*
b) If f is a Lipschitz function on $\mathbb{R}$, $(\omega^R, \omega^I) \mapsto \mathrm{tr}(f(\mathbf{X}_A(\omega)))$ is a Lipschitz function on $\mathbb{R}^{N^2}$ with Lipschitz constant bounded by $2a|f|_{\mathcal{L}}$. If f is differentiable, we more precisely have

$$\sum_{1\le i\le j\le N} \left(\partial_{\omega_{ij}^R} \mathrm{tr}(f(\mathbf{X}_A))\right)^2 + \sum_{1\le i<j\le N} \left(\partial_{\omega_{ij}^I} \mathrm{tr}(f(\mathbf{X}_A))\right)^2 \le 4a^2 |f|_{\mathcal{L}}^2.$$

In Theorem 1.1, we proved that for any compactly supported family of measures $(P_{i,j}, i \le j)$, concentration was holding for convex functions, and obtained a sub-Gaussian speed for this concentration. Next, we push the argument to obtain concentration for non convex functions, up to a loss in the accuracy of the speed. Towards this end, let

$$||f||_{\mathcal{L}} = ||f||_\infty + |f|_{\mathcal{L}}, \quad \mathcal{F}_{\mathrm{lip}} = \{f : ||f||_{\mathcal{L}} \le 1\},$$

and for $\mathcal{K} \subset \mathbb{R}$ compact, let

$$\mathcal{F}_{\mathrm{lip},\mathcal{K}} := \{f : \mathrm{supp}(f) \subset \mathcal{K},\ f \in \mathcal{F}_{\mathrm{lip}}\}.$$

Theorem 1.3 *Let $(P_{i,j}, i \le j)$ be compactly supported probability measures on $K \subset\subset \mathbb{R}$. Fix $\delta_1(N) = 8|K|\sqrt{\pi}a/N$, $S = \max_{z\in K} |z|^2$ and $M = \sqrt{8Sa^2}$.*
a) Let $\mathcal{K} \in \mathbb{R}$ be compact. Then, for any $\delta \ge 4\sqrt{|\mathcal{K}|\delta_1(N)}$,

$$\begin{aligned} &\mathbb{P}^N\Big(\sup_{f\in\mathcal{F}_{\mathrm{lip},\mathcal{K}}} |\mathrm{tr}_N(f(\mathbf{X}_A)) - \mathbb{E}^N[\mathrm{tr}_N(f(\mathbf{X}_A))]| \ge \delta\Big) \\ &\qquad \le \frac{32|\mathcal{K}|}{\delta} \exp\left(-N^2 \frac{1}{16|K|^2a^2}\Big[\frac{\delta^2}{16|\mathcal{K}|} - \delta_1(N)\Big]^2\right). \end{aligned} \tag{1}$$

b) For any $\delta > (128(M+\sqrt{\delta})\delta_1(N))^{2/5}$,

$$\begin{aligned} &\mathbb{P}_A^N\left(\sup_{f\in\mathcal{F}_{\mathrm{lip}}} |\mathrm{tr}_N(f(\mathbf{X}_A)) - \mathbb{E}^N[\mathrm{tr}_N(f(\mathbf{X}_A))]| \ge \delta\right) \\ &\qquad \le \frac{128(M+\sqrt{\delta})}{\delta^{3/2}} \exp\left(-N^2 \frac{1}{16|K|^2a^2}\Big[\frac{\delta^{5/2}}{128(M+\sqrt{\delta})} - \delta_1(N)\Big]^2\right). \end{aligned}$$

The results above have direct implication on concentration for the empirical measure of eigenvalues with respect to the Wasserstein[2] distance, given for any probability measures (μ, ν) on $\mathbb{R}$ by

$$d(\mu,\nu) = \sup_{f\in\mathcal{F}_{\mathrm{lip}}} \Big|\int f d\mu - \int f d\nu\Big|.$$

Let $\hat{\mu}^N$ denote the empirical measure of the eigenvalues of $\mathbf{X}_A$, that is $\mathrm{tr}_N(f(\mathbf{X}_A)) = \int f(x)\hat{\mu}^N(dx)$ for every bounded Borel f. We have the following immediate corollary.

[2] also called Monge-Kantorovich-Rubinstein, see the historical comments in [11, Page 341–342].

Corollary 1.4 *a) With the assumptions and notations of Theorem 1.3 b), we have for* $\delta > (128(M+\sqrt{\delta})\delta_1(N))^{2/5}$,

$$\mathbb{P}^N(d(\hat{\mu}^N, \mathbb{E}^N\hat{\mu}^N) > \delta) \leq \frac{128(M+\sqrt{\delta})}{\delta^{3/2}} \exp\left(-N^2 \frac{1}{16|K|^2a^2}[\frac{\delta^{5/2}}{128(M+\sqrt{\delta})} - \delta_1(N)]^2\right).$$

b) Assume that $(P_{ij}, 1 \leq i,j \leq N)$ *satisfy the logarithmic Sobolev inequality with uniform constant* c. *Then, there exist positive universal constants* C_1 *and* C_2 *so that for any* $\delta > 0$,

$$\mathbb{P}^N\left(d(\hat{\mu}^N, \mathbb{E}^N(\hat{\mu}^N)) \geq \delta\right) \leq \frac{C_1}{\delta^{3/2}} \exp\Big\{-\frac{C_2}{ca^2}N^2\delta^5\Big\}.$$

Next, we also get the following extension.

Theorem 1.5 *Assume that* $(P_{ij}, 1 \leq i,j \leq N)$ *satisfy the logarithmic Sobolev inequality with uniform constant* c. *Then, for any probability measure* μ *on* $\mathbb{R}$, *for any* $\delta > 0$,

$$\mathbb{P}^N\left(d(\hat{\mu}^N, \mu) \geq \mathbb{E}^N\left(d(\hat{\mu}^N, \mu)\right) + \delta\right) \leq \exp\Big\{-\frac{N^2}{8ca^2}\delta^2\Big\}.$$

We remark here that we restricted ourselves in this paper to functionals of the spectral measure $\hat{\mu}^N$ of the matrix $\mathbf{X}_A$. However, concentration inequalities can as well be obtained to other quantities such as the spectral radius of $\mathbf{X}_A$ which is easily seen to be a convex Lipschitz function of the entries too.

1.2 Concentration of the spectral measure for classical matrices

In the previous section, we obtained concentration of the marginals of the spectral measure for inhomogeneous Wigner's matrices for which we do not know a priori that $\mathbb{E}[\mathrm{tr}_N(f(\mathbf{X}_A))]$ converges. Here, we shall specialize our discussion to some classical models for which such a convergence is already known. These include Wigner's matrices (with non necessarily Gaussian entries), band matrices, diluted matrices, and Wishart's ensembles including inhomogeneous Wishart matrices. In such models, the results of Section 1.1 will allow us to deduce from already known L^1 convergence, an exponential speed for this convergence. Hereafter, we shall state results under the natural normalization

$$\int x dP_{ij}(x) = 0, \qquad \int |x|^2 dP_{ij}(x) = 1. \tag{2}$$

Wigner's matrices

First, let us concentrate on Wigner's matrices. In this case, we have $A_{ij} = 1$ for all i,j, and we write $\mathbf{X}_A = \mathbf{X}_1$. Then, it is well known that if the P_{ij}'s satisfy the moments conditions (see [2] for instance)

$$\sup_{ij} \int |x|^4 dP_{ij}(x) < \infty, \tag{3}$$

one has

$$\lim_{N\to\infty} \mathbb{E}^N(\mathrm{tr}_N(f)) = \int f(x) d\sigma(x)$$

with σ the semicircle law

$$\sigma(dx) = \frac{1}{2\pi}\sqrt{4-x^2}\,1_{|x|\leq 2}dx. \tag{4}$$

Here, we shall restrict ourselves according to the previous section (see the assumptions of Theorem 1.1) to entries with all moments finite. Further, Bai (see [2] for real entries and [3, Theorem 3.6] for the complex case) proved that if F_N is the distribution function $F_N(x) = \mathbb{E}^N\left(\mathrm{tr}_N(1\!\!1_{\mathbf{X}_1 \leq x})\right)$, and F is the distribution function of the semicircle law (4), then there exists a finite constant C so that for any $N \in \mathbb{N}$,

$$\sup_{x \in \mathbb{R}} |F_N(x) - F(x)| \leq C N^{-\frac{1}{4}}.$$

As a direct consequence, it is not hard to verify that if f is differentiable with $f' \in L^1(\mathbb{R})$,

$$\delta_N(f) := |\mathbb{E}^N\left[\mathrm{tr}_N(f(\mathbf{X}_1))\right] - \int f(x) d\sigma(x)| \leq C |f'|_{L^1(\mathbb{R})} N^{-\frac{1}{4}}.$$

Recall that it is known since Wigner [30] that for any $k \in \mathbb{N}$, $\mathbb{E}^N\left[\mathrm{tr}_N((\mathbf{X}_1^{2k}))\right]$ can be bounded independently of N. For general Lipschitz functions, one can use an approximating scheme to see that for any $\epsilon > 0$, there exists a finite constant $C(\epsilon) > 0$ such that

$$\delta_N(f) \leq C(\epsilon) ||f||_{\mathcal{L}} N^{-\frac{1}{4}+\epsilon}.$$

We can therefore deduce from Theorem 1.1 the

Corollary 1.6 *a) Under the assumptions of Theorem 1.1 a) and (2), for any Lipschitz function f, for any $\delta > \delta_2(N) := 8|K|\sqrt{\pi}a|f|_{\mathcal{L}}/N + \delta_N(f) > 0$,*

$$\mathbb{P}^N\left(\left|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \int f(x) d\sigma(x)\right| \geq \delta\right) \leq 4 e^{-\frac{1}{16|K|^2 a^2 |f|_{\mathcal{L}}^2} N^2 (\delta - \delta_2(N))^2}.$$

b) Under the assumptions of Theorem 1.1 b) and (2), for any Lipschitz function f, for any $\delta > \delta_N(f)$,

$$\mathbb{P}^N\left(\left|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \int f(x) d\sigma(x)\right| \geq \delta\right) \leq 2 e^{-\frac{1}{8ca^2 |f|_{\mathcal{L}}^2} N^2 (\delta - \delta_N(f))^2}.$$

Of course, Corollary 1.4 and Theorem 1.5 also provide exponential convergence of Wasserstein's distance between the spectral measure of Wigner's matrices and the semicircle law. We emphasize that they apply both for real and complex Wigner matrices.

Now we turn to more interesting consequences of the possible inhomogeneity in the matrices studied in Section 1.1 and first to the most natural

Band matrices Let us recall that band matrices are matrices of type $\mathbf{X}_A$ with $A_{i,j}$ null for $|i-j|$ big enough. More generally, one can consider the sometimes called generalized deformed Wigner ensemble described by a sequence $\mathbf{X}_{A^N} \in \mathcal{M}_{N \times N}^{sa}(\mathbb{C})$, $N \in \mathbb{N}$, so that there exists a non random bounded continuous function ϕ so that

$$\lim_{N \to \infty} \sup_{u \in [0,1]} \int_0^1 du \Big| \sum_{i,j} A_{ij}^N 1\!\!1_{t \in [i/N, i+1/N[} 1\!\!1_{u \in [j/N, j+1/N[} - \phi(t,u)\Big| = 0.$$

We denote by $\mathbf{X}_\phi$ the matrix $\mathbf{X}_A$ for such a choice of A. It was observed since [7, 10, 17, 22, 13] under various hypotheses that the empirical measure of $\mathbf{X}_\phi$ converges when the $(P_{ij}, 1 \leq i \leq j \leq N)$ satisfy (2) and have some uniformly bounded moments. Let us recall the result from [17, section 8.3] for real symmetric $\mathbf{X}_\phi$ (see [5, 22, 13] for Hermitian matrices) which as far

as we know have the weakest hypotheses. Assuming that $\sup_{i,j} P_{i,j}[|x|^3]$ is finite, they found that if k_ϕ is the unique analytic function on $[0,1] \times \mathbb{C}\backslash\mathbb{R}$, so that

$$k_\phi(x,z) = (z - K_\phi(k_\phi(.,z))(x))^{-1}$$

where K_ϕ is the operator in $L^2([0,1])$ with kernel ϕ, for any $z \in \mathbb{C} \setminus \mathbb{R}$,

$$\lim_{N\to\infty} \mathbb{E}^N[\mathrm{tr}_N\left((z - \mathbf{X}_\phi)^{-1}\right)] = m_\phi(z) := \int_0^1 k_\phi(x,z)dx. \tag{5}$$

Let σ_ϕ be the unique probability measure with Stieljes transform m_ϕ. Let us recall how the convergence of $\mathbb{E}^N(\mathrm{tr}_N(f(\mathbf{X}_\phi)))$ for Lipschitz functions f can be deduced from the convergence of the Stieljes transform. The main observation is that if m is the Stieljes transform of a probability measure μ, $m(z) = \int (z-x)^{-1} d\mu(x)$, $z \in \mathbb{C}\backslash\mathbb{R}$, then, with $z = u + iv$ and $\Im(z) = v$ denoting the imaginary part of z,

$$\pi^{-1}\Im(m(z)) = \int_{-\infty}^{+\infty} \frac{v}{\pi\left((x-u)^2 + v^2\right)} \mu(dx) = \frac{dP_v * \mu}{du}(u) \tag{6}$$

with P_v the Cauchy law with parameter $v \neq 0$. Thus, for $f \in \mathcal{F}_{\mathrm{lip}} \cap L^1(\mathbb{R})$,

$$\begin{aligned}
&|\mathbb{E}^N[\mathrm{tr}_N(f(\mathbf{X}_\phi))] - \int f d\sigma_\phi| \\
&\leq |\int \mathbb{E}^N[\mathrm{tr}_N(f(\mathbf{X}_\phi + u))]P_v(du) - \int f dP_v * \sigma_\phi| + 4||f||_{\mathcal{L}} \int (|u| \wedge 1) P_v(du) \\
&= \pi^{-1}|\int f(u)\Im\left(\mathbb{E}^N[\mathrm{tr}_N((u + iv - \mathbf{X}_\phi)^{-1})] - m_\phi(u+iv)\right) du| + 4||f||_{\mathcal{L}} \int (|u| \wedge 1) P_v(du)
\end{aligned} \tag{7}$$

It is now clear from (5) that the first term in the right hand side of (7) goes to zero as N goes to infinity by dominated convergence for $v \neq 0$ and then that the second is as small as one wishes when v goes to zero. Hence, using the a-priori moment bounds on $\mathbf{X}_\phi$, for any $f \in \mathcal{F}_{\mathrm{lip}}$,

$$\lim_{N\to\infty} \mathbb{E}^N[\mathrm{tr}_N(f(\mathbf{X}_\phi))] = \int f d\sigma_\phi.$$

Further, this convergence can be seen to be uniform over $\mathcal{F}_{\mathrm{lip}}$ by considering more carefully the first integral in the right hand side of (7).
As a conclusion, for band-matrices, not only Theorems 1.1, 1.3, 1.5 and Corollary 1.4 hold but actually one has also the convergence of $\mathbb{E}^N[\mathrm{tr}_N f(\mathbf{X}_A)]$ (for $f \in \mathcal{F}_{\mathrm{lip}}$) and of $\mathbb{E}^N \hat{\mu}^N$.

In the same spirit, one can consider
Diluted random matrices Diluted random matrices appear in diluted models of spin glasses. They are described as matrices $\mathbf{X}_A$ with A now random with $\{0,1\}$-valued entries. Assume for simplicity that $\{A_{ij}, 1 \leq i \leq j \leq N\}$ are i.i.d with law $Q = p\delta_0 + (1-p)\delta_1$ (p eventually depending on N) and denote $Q^N = \otimes_{1\leq i\leq j\leq N} Q$ with E^N the corresponding expectation. Convergence of the empirical measure of $\mathbf{X}_A$ for such random matrices A is discussed in [17], p.193-197. We here want to stress the following concentration result

Corollary 1.7 *Assume that the $(P_{ij}, i \leq j, i, j \in \mathbb{N})$ are uniformly compactly supported, that is that there exists a compact set $K \subset \mathbb{C}$ so that for any $1 \leq i \leq j \leq N$, $P_{ij}(K^c) = 0$. Set*

$L = \max\{\sup_{x\in K} x^2, |K|^2\}$. *Assume f is convex and Lipschitz. Then, for any $\delta > \delta_3(N) := 4(|K| + \sup_{x\in K}|x|)\sqrt{\pi}|f|_{\mathcal{L}}/N > 0$,*

$$Q^N \otimes \mathbb{P}^N \left(|\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - E^N \mathbb{E}^N [\mathrm{tr}_N(f(\mathbf{X}_A))]| \geq \delta \right) \leq 8e^{-\frac{1}{64L|f|^2_{\mathcal{L}}} N^2(\delta - \delta_3(N))^2}.$$

Concentration for the Wasserstein distance can be derived as in Corollary 1.4.

We now turn to the well known

Wishart's matrices Let us recall that if $\mathbf{Y}$ is a $N \times M$ matrix, $N \leq M$, with independent entries $\omega_{kl} = (\omega^R_{kl} + i\omega^I_{kl})$ of law P_{kl}, $\mathbf{Z} = \mathbf{Y}\mathbf{Y}^*$ is a so-called Wishart's matrix. We let throughout $\mathbb{P}^{N,M} = \otimes_{1\leq k\leq N, 1\leq l\leq M} P_{kl}$. For the sake of completeness, let us consider inhomogeneous Wishart matrices given, for a diagonal real matrix $\mathbf{R} = (\lambda_1, ..\lambda_M)$ with $\lambda_i \geq 0$ by $\mathbf{Z} = \mathbf{YRY}^*$. To deduce the concentration of the spectral measure for such matrices from Theorem 1.1, note that if we consider

$$\begin{aligned}
A_{ij} &= 0 \quad \text{for } 1 \leq i \leq N, 1 \leq j \leq N \\
A_{ij} &= 0 \quad \text{for } M+1 \leq i \leq M+N, M+1 \leq j \leq N+M \\
A_{ij} &= \sqrt{\lambda_{i-M}} \quad \text{for } N+1 \leq i \leq M+N, 1 \leq j \leq N \\
A_{ij} &= \sqrt{\lambda_{j-M}} \quad \text{for } 1 \leq i \leq N, N+1 \leq j \leq N+M
\end{aligned}$$

then $A = A^*$ and if we consider $\mathbf{X}_A \in \mathcal{M}^{sa}_{(N+M)\times(N+M)}(\mathbb{C})$ constructed as in the previous section, $\mathbf{X}_A$ can be written as

$$\begin{pmatrix} 0 & \mathbf{YR}^{\frac{1}{2}} \\ \mathbf{R}^{\frac{1}{2}}\mathbf{Y}^* & 0 \end{pmatrix}.$$

Now, it is straightforward to see that $(\mathbf{X}_A)^2$ is equal to

$$\begin{pmatrix} \mathbf{YRY}^* & 0 \\ 0 & \mathbf{R}^{\frac{1}{2}}\mathbf{Y}^*\mathbf{YR}^{\frac{1}{2}} \end{pmatrix}$$

In particular, for any measurable function f,

$$\mathrm{tr}(f((\mathbf{X}_A)^2)) = 2\mathrm{tr}(f(\mathbf{YRY}^*)) + (M-N)f(0).$$

It is therefore a direct consequence of Theorem 1.1 that

Corollary 1.8 *With $(Y_{kl}, 1 \leq k \leq N, 1 \leq l \leq M)$ independent random variables, and with $\mathbb{P}^{N,M}$ as above, we let $\mathbf{R}$ be a non negative diagonal matrix with finite spectral radius ρ. Set $\mathbf{Z} = \mathbf{YRY}^*$. Then,*
a) If the $(P_{kl}, 1 \leq k \leq N, 1 \leq l \leq M)$ are supported in a compact set K, for any function f so that $g(x) = f(x^2)$ is convex and has finite Lipschitz norm $|g|_{\mathcal{L}} \equiv |||f|||_{\mathcal{L}}$, for any $\delta > \delta_0(N+M) := 4|K|\sqrt{\pi\rho}|||f|||_{\mathcal{L}}/(N+M)$,

$$\mathbb{P}^{N,M}\left(|\mathrm{tr}_N(f(\mathbf{Z})) - \mathbb{E}^{N,M}(\mathrm{tr}_N(f(\mathbf{Z})))| > \delta \frac{M+N}{N} \right) \leq 4e^{-\frac{1}{4|K|^2\rho|||f|||^2_{\mathcal{L}}}(\delta - \delta_0(N+M))^2(N+M)^2}.$$

b) If the $(P_{kl}, 1 \leq k \leq N, 1 \leq l \leq M)$ satisfy the logarithmic Sobolev inequality with uniformly bounded constant c, the above result holds for any Lipschitz functions $g(x) = f(x^2)$: for any $\delta > 0$,

$$\mathbb{P}^{N,M}\left(|\mathrm{tr}_N f(\mathbf{Z})) - \mathbb{E}^{N,M}(\mathrm{tr}_N f(\mathbf{Z}))| > \delta \frac{M+N}{N} \right) \leq 2e^{-\frac{1}{2c\rho|||f|||^2_{\mathcal{L}}}\delta^2(N+M)^2}.$$

The proof is in fact straightforward since, with the above remarks, one should see $f(\mathbf{Z})$ as $g(\mathbf{X}_A) = f((\mathbf{X}_A)^2)$ and thus control the Lipschitz norm of g and the convexity of g.

Remarks

1. The interest in Corollary 1.8 is in the case where N and M are large and N/M remains bounded and bounded away from zero.

2. For the convergence of the expected empirical measure of Wishart matrices, see [20], [23], and for rates, see [2].

3. Exponential convergence for the Wasserstein distance follows from our results by noting that the approximating set of functions in the proof of Theorem 1.3 below used convex increasing functions.

4. The results presented above extend to the case where $\mathbf{R}$ is self-adjoint non negative but not diagonal. Indeed, the key Lemma 1.2 generalizes as follows. Let Y, Y' denote $N \times M$ matrices with complex entries, with $||Y - Y'||^2 = \sum_{1 \le k \le N, 1 \le l \le M} |Y_{kl} - Y'_{kl}|^2$, writing $\mathbf{R} = U\widetilde{\mathbf{R}}U^*$ with U unitary, one has for $f \in \mathcal{F}_{\text{lip}}$ that

$$\mathrm{tr} f(Y\mathbf{R}Y^*) = \mathrm{tr} f(YU\widetilde{\mathbf{R}}U^*Y^*) = \mathrm{tr} f(U^*YU\widetilde{\mathbf{R}}U^*Y^*U)$$

Hence, with ρ denoting the spectral radius of $\mathbf{R}$, Lemma 1.2.b) and the observations above Corollary 1.8 imply

$$|\mathrm{tr} f(Y\mathbf{R}Y^*) - \mathrm{tr} f(Y'\mathbf{R}(Y')^*)| \le \frac{\sqrt{\rho}}{2}||U^*YU - U^*Y'U|| = \frac{\sqrt{\rho}}{2}||Y - Y'||\,.$$

Hence, viewed as a function of the entries of Y, $f(Y\mathbf{R}Y^*)$ is Lipschitz of constant $\sqrt{\rho}/2$. A similar argument applies to the convexity considerations.

1.3 Concentration for non-commutative functionals

In this section, we present concentration inequalities for non-commutative functionals encountered in free probability theory. Let $(\mathbf{X}^1_{A_1}, \ldots, \mathbf{X}^n_{A_n})$ be independent, $N \times N$ self adjoint inhomogeneous random matrices as in Section 1.1, with

$$(\mathbf{X}^p_{A_p})_{ij} = \frac{1}{\sqrt{N}}(\omega^p)_{ij}(A_p)_{ij}\,,$$

with the A_p self adjoint and the entries $(\omega^p)_{ij} = (\omega^p)^R_{ij} + \sqrt{-1}(\omega^p)^I_{ij}$ of law P^p_{ij} such that

$$P^p_{ij}((\omega^p)_{ij} \in \bullet) = \int \mathbb{1}_{u+iv \in \bullet} P^{p,R}_{ij}(du) P^{p,I}_{ij}(dv)\,.$$

We let $\mathbb{P}^{N,n} = \otimes_{1 \le i \le j \le N, 1 \le p \le n}(P^{p,R}_{ij} \otimes P^{p,I}_{ij})$, with respective expectation denoted by $\mathbb{E}^{N,n}$. In free probability theory, the law of $(\mathbf{X}^1_{A_1}, \ldots, \mathbf{X}^n_{A_n})$ is determined by the family $\mathrm{tr}[Q(\mathbf{X}^1_{A_1}, \ldots, \mathbf{X}^n_{A_n})]$ where Q are all possible (non-commutative) polynomials in n variables. In the particular case that $A_1 = \ldots = A_n = I$ and $P^p_{ij}(z) = 0$, $P^{p,R}_{ij}(x^2) = P^{p,I}_{ij}(x^2) = 1$, the law of $(\mathbf{X}^1_{A_1}, \ldots, \mathbf{X}^n_{A_n})$ is known to converge to the law of n free semicircular variables. When P^p_{ij} are Gaussian, a central limit theorem was obtained in [8]. In order to state a large deviations upper bound

(still in the Gaussian case), one cannot look at polynomials because these do not possess good enough exponential moments. Instead, in their study of large deviations upper bounds, T. Cabanal-Duvillard and one of the authors restricted themselves in [9] to the (separating) family of analytic, bounded functionals in the complex vector field generated by the family

$$\mathcal{F}_{\rm NC} = \Big\{ F(X_1, \dots, X_n) = \prod_{l=1}^{k} (z_l - \sum_{r=1}^{n} \alpha_r^l X_r)^{-1}, \quad z_l \in \mathbb{C} \setminus \mathbb{R}, \alpha_r^l \in \mathbb{R}, k \in \mathbb{N} \Big\} .$$

However, the machinery developed in [9] is restricted to Gaussian entries. Here, with a less sharp bound, we provide the following concentration inequality.

Theorem 1.9 *Assume the laws $P_{ij}^{p,R}, P_{ij}^{p,I}$ satisfy the logarithmic Sobolev inequality with uniform constant c. Then, for any $F \in \mathcal{F}_{\rm NC}$, and any $\delta > 0$,*

$$\mathbb{P}^{N,n}(|{\rm tr}_N F(\mathbf{X}_{A_1}^1, \dots, \mathbf{X}_{A_p}^p) - \mathbb{E}^{N,n} {\rm tr}_N F(\mathbf{X}_{A_1}^1, \dots, \mathbf{X}_{A_p}^p)| > \delta) \leq 2 \exp\Big\{ - \frac{\delta^2 N^2}{16 c a^2 ||F||_{\mathcal{L}}} \Big\} ,$$

where, if $F(X_1, \dots, X_n) = \prod_{l=1}^{k} (z_l - \sum_{r=1}^{n} \alpha_r^l X_r)^{-1}$,

$$||F||_{\mathcal{L}}^2 = \left[\sum_{p=1}^{n} \left(\sum_{l=1}^{k} |\alpha_l^p| \right) \left(\sum_{l=1}^{k} |\alpha_l^p| \cdot |\Im(z_l)|^{-2} \right) \right] \prod_{l=1}^{k} |\Im(z_l)|^{-2} .$$

2 Proofs

Let us first assume Lemma 1.2 and give the

Proof of Theorem 1.1

a) For a real valued random variable v, denote by M_v the median of v, i.e.

$$M_v = \sup\{t : P(v \leq t) \leq 1/2\} .$$

Following Talagrand (see [25]–[28] and Theorem 6.6 in [27]), for any $F : \mathbb{R}^{N^2} \to \mathbb{R}$ Lipschitz with Lipschitz constant bounded by one, if $\mathbb{P}^N$ is supported on $[-1, 1]^{N^2}$ and F is a convex function of (ω^R, ω^I), for any $\delta > 0$,

$$\mathbb{P}^N \left(|F - M_F| \geq \delta \right) \leq 4 e^{-\frac{\delta^2}{16}} .$$

Hence, taking $F(\omega^R, \omega^I) = {\rm tr}(f(\mathbf{X}_A(\omega)))$, F is convex and Lipschitz with Lipschitz norm bounded by $2a|f|_{\mathcal{L}}$ according to Lemma 1.2, and one gets by homothety that, for any $\mathbb{P}^N$ as in the statement of the theorem,

$$\mathbb{P}^N \left(|F - M_F| \geq \delta \right) \leq 4 e^{-\frac{\delta^2}{16 |K|^2 a^2 |f|_{\mathcal{L}}^2}} . \tag{8}$$

In particular,

$$|\mathbb{E}^N {\rm tr}_N f - \frac{M_F}{N}| \leq \mathbb{E}^N |{\rm tr}_N f - \frac{M_F}{N}| \leq 4 \int_0^\infty e^{-\frac{\delta^2}{16 |K|^2 a^2 |f|_{\mathcal{L}}^2}} d\delta = \delta_0(N) .$$

Substituting back into (8) yields the first part of Theorem 1.1.

b). In case where the $(P_{ij}^R, P_{ij}^I, 1 \leq i \leq j \leq N)$ satisfy a log-Sobolev inequality with uniformly bounded constants c, by the product property of the logarithmic Sobolev inequality, for any $F : \mathbb{R}^{N^2} \to \mathbb{R}$ differentiable, we have

$$\int F^2 \log \frac{F^2}{\int F^2 d\mathbb{P}^N} d\mathbb{P}^N \leq 2c \int ||\nabla F||^2 d\mathbb{P}^N,$$

where

$$||\nabla F||^2 = \sum_{1 \leq i \leq j \leq N} (\partial_{\omega_{ij}^R} F(\omega))^2 + \sum_{1 \leq i < j \leq N} (\partial_{\omega_{ij}^I} F(\omega))^2 .$$

Therefore, we can follow Herbst's argument (see Ledoux [19], proposition 2.3) to see that

$$\mathbb{P}^N \left(|F - \mathbb{E}^N(F)| > \delta \right) \leq 2e^{-\frac{\delta^2}{2ca^2 ||(||\nabla F||)||_\infty^2}}.$$

Taking $F(\omega) = \mathrm{tr}(f(\mathbf{X}_A(\omega)))$ gives, with Lemma 1.2.b),

$$\mathbb{P}^N \left(|\mathrm{tr}(f(\mathbf{X}_A)) - \mathbb{E}^N(\mathrm{tr}(f(\mathbf{X}_A)))| > \delta \right) \leq 2e^{-\frac{\delta^2}{8ca^2 ||f'||_\infty^2}}$$

for any differentiable functions f. The generalization to Lipschitz functions is obtained by approximations, the set of differentiable functions being dense in the set of Lipschitz functions for the Lipschitz norm.

□

Very similarly, we can give the

Proof of Corollary 1.7 : Indeed, again assuming Lemma 1.2, we see that if f is convex Lipschitz, $(A_{ij})_{1 \leq i \leq j \leq N} \to \mathbb{E}^N[\mathrm{tr}_N(f(\mathbf{X}_A))]$ is convex Lipschitz with Lipschitz constant bounded by $2 \sup_{x \in K} |x| |f|_{\mathcal{L}}$. Hence, applying Talagrand's result (8) twice, we obtain

$$\mathbb{P}^N \left(|\mathrm{tr}_N(f(\mathbf{X}_A)) - \mathbb{E}^N(\mathrm{tr}_N(f(\mathbf{X}_A)))| \geq \delta + \frac{8|K|\sqrt{\pi}|f|_{\mathcal{L}}}{N} \right) \leq 4e^{-\frac{1}{16|K|^2 |f|_{\mathcal{L}}^2} N^2 \delta^2} \tag{9}$$

and

$$\begin{aligned} Q^N \Big(|E^N(\mathrm{tr}_N(f(\mathbf{X}_A))) - E^N \mathbb{E}^N(\mathrm{tr}_N(f(\mathbf{X}_A)))| \geq \delta + \frac{8 \sup_{x \in K} |x| \sqrt{\pi} |f|_{\mathcal{L}}}{N} \Big) \\ \leq 4 \exp \Big\{ - \frac{1}{16 \sup_{x \in K} |x|^2 |f|_{\mathcal{L}}^2} N^2 \delta^2 \Big\}. \end{aligned} \tag{10}$$

Combining (9) and (10) give Corollary 1.7.

□

Thus, let us prove the key Lemma 1.2.

Proof of Lemma 1.2.a) Note that for any $Y, X \in \mathcal{M}_{N \times N}^{sa}(\mathbb{C})$,

$$f(Y) - f(X) = \int_0^1 Df(X + \eta(Y - X)) \sharp (Y - X) d\eta$$

where

$$Df(X) \sharp (Y) = \lim_{\epsilon \downarrow 0} \epsilon^{-1} (F(X + \epsilon Y) - F(X)).$$

For polynomial functions f, the non-commutative derivation D can easily be computed and one finds in particular that for any $p \in \mathbb{N}$,

$$Y^p - X^p = \int_0^1 \left(\sum_{k=0}^{p-1} (X + \eta(Y-X))^k (Y-X)(X+\eta(Y-X))^{p-k-1} \right) d\eta. \tag{11}$$

For such a polynomial function, and taking the trace, one deduces that

$$\begin{aligned} \mathrm{tr}(X^p) - \mathrm{tr}((\frac{X+Y}{2})^p) &= \int_0^1 p\mathrm{tr}\left((\frac{X+Y}{2} + \eta\frac{X-Y}{2})^{p-1} \frac{X-Y}{2} \right) d\eta\,, & (12) \\ \mathrm{tr}(Y^p) - \mathrm{tr}((\frac{X+Y}{2})^p) &= \int_0^1 p\mathrm{tr}\left((\frac{X+Y}{2} - \eta\frac{X-Y}{2})^{p-1} \frac{Y-X}{2} \right) d\eta\,. & (13) \end{aligned}$$

Hence, by further use of (11), (12) and (13) yield

$$\begin{aligned} \Delta &\equiv \mathrm{tr}(X^p) + \mathrm{tr}(Y^p) - 2\mathrm{tr}((\frac{X+Y}{2})^p) \\ &= \frac{p}{2} \sum_{k=0}^{p-2} \int_0^1 \int_0^1 \eta d\eta d\theta \mathrm{tr}\left((X-Y) Z_{\eta,\theta}^k (X-Y) Z_{\eta,\theta}^{p-2-k} \right) \end{aligned} \tag{14}$$

with

$$Z_{\eta,\theta} \equiv \frac{X+Y}{2} - \eta\frac{X-Y}{2} + \eta\theta(X-Y).$$

Next, for fixed $\eta, \theta \in [0,1]^2$, $X, Y \in \mathcal{M}_{N\times N}^{sa}(\mathbb{C})$, $Z_{\eta,\theta} \in \mathcal{M}_{N\times N}^{sa}(\mathbb{C})$, and we can find a unitary matrix $U_{\eta,\theta}$ and a diagonal matrix $D_{\eta,\theta}$ with real diagonal elements $(\lambda_{\eta,\theta}^1, .., \lambda_{\eta,\theta}^N)$ so that $Z_{\eta,\theta} = U_{\eta,\theta} D_{\eta,\theta} U_{\eta,\theta}^*$. Let $W_{\eta,\theta} = U_{\eta,\theta}^* (X-Y) U_{\eta,\theta}$. Then, the right hand side of (14) is given by

$$\begin{aligned} \Delta &= \frac{p}{2} \sum_{k=0}^{p-2} \int_0^1 \int_0^1 \eta d\eta d\theta \mathrm{tr}\left(W_{\eta,\theta} D_{\eta,\theta}^k W_{\eta,\theta} D_{\eta,\theta}^{p-2-k} \right) \\ &= \frac{p}{2} \int_0^1 \int_0^1 \eta d\eta d\theta \left(\sum_{k=0}^{p-2} \sum_{1\le i,j\le N} (\lambda_{\eta,\theta}^i)^k (\lambda_{\eta,\theta}^j)^{p-2-k} |W_{\eta,\theta}^{ij}|^2 \right). \end{aligned} \tag{15}$$

But

$$\sum_{k=0}^{p-2} (\lambda_{\eta,\theta}^i)^k (\lambda_{\eta,\theta}^j)^{p-2-k} = \frac{(\lambda_{\eta,\theta}^i)^{p-1} - (\lambda_{\eta,\theta}^j)^{p-1}}{\lambda_{\eta,\theta}^i - \lambda_{\eta,\theta}^j} = (p-1) \int_0^1 (\alpha\lambda_{\eta,\theta}^j + (1-\alpha)\lambda_{\eta,\theta}^i)^{p-2} d\alpha.$$

Hence, substituting in (15) gives,

$$\Delta = \frac{1}{2} \sum_{1\le i,j\le N} \int_0^1 \int_0^1 \int_0^1 d\alpha\eta d\eta d\theta |W_{\eta,\theta}^{ij}|^2 f''(\alpha\lambda_{\eta,\theta}^j + (1-\alpha)\lambda_{\eta,\theta}^i) \tag{16}$$

for the polynomial function $f(x) = x^p$. Now, with (X, Y) fixed, the eigenvalues $(\lambda^1_{\eta,\theta}, .., \lambda^N_{\eta,\theta})$ and the entries of $W_{\eta,\theta}$ are uniformly bounded. Hence, by Runge's theorem, we can deduce by approximation that (16) holds for any twice continuously differentiable function f. As a consequence, for any such convex function, $f'' \geq 0$ and

$$\Delta = \frac{1}{2}\mathrm{tr}(f(X)) + \frac{1}{2}\mathrm{tr}(f(Y)) - \mathrm{tr}(f(\frac{X+Y}{2})) \geq 0.$$

The generalization to arbitrary convex functions is then obtained by approximations. □

Proof of Lemma 1.2.b) Let us first consider a bounded continuously differentiable function f and show that $\omega \to \mathrm{tr}(f(\mathbf{X}_A(\omega)))$ is differentiable and bound the Euclidean norm of its gradient. More precisely, we shall prove that

$$\sum_{1\leq i\leq j\leq N} \left(\partial_{\omega^R_{ij}} \mathrm{tr} f(\mathbf{X}_A(\omega))\right)^2 + \sum_{1\leq i<j\leq N} \left(\partial_{\omega^I_{ij}} \mathrm{tr} f(\mathbf{X}_A(\omega))\right)^2 \leq 4a^2 |f|^2_{\mathcal{L}} \tag{17}$$

It is not hard to verify that

$$\partial_{\omega^R_{ij}} \mathrm{tr} f(\mathbf{X}_A) = \frac{1}{\sqrt{N}} \mathrm{tr}(f'(\mathbf{X}_A)\Delta_{ij}) \in \mathbb{R} \tag{18}$$

with $(\Delta_{ij})_{kl} = a_{kl}$ if $(kl) = (ij)$ or (ji) and $(\Delta_{ij})_{kl} = 0$ otherwise. In fact, this is a consequence of (11) for polynomial functions and can then be obtained for continuously differentiable functions by approximations. In other words, (18) shows that

$$\partial_{\omega^R_{ij}} \mathrm{tr} f(\mathbf{X}_A) = \begin{cases} \frac{1}{\sqrt{N}}[a_{ij} f'(\mathbf{X}_A)_{ij} + a_{ji} f'(\mathbf{X}_A)_{ji}], & \text{if } i \neq j \\ \frac{a_{ii}}{\sqrt{N}} f'(\mathbf{X}_A)_{ii}, & \text{if } i = j. \end{cases}$$

Hence,

$$\begin{aligned} \sum_{1\leq i\leq j\leq N} \left(\partial_{\omega^R_{ij}} \mathrm{tr} f(\mathbf{X}_A)\right)^2 &\leq \frac{2}{N} \sum_{1\leq i,j\leq N} |a_{ij} f'(\mathbf{X}_A)_{ij}|^2 \\ &\leq \frac{2a^2}{N} \sum_{1\leq i,j\leq N} |f'(\mathbf{X}_A)_{ij}|^2 \\ &= 2a^2 \mathrm{tr}_N\left(f'(\mathbf{X}_A) f'(\mathbf{X}_A)^*\right). \end{aligned} \tag{19}$$

But if $(\lambda_1, .., \lambda_N) \in \mathbb{R}^N$ denotes the eigenvalues of $\mathbf{X}_A$,

$$\mathrm{tr}_N\left(f'(\mathbf{X}_A) f'(\mathbf{X}_A)^*\right) = \frac{1}{N}\sum_{i=1}^N f'(\lambda_i)^2 \leq ||f'||^2_\infty$$

so that with (19), we conclude

$$\sum_{1\leq i\leq j\leq N} \left(\partial_{\omega^R_{ij}} \mathrm{tr} f(\mathbf{X}_A)\right)^2 \leq 2a^2 ||f'||^2_\infty.$$

The same argument applies for the derivatives with respect to ω^I_{ij}, and we find

$$|\mathrm{tr}(f(\mathbf{X}_A(\omega))) - \mathrm{tr}(f(\mathbf{X}_A(\omega')))| \leq 2a|f|_{\mathcal{L}} ||\omega - \omega'||$$

according to integration by parts formula and Cauchy-Schwartz 's inequality. Note that this last result for differentiable functions extends to Lipschitz functions by approximations. □

Proof of Theorem 1.3 a) Without loss of generality, we assume that $\min\{x : x \in \mathcal{K}\} = 0$. We will approximate a function $f \in \mathcal{F}_{\mathrm{lip},\mathcal{K}}$ by a combination of a finite number of convex functions from $\mathcal{F}_{\mathrm{lip}}$. Towards this end, fix $\Delta = \delta/4$, and define the function

$$g(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 \leq x \leq \Delta \\ \Delta, & x \geq \Delta\,. \end{cases}$$

We note that while $g(\cdot)$ is not convex, it is given as the sum of two convex functions in $\mathcal{F}_{\mathrm{lip}}$. Define recursively $f_\Delta(x) = 0$ for $x \leq 0$ and

$$f_\Delta(x) = \sum_{i=0}^{\lceil x/\Delta \rceil} (2\mathbb{1}_{f((i+1)\Delta) > f_\Delta(i\Delta)} - 1) g(x - i\Delta)\,.$$

Note that $||f - f_\Delta|| < \Delta$, and at most $2|\mathcal{K}|/\Delta$ different convex functions $\{h_k\}$ from $\mathcal{F}_{\mathrm{lip}}$ were used in this approximation, regardless of the particular function $f \in \mathcal{F}_{\mathrm{lip},\mathcal{K}}$. Thus,

$$\begin{aligned} &\mathbb{P}^N \left(\sup_{f \in \mathcal{F}_{\mathrm{lip},\mathcal{K}}} |\mathrm{tr}(f(\mathbf{X}_A)) - \mathbb{E}^N(\mathrm{tr}(f(\mathbf{X}_A)))| > \delta \right) \\ &\quad \leq \frac{2|\mathcal{K}|}{\Delta} \sup_k \mathbb{P}^N \left(|\mathrm{tr}(h_k(\mathbf{X}_A)) - \mathbb{E}^N(\mathrm{tr}(h_k(\mathbf{X}_A)))| > \frac{\Delta(\delta - 2\Delta)}{2|\mathcal{K}|} \right) \\ &\quad \leq \frac{8|\mathcal{K}|}{\Delta} \exp\left(-N^2 \frac{1}{16|K|^2 a^2} [(\frac{\Delta}{2|\mathcal{K}|}(\delta - 2\Delta) - \delta_1(N))_+]^2 \right)\,, \end{aligned}$$

where we have used Theorem 1.1 in the last inequality. Substituting $\Delta = \delta/4$ yields the conclusion.
b) Note that since the $P_{i,j}$'s are compactly supported in K, we have

$$\mathrm{tr}_N(\mathbf{X}_A^2) = N^{-2} \sum_{i,j} |A_{ij}\omega_{ij}|^2 \leq Sa^2.$$

Thus, for any $M_1 > 0$, $\mathrm{tr}_N(\mathbb{1}_{|\mathbf{X}_A| > M_1}) \leq Sa^2/M_1^2$. Let now

$$f^{M_1}(x) = \begin{cases} f(x), & |x| < M_1 \\ f(M_1) - \mathrm{sign}(f(M_1))(x - M_1), & M_1 < x < M_1 + |f(M_1)| \\ f(-M_1) + \mathrm{sign}(f(-M_1))(x + M_1), & -M_1 - |f(-M_1)| < x < -M_1 \\ 0, & \text{otherwise}\,. \end{cases}$$

Then,

$$|\mathrm{tr}_N f(\mathbf{X}_A) - \mathbb{E}^N \mathrm{tr}_N f(\mathbf{X}_A)| \leq |\mathrm{tr}_N f^{M_1}(\mathbf{X}_A) - \mathbb{E}^N \mathrm{tr}_N f^{M_1}(\mathbf{X}_A)| + 4\frac{Sa^2}{M_1^2}\,.$$

Take now $M_1 = M/\sqrt{\delta}$. Then, using that $4Sa^2/M^2 \leq 1/2$, and setting $\mathcal{K} = [-(M_1+1), (M_1+$

1)], one sees that $f^{M_1} \in \mathcal{F}_{\mathrm{lip},\mathcal{K}}$ and

$$\begin{aligned}
&\mathbb{P}^N\Big(\sup_{f\in\mathcal{F}_{\mathrm{lip}}} |\mathrm{tr}_N f(\mathbf{X}_A) - \mathbb{E}^N \mathrm{tr}_N f(\mathbf{X}_A)| > \delta\Big) \\
&\quad \le \mathbb{P}^N\Big(\sup_{g\in\mathcal{F}_{\mathrm{lip},\mathcal{K}}} |\mathrm{tr}_N g(\mathbf{X}_A) - \mathbb{E}^N \mathrm{tr}_N g(\mathbf{X}_A)| > \delta/2\Big) \\
&\quad \le \frac{128(M+\sqrt{\delta})}{\delta^{3/2}} \exp\left(-N^2 \frac{1}{16|K|^2a^2}\Big[\frac{\delta^{5/2}}{128(M+\sqrt{\delta})} - \delta_1(N)\Big]^2\right),
\end{aligned}$$

where part a) of the theorem was used in the last inequality. □

Proof of Corollary 1.4 The first part is immediate from Theorem 1.3. The second part follows from Theorem 1.1 using the same approximation by a fixed basis of Lipschitz functions as in Theorem 1.3. □

Proof of Theorem 1.5 This type of result is rather common in this field and comes from the great generality of concentration inequalities. In fact,

$$F(\omega) = d(\mathrm{tr}_N, \mu)(\omega) = \sup_{f\in\mathcal{F}_{\mathrm{lip}}} |\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \int f d\mu|$$

can be seen to be Lipschitz since for any $\omega, \omega' \in \mathbb{R}^{\frac{N(N+1)}{2}}$,

$$\begin{aligned}
F(\omega) &= \sup_{f:||f||_{\mathcal{L}}\le 1} |\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \mathrm{tr}_N(f(\mathbf{X}_A(\omega'))) + \mathrm{tr}_N(f(\mathbf{X}_A(\omega'))) - \int f d\mu| \\
&\le F(\omega') + \sup_{f:||f||_{\mathcal{L}}\le 1} |\mathrm{tr}_N(f(\mathbf{X}_A(\omega))) - \mathrm{tr}_N(f(\mathbf{X}_A(\omega')))| \\
&\le F(\omega') + 2a||\omega - \omega'||
\end{aligned}$$

where we have used in the last line lemma 1.2.b). By symmetry, we deduce that F is Lipschitz. Since Herbst's argument used in the proof of Theorem 1.1 extends to Lipschitz functions by density, it immediately gives Theorem 1.5. □

Proof of Theorem 1.9 : As in the proof of Theorem 1.1.b), we only need to show that, if $F(X_1, \ldots, X_n) = \prod_{l=1}^k (z_l - \sum_{r=1}^n \alpha_r^l X_r)^{-1}$, $G : (\mathbb{R}^{N^2})^n \mapsto \mathbb{C}$ given by

$$G(\omega^1, .., \omega^n) := \mathrm{tr}_N\left(F(\mathbf{X}_{A_1}^1(\omega_1), .., \mathbf{X}_{A_n}^n(\omega_n))\right)$$

is Lipschitz and bound its Lipschitz norm. In fact, G is differentiable and for any $1 \le i \le j \le N$, $1 \le p \le n$,

$$\partial_{\omega_{ij}^{p,R}} G(\omega^1, .., \omega^n) = \sum_{l=1}^k \alpha_p^l \mathrm{tr}_N\left(\prod_{m=1}^l (z_m - \sum_{r=1}^n \alpha_r^m X_r)^{-1} \Delta_{ij} \prod_{m=l}^k (z_m - \sum_{r=1}^n \alpha_r^m X_r)^{-1}\right)$$

with $(\Delta_{ij})_{kl} = a_{kl}^p$ if $(kl) = (ij)$ or (ji) and $(\Delta_{ij})_{kl} = 0$ otherwise. Thus, as in the proof of

Lemma 1.2.b), we deduce by Jensen's inequality

$$\sum_{1\le i\le j\le N} \left|\partial_{\omega_{ij}^{p,R}} G(\omega^1,..,\omega^n)\right|^2$$
$$\le 2a^2 \left(\sum_{l=1}^k |\alpha_p^l|\right) \sum_{l=1}^k |\alpha_p^l| \mathrm{tr}_N \left|\prod_{m=l}^k (z_m - \sum_{r=1}^n \alpha_r^m X_r)^{-1} \prod_{m=1}^l (z_m - \sum_{r=1}^n \alpha_r^m X_r)^{-1}\right|^2$$

where for any $N\times N$ matrix M, $|M|^2 = MM^*$. Noticing that the operator norm of $(z-M)^{-1}$ is bounded by $\Im(z)^{-1}$ for any self-adjoint matrix M, we conclude

$$\sum_{1\le i\le j\le N} \left|\partial_{\omega_{ij}^{p,R}} G(\omega^1,..,\omega^n)\right|^2 \le 2a^2 \left(\sum_{l=1}^k |\alpha_p^l|\right) \sum_{l=1}^k |\alpha_p^l| \prod_{m=1}^k |\Im(z_m)|^{-2} |\Im(z_l)|^{-2}$$

By similar computations for the derivations with respect to the $\omega_{ij}^{p,I}$ and summing over p, we complete the proof of Theorem 1.9. □

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